数理解析研究所講究録
第 92 巻 1970 年 1-9

CAUCHY PROBLEM FOR THE NON-LINEAR TRANSFER OF RADIATION IN FINITE ATMOSPHERES WITH TWO-LEVEL ATOMS

by

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Summary

In this paper the non-linear scattering processes in an inhomogeneous, finite, and plane-parallel atmosphere consisting of two-level atoms are discussed. Stimulated emission and redistribution in frequency of radiation energy are allowed for. With the aid of the method of self-consistent optical depths, the non-linear transfer problem is reduced to the solution of linear integral equation, and furthermore, the initial value method based on the invariant imbedding principle is used to convert the two-point boundary value problem to the Cauchy problem.

I. Introduction

Recently, much attentions of astrophysicists in the field of radiative transfer have been focussed on the non-linear LTE problems in the formation of stellar spectra (cf. Thomas (1965); Avrett, Gingerich, and Whitney (1965); Hummer and Rybicki (1967); Jeffries (1968)). Because in the real stellar atmospheric conditions the radiation fields depend not only upon the local optical properties of the media, but also on the radiation intensity impinging on the point under consideration.

In a manner similar to that in radiative transfer in homogeneous finite atmospheres, the invaraint imbedding technique (cf. Bellman and Kalaba (1956), Sobolev (1956), and Ueno(1960)), and the combined operations method (cf. Busbridge (1961)) have also been applied to the problems of diffuse reflection and transmission of light in an inhomogeneous finite atmosphere.

Furhtermore, methods using differential equations such as the generalized Riccati transformation (cf. Rybicki and Usher (1966))and the discretization method (cf. Feautrier (1964)) have been presented for the interpretation of line formation in stellar atmospheres,whereas the kernel approximation method (cf. Avrett and Loeser (1966)) and the flux derivative method (cf. Athay and Skumanich (1967)) have been used as an alternative integral equation approach. On the other hand, the initial value method has been applied to the theory of Fredholm integral equation with kernels reducible to symmetric kernels. (cf. Kagiwada , Kalaba,and Schumitzky (1967))

In Byurakan school the self-consistent optical depth method introduced by Ambarzumian (1964) makes it possible in some cases to linearize the problems of non-linear scattering processes with the aid of real optical depth as the independent variable. In such a non-linear theory the real optical depth of a point under discussion depends upon the radiation field and can be determined only after the entire problem has been solved (cf. Engibaryan (1965); Engibaryan (1966); Terebizh (1967)).

In this paper, with the aid of the self-consistent optical depth method the non-linear transfer problem is reduced to the solution of Fredholm integral equation and furthermore the initial value method based on the invariant imbedding is used to convert the two-point boundary value problem to the Cauchy problem.

II. The equation of transfer

Consider a plane-parallel, isotropically and non-coherently scattering atmosphere of the geometrical thickness Z_0 with the internal source distribution $B(\tau)$. We inquire into the non-linear scattering processes of radiation, whose excited and ground states are respectively denoted by 1 and 2 . In other words, we consider solutions of the restricted two-level problem of line formation with complete redistribution.

Let I_ν be the specific intensity of the radiation along the direction given by the element of length ds . Assuming the frequency-independent source function S , the equation of transfer appropriate to this case takes the form

$$\frac{dI_\nu}{ds} = -k_\nu(x)\, n_1 \left(1 - \frac{n_2}{n_1}\frac{g_1}{g_2}\right)(I_\nu - S)\,, \tag{1}$$

where $k_\nu(x)$ is the atomic absorption coefficient expressed in terms of the dimensionless frequency x, $n_i(i=1,2)$ and $g_i(i=1,2)$ represent respectively the populations of i-th levels and the corresponding statistical weights, and

$$S = \sigma\left(\frac{n_1}{n_2}\frac{g_2}{g_1} - 1\right)^{-1}, \tag{2}$$

$$\sigma = 2h\nu^3/c^2\,. \tag{3}$$

The total number density, (n_1+n_2), is assumed known, whereas n_1 and n_2 are inquired into. The frequency ν is the transition frequency between the two levels.

