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REGULARITY OF HYPERFUNCTION SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS

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§1. The theory of hyperfunctions seems proving its usefulness in analysis thanks to the works by Profs. Martineau and Komatsu, and a number of young mathematicians.

Hyperfunctions are defined as follows in case of dimension 1. Take real axis $\mathbb{R}$ embedded in the complex plane $\mathbb{C}$. Take an open set I of $\mathbb{R}$ and take an open set D of $\mathbb{C}$ that contains I as a closed set. Such D will be called a complex neighborhood of I. Then by definition the space of hyperfunctions on I, $\mathcal{B}(I)$, is the quotient group of spaces of holomorphic functions on $D-I$ and D:

$$\mathcal{B}(I) = \mathcal{O}(D-I)/\mathcal{O}(D) .$$

Incidentally, this is equivalent to saying that $\mathcal{B}(I)$ is the 1^{st} cohomology group of D relative to $D-I$, with $\mathcal{O}$ as its coefficient group: $\mathcal{B}(I) = H^1(D \bmod D-I, \mathcal{O})$; namely, a hyperfunction is nothing but a 1^{st} relative cohomology class of $D \bmod D-I$ of holomorphic functions. It is shown that (i) $\mathcal{B}(I)$ is inherent to I, being independent of the choice of its complex neighborhood D, (ii) The presheaf $I \mapsto \mathcal{B}(I)$ is a sheaf: $\mathcal{B}(I) = \Gamma(I, \mathcal{B})$ and, (iii) The sheaf $\mathcal{B}$ is flabby.

Now the definition of hyperfunctions on an oriented real analytic manifold M of arbitrary dimension n: Let X be a complex neighborhood or a complexification of M. Then the space of hyperfunctions on M is:

$$\mathcal{B}(M) =_{def} H^n(X \bmod X-M, \mathcal{O}) ,$$

with $\mathcal{O}$ = sheaf of holomorphic functions on X. Again, our notion of hyperfunctions enjoys the properties just mentioned: $\mathcal{B}(M)$ is inherent to M; the presheaf $U \mapsto \mathcal{B}(U)$ is a flabby sheaf on M. $\mathcal{B}$ naturally contains as a subsheaf the sheaf of distributions and hence the sheaf $\mathcal{A}$ of real analytic functions on M which is the restriction of sheaf $\mathcal{O}$ onto M.

§2. Recently there has been a new development of hyperfunction theory ([1], [2], [3]) which makes it possible to describe and analyse in detail the structure of a hyperfunction by means of the cotangential <u>sphere</u> bundle S^*M. Here S^*M is the quotient space of T^*M — (the zero section) divided by $\mathbb{R}^+$, the group of <u>positive</u> real numbers. (It is important that we deal with the S^*M constructed in this way, and not with the cotangential <u>projective</u> bundle obtained by division by all non-zero real numbers.) S^*M is a (2n-1)-dimensional manifold equipped with a projection map π: $S^*M \to M$ whose fibers are (n-1)-spheres S^{n-1}.

Consider first the case $n = 1$. Here each fiber of π: $S^*M \to M$ is a 0-sphere S^0 which consists just of 2 points. Hence $S^*M = M \sqcup M$, the direct union or the disjoint union of 2 copies of M. On the other hand an implication of our definition of hyperfunctions of dimension 1 is that a hyperfunction is expressed as a sum of two 'ideal' boundary values of a holomorphic function $\varphi \in \mathcal{O}(D-I)$: $f(x) = \varphi(x+i0)+(-\varphi(x-i0))$. Hence, if we denote with $\tilde{\mathcal{A}}_+$ and $\tilde{\mathcal{A}}_-$ the sheaves over $\mathbb{R}$ consisting of 'ideal' boundary values from the upper and the lower half plane respectively, we have

$$\mathcal{B} = \tilde{\mathcal{A}}_+ + \tilde{\mathcal{A}}_- \quad , \quad \text{and} \quad \tilde{\mathcal{A}}_+ \cap \tilde{\mathcal{A}}_- \simeq \mathcal{A}$$

or equivalently, $0 \to \mathcal{A} \to \tilde{\mathcal{A}}_+ \oplus \tilde{\mathcal{A}}_- \to \mathcal{B} \to 0$ where $\mathcal{A}$ denotes the sheaf of real analytic functions on $\mathbb{R}$. This exact sequence yields at once

$$\mathcal{B}/\mathcal{A} \simeq \tilde{\mathcal{A}}_+/\mathcal{A} \oplus \tilde{\mathcal{A}}_-/\mathcal{A} \, .$$

This means the sheaf $\mathcal{B}/\mathcal{A}$ that measures the degree of irregularity of hyperfunctions, can be decomposed into two independent components. If one restricts the above observations to an appropriate subsheaf of $\mathcal{B}$, say the sheaf of locally L_p functions with $p > 1$, then we have a decomposition of the sheaf of such functions into two components; and this is what is known as the function space of class H_p in Fourier analysis. The case $p = 1$ is excluded because this class of functions (or rather, of hyperfunctions) is not stable under this decomposition. Similarly the sheaves $\mathcal{E}$ (C^∞ functions) and $\mathcal{D}'$ (distributions) are both subsheaves of $\mathcal{B}$ which are stable under this decomposition; we can talk about decomposition of the quotient sheaves $\mathcal{E}/\mathcal{A}$, $\mathcal{D}'/\mathcal{A}$ and hence, also about decomposition of $\mathcal{D}'/\mathcal{E} = (\mathcal{D}'/\mathcal{A})/(\mathcal{E}/\mathcal{A})$.

All these things are elementary. We mention however the following points: 1st, it is the quotient sheaf $\mathcal{B}/\mathcal{A}$, and not the sheaf $\mathcal{B}$ itself, which is subject to a decomposition in a natural way independent of the choice of coordinate system. 2nd, flabbiness of $\mathcal{B}$ together with cohomological trivialness of $\mathcal{A}$ immediately implies flabbiness of $\mathcal{B}/\mathcal{A}$. 3rd, in higher dimensions, S*M has a connected fiber of sphere. This means that decomposition of the sheaf $\mathcal{B}/\mathcal{A}$ is not described as a mere direct sum. We need a new language to describe it; and this new language is provided by the notion of <u>direct images</u> of sheaves.

Let $\mathcal{G}$ be a sheaf on a space Y and let $f: Y \to X$ be a morphism or a continuous map. Then the (0-th) direct image of $\mathcal{G}$ is by definition the sheaf $f_*\mathcal{G}$ over X characterized by the formula $\Gamma(U, f_*\mathcal{G}) = \Gamma(f^{-1}U, \mathcal{G})$ valid for every open set U of X. Since the functor $f_*: \mathcal{G} \mapsto f_*\mathcal{G}$ is left exact, it is natural to introduce the (q-th) right derived functor $\mathbb{R}^q f_*$ of f_*, and this is nothing but to introduce the sheaf $\mathbb{R}^q f_*\mathcal{G} = \mathcal{H}^q_f\mathcal{G}$ over X called q-th direct image of $\mathcal{G}$ which is obtained from the presheaf $U \mapsto H^q(f^{-1}U, \mathcal{G})$. We shall say that the map f is purely r-dimensional with respect to $\mathcal{G}$ if $\mathbb{R}^q f_*\mathcal{G} = 0$ unless $q = r$. For instance f is purely 0-dimensional with respect to any flabby sheaf over Y, and the 0-th direct image is again flabby.