Put

$$ds = dz/v\,, \tag{4}$$

where z is the geometrical depth and v is the cosine of the inclination with respect to the outward normal.

Eq.(1) should be solved subject to the boundary conditions

$$I_\nu(0,+v) = 0\,, \qquad I_\nu(z_0,-v) = 0\,, \tag{5}$$

where $0 < v \leq 1$, i.e., no incident radiation upon the upper and lower boundary surfaces.

Writing

$$k_\nu(x) = k_0\, k(x)\,, \qquad k_0 = \frac{h\nu}{4\pi} B_{12}\,, \tag{6}$$

and

(7) $$x = (\nu - \nu_0)/\Delta\nu_D ,$$

where k_0 is the atomic absorption coefficient at the center of the line , ν_0 is the frequency at the line center, and $\Delta\nu_D$ is the Doppler width, which may have the thermal and turbulent components, i.e.

(8) $$\Delta\nu_D = \nu_0 \left(\frac{2kT}{mc^2} + \frac{V^2}{c^2} \right)^{1/2} .$$

The real optical depth at the line center is

(9) $$\tau = \int_0^{z_0} k_0 \left(1 - \frac{n_2}{n_1}\frac{g_1}{g_2}\right) n_1 \, dz .$$

On the other hand, the limiting optical thickness $\bar{\tau}_0$ is given by

(10) $$\bar{\tau}_0 = \int_0^{z_0} k_0 (n_1 + n_2) dz = \tau_0 + \gamma \int_0^{\tau_0} S(t, \tau_0) dt ,$$

where

(11) $$\gamma = \sigma^{-1} \left(1 + \frac{g_2}{g_1} \right) ,$$

and $S(t, \tau_0)$ is the required source function. The limiting optical depth of a point is its real optical depth in the case when all the intensities tend to zero.

Eq.(9) shows that the real optical thickness τ_0 at $z=z_0$ can not be determined until we get the source function, whereas the value of the limiting optical thickness $\bar{\tau}_0$ is given.

Let Einstein coefficients for absorption, stimulated and spontaneous emissions be respectively denoted by B_{12}, B_{21}, and A_{21}, and furthermore let the rate of inelastic and super-elastic collisional transitions be respectively denoted by C_{12} and C_{21}.

Starting with the statistical equilibrium equation for the two level atoms

(12) $$n_2 (A_{21} + B_{21}\bar{J} + C_{21}) = n_1 (B_{12}\bar{J} + C_{12}),$$

where

(13) $$\bar{J} = \frac{1}{4\pi} \int d\omega \int d\nu \, k(x) I_\nu ,$$

we get the required Milne integral equation governing the line-source function

(14) $$S(\tau,\tau_0) = (1-\lambda(\tau))B(\tau) + \frac{\lambda(\tau)}{2}\int_0^{\tau_0} K(|t-\tau|)S(t,\tau_0)dt \, ,$$

where

(15) $$\lambda(\tau) = \left(1 + \frac{C_{21}}{A_{21}}\left(1-e^{-h\nu/kT}\right)\right)^{-1} ,$$

and

(16) $$B(\tau) = \sigma\left(e^{h\nu/kT} - 1\right)^{-1} .$$

In the restricted two-level problem λ and B depend only on the kinetic temperature and the electron density.

Clearly, the solution of eq.(14) is equal to that of the transfer equation (1) with the boundary conditions (5).

In eq.(14) the kernel K(t) is given by

(17) $$K(t) = A\int_{-\infty}^{\infty} k^2(x)E_1(k(x)t)\,dx \qquad (t>0) ,$$

where k(x) is given by eq.(6) and $E_1(t)$ is the first exponential integral for positive argument t,

(18) $$E_1(t) = \int_0^1 e^{-t/y}\frac{dy}{y} \qquad (t>0) .$$

In eq.(17) A is the normalization constant

(19) $$A\int_{-\infty}^{\infty} k(x)\,dx = 1 .$$

III. Cauchy system for the Milne integral equation

Putting

(20) $$J(\tau,\tau_0) = S(\tau,\tau_0)/\lambda(\tau) ,$$

and

(21) $$L_\tau\{f(t)\} = \frac{1}{2}\int_0^{\tau_0} K(|t-\tau|)\lambda(t)f(t)\,dt ,$$

from eq.(14) we have

(22) $$[1-L]_\tau\{J(t,\tau_0)\} = \overline{B}(\tau) ,$$

where 1 is an identity operator, and

(23) $$\bar{B}(\tau) = (1-\lambda(\tau))B(\tau)/\lambda(\tau),$$

and λ and B are tabular functions of τ.