§3. Now the decomposition of $\mathcal{B}/\mathcal{A}$ is attained in the following manner. A sheaf over S*M, which we shall call sheaf $\mathcal{C}$, will be constructed in a natural manner (as will be described in §4) and in such a way that the 0-th direct image of $\mathcal{C}$ by the projection map $\pi: S^*M \to M$ is canonically isomorphic to $\mathcal{B}/\mathcal{A}$. In other words, we have an exact sequence of natural homomorphisms

$$0 \longrightarrow \mathcal{A} \xrightarrow{\alpha} \mathcal{B} \xrightarrow{\beta} \pi_*\mathcal{C} \longrightarrow 0 .$$

Moreover, the map π is purely 0-dimensional with respect to $\mathcal{C}$, and the image $\pi_*\mathcal{C}$ is a flabby sheaf on M. Far beyond these facts, M. Kashiwara established the decisive result that the sheaf $\mathcal{C}$ itself is a flabby sheaf.

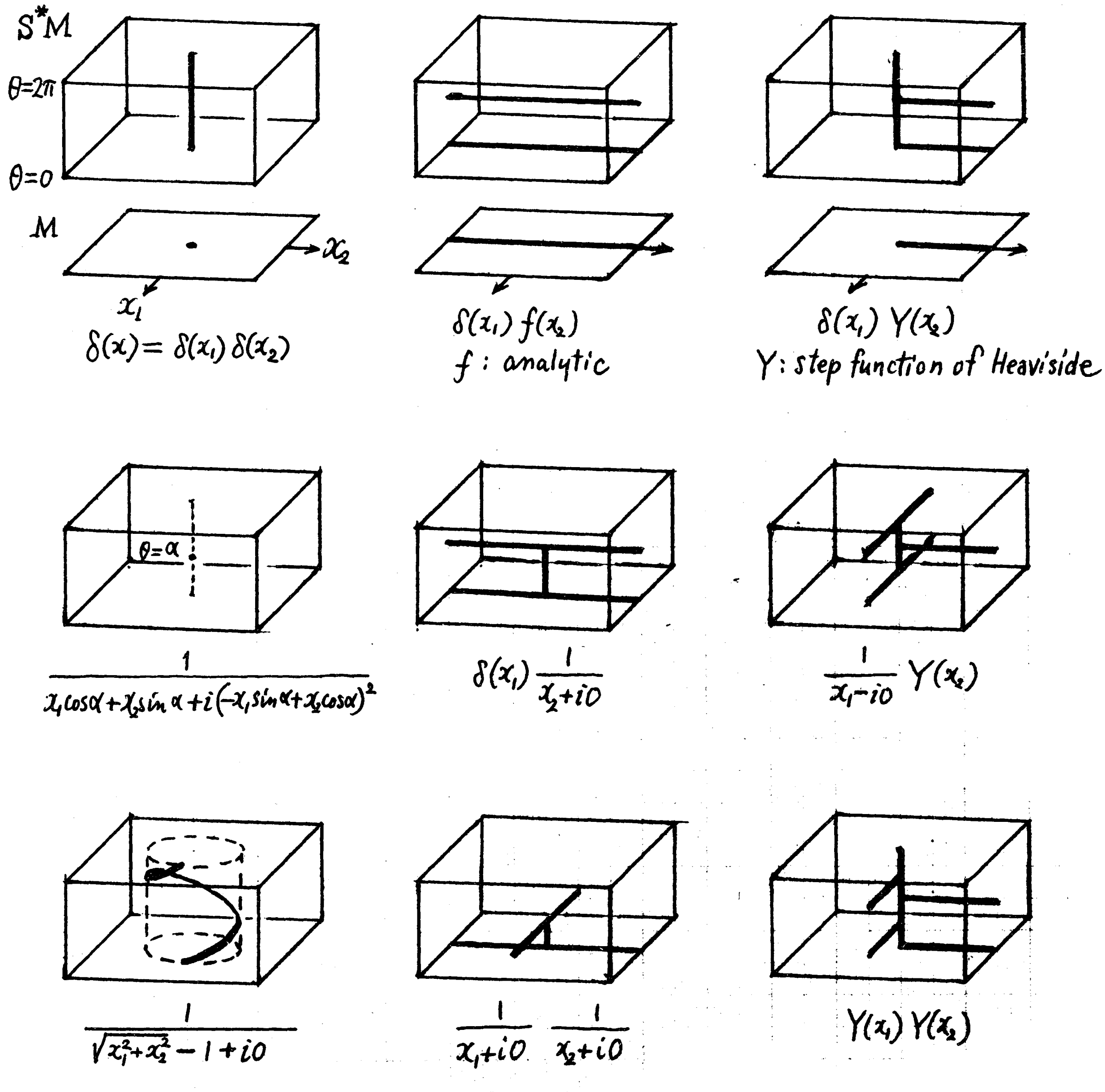
Taking the cross sections of the above sequence we have an exact sequence