In a manner similar to that given by Kagiwada, Kalaba, and Schumitzky (1968), we shall derive the initial value problem for the integral equation (22).

On differentiating eq.(22) with respect to τ_0, we get

(24) $$J_{\tau_0}(\tau,\tau_0) = \frac{1}{2}K(\tau_0-\tau)\lambda(\tau_0)J(\tau_0,\tau_0) + L_\tau\{J_{\tau_0}(t,\tau_0)\}.$$

Introduce the $\Phi(\tau,\tau_0)$-function as the solution of the integral equation

(25) $$\Phi(\tau,\tau_0) = K(\tau_0-\tau) + L_\tau\{\Phi(t,\tau_0)\}.$$

The comparison of eqs.(24) and (25) provides us with

(26) $$J_{\tau_0}(\tau,\tau_0) = \frac{1}{2}\lambda(\tau_0)J(\tau_0,\tau_0)\Phi(\tau,\tau_0).$$

This is a required differential equation expressed in terms of $J(\tau_0,\tau_0)$ and $\Phi(\tau,\tau_0)$.

Eq.(25) is rewritten in the form

(27) $$\Phi(\tau,\tau_0) = A\int_{-\infty}^{\infty} k^2(x)E_1(k(x)(\tau_0-\tau))dx + L_\tau\{\Phi(t,\tau_0)\}.$$

Introduce the auxiliary equation

(28) $$M(\tau,\tau_0;V,x) = k(x)e^{-k(x)(\tau_0-\tau)/V} + L_\tau\{M(t,\tau_0;V,x)\}.$$

Then, $\Phi(\tau,\tau_0)$ may be expressed in terms of M-function

(29) $$\Phi(\tau,\tau_0) = \int_{-\infty}^{\infty}\int_0^1 A\,k(x)\,M(\tau,\tau_0;V,x)\,dx\,dV/V.$$

On differentiating eq.(28) with respect ro τ_0, we get

(30) $$M_{\tau_0}(\tau,\tau_0;V,x) = -\frac{k^2(x)}{V}e^{-k(x)(\tau_0-\tau)/V} + \frac{1}{2}\lambda(\tau_0)K(\tau_0-\tau)M(\tau_0,\tau_0;V,x) + L_\tau\{M_{\tau_0}(t,\tau_0;V,x)\}.$$

Recalling eq.(25), eq.(30) is converted into the following expression

(31) $$[1-L]_\tau\{f(t)\}=0$$

where

(32) $$f(t) = M_{\tau_0}(\tau,\tau_0;v,x) + \frac{k(x)}{v}M(\tau,\tau_0;v,x) - \frac{1}{2}\lambda(\tau_0)\Phi(\tau,\tau_0)M(\tau_0,\tau_0;v,x).$$

Since f(t) is a solution of the homogeneous Milne type integral equation, we have

(33) $$M_{\tau_0}(\tau,\tau_0;v,x) = -\frac{k(x)}{v}M(\tau,\tau_0;v,x) + \frac{1}{2}\lambda(\tau_0)\Phi(\tau,\tau_0)M(\tau_0,\tau_0;v,x).$$

The expression for $M(\tau_0, \tau_0 ; v,x)$ takes the form

(34) $$M(\tau_0,\tau_0;v,x) = k(x) + \frac{A}{2}\int_{-\infty}^{\infty}k^2(y)dy\int_0^1\frac{du}{u}\int_0^{\tau_0}\lambda(t)e^{-k(y)(\tau_0-t)/u}M(t,\tau_0;v,x)dt.$$

Introducing the scattering function

(35) $$R(\tau_0;v,u;x,x_0) = k(x)\int_0^{\tau_0}\lambda(t)e^{-k(x)(\tau_0-t)/v}M(t,\tau_0;u,x_0)dt,$$

and interchanging the orders of integration in eq.(34), we obtain

(36) $$M(\tau_0,\tau_0;v,x) = k(x) + \frac{A}{2}\int_{-\infty}^{\infty}k(y)dy\int_0^1 R(\tau_0;v,u;x,y)\frac{du}{u}.$$

The differentiation of eq.(35) with respect to τ_0 provides us with

(37) $$R_{\tau_0}(\tau_0;v,u;x,x_0) = -\left(\frac{k(x)}{v}+\frac{k(x_0)}{u}\right)R + k(x)k(x_0)\lambda(\tau_0)X(\tau_0;v,x)X(\tau_0;u,x_0),$$

where

(38) $$X(\tau_0;v,x) = 1 + \frac{A}{2}\int_{-\infty}^{\infty}\frac{k^2(y)}{k(x)k(y)}dy\int_0^1 R(\tau_0;u,v;y,x)\frac{du}{u} = M(\tau_0,\tau_0;v,x)/k(x).$$

Except for the notation, eq.(37) reduces to that given by Bellman, Kalaba and Ueno (1962).

The initial condition comes from eq.(35) such that

(39) $$R(0;v,u;x,x_0) = 0.$$

Furthermore, it should be mentioned that eq.(37) is derived under the assumption of the principle of reciprocity

(40) $$R(\tau_0;v,u;x,x_0) = R(\tau_0;u,v;x_0,x).$$

Via eq.(36), $M(\tau_0;\tau_0;v,x)$ is expressed in terms of S-function. Then, the solution, $M(\tau,\tau_0;v,x)$, of the auxiliary equation is computed by eq.(33), allowing for eq.(29). Hence, the evaluation of the modified source function $J(\tau,\tau_0)$ with the aid of eq.(26) needs the determination of $J(\tau_0,\tau_0)$.

Recalling eq.(22), we have

(41) $$J(\tau_0,\tau_0) = \bar{B}(\tau_0) + \frac{A}{2}\int_{-\infty}^{\infty}\int_0^1 e(\tau_0,v,x)k^2(x)dx\frac{dv}{v},$$

where the emergence function

(42) $$e(\tau_0, \nu, x) = \int_0^{\tau_0} e^{-k(x)(\tau_0 - t)/\nu} \lambda(t) J(t,\tau_0) dt.$$

On the other hand, comparing eqs.(22) and (25), we get

(43) $$\int_0^{\tau_0} \lambda(t) J(t,\tau_0) K(\tau_0 - t) dt = \int_0^{\tau_0} \Phi(t,\tau_0) \bar{B}(t) \lambda(t) dt.$$

Then, we have an alternative expression of the emergence function as below:

(44) $$e(\tau_0, \nu, x) = \int_0^{\tau_0} \lambda(t) \bar{B}(t) M(t,\tau_0;\nu,x) dt.$$

Once the R-function has been given by eq.(37), making use of eqs.(20), (26),(29),(33),(41), and (44), the source function $S(\tau,\tau_0)$ under consideration can be determined.

Assuming that the particles have Maxwellian velocity distributions corresponding to a kinetic temperature T, the collisional transition rates are provided by

(45) $$C_{12} = n_2^* C_{21} / n_1^*$$

where $n_i^*(i=1,2)$'s represent equilibrium number densities according to Boltzmann equation

(46) $$(n_2^* / n_1^*) = (g_2/g_1) e^{-h\nu/kT}$$

On inserting the thermodynamical equilibrium approximation given by eq.(46) into eq.(9), we can approximate the real optical thickness τ_0. Then, via eqs, (20), (26), (29), (33), (41), and (44), we can get the first approximate value of the source functions $S(\tau,\tau_0)$, which is used for the evaluation of the amended real optical thickness via eq.(10). Hence, such an iterative procedure will be continued until the satisfactory coincidence of the real optical thickness.

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