$$0 \longrightarrow \mathcal{A}(M) \xrightarrow{\alpha} \mathcal{B}(M) \xrightarrow{\beta} \Gamma(M, \pi_*\mathcal{C}) \longrightarrow 0 .$$

The third term is rewritten as $\Gamma(S^*M, \mathcal{C})$ by the definition of direct image. Hence, for each hyperfunction $u \in \mathcal{B}(M)$, the image βu (i.e. the residue class of u modulo analytic functions) may be viewed either as a section of $\pi_*\mathcal{C}$ over M or as a section of $\mathcal{C}$ over S*M. Accordingly, the notion of support of βu also admits two interpretations, either as a closed set of M or as that of S*M. The former is the singular support of u in the customary sense (in notation: $S.S._M u$) while the latter is that of u in a sharpened sense (in notation: $S.S._C u$). Clearly $S.S._M u = \pi(S.S._C u)$ and $\pi^{-1}(S.S._M u) \supset S.S._C u$, and flabbiness of $\mathcal{C}$ implies that any closed set of S*M can actually appear as a singular support of some hyperfunction. Thus $S.S._C u$ gives us more detailed information about the irregularity of u than $S.S._M u$. If $\bar{u}$ denotes the complex conjugate of u , then $S.S._C \bar{u} = (S.S._C u)^a$ where $\underline{a}$ signifies antipodal points on S^*M.

A few examples of $S.S._C u$ in the case of $n = 2$ are illustrated in Fig. 1.

Fig. 1



As is shown in [3] one can develop a calculus on the sheaf $\mathcal{C}$ with applications to the calculus for hyperfunctions. For example:

<u>Multiplication</u>. —— The product of 2 hyperfunctions $u_1, u_2 \in \mathcal{B}(M)$ is well-defined if $S.S._C\, u_1$ and $(S.S._C\, u_2)^a$ are disjoint to each other. For example, $\delta(x_1)\delta(x_2)$ or more generally, $f_1(x_1)f_2(x_2)$ is well-defined.

<u>Specialization</u> (or <u>restriction</u>). —— Let N be an oriented submanifold of M. The conormal sphere bundle $S^*_N M$ is naturally considered to be a submanifold of S^*M. Then the specialization $f|_N \in \mathcal{B}(N)$ is always well-defined for a hyperfunction $f \in \mathcal{B}(M)$ whose singular support $S.S._C\, f$ is disjoint to $S^*_N M$. ([3])

For example, if N is a hypersurface which is non-characteristic with respect to a differential operator P, and if $u \in \mathcal{B}(M)$ satisfies $Pu \in \mathcal{A}(M)$, then by the theorem below u as well as any (higher) derivatives of u can be specialized onto N. This means that the notion of initial data makes sense for a hyperfunction solution of linear differential equation.

Generally speaking, if $f\colon N \to M$ is a morphism between oriented real analytic manifolds of dimension n' and n respectively, and if $\rho\colon S^*M\times_M N - S^*_N M \to S^*N$ and $\sigma\colon S^*M\times_M N - S^*_N M \to S^*M$ denote 2 morphisms naturally induced by f, then we have as a generalization of the notion of specialization the following sheaf homomorphism over S^*N

$$f^* : \rho_!(\sigma^{-1}\mathcal{C}_M) \to \mathcal{C}_N ,$$

where $\rho_!$ stands for the 0-th direct image with <u>proper support</u>. We have, on the other hand, the following sheaf homomorphism over S*M as the <u>integration</u> in $\mathcal{C}$:

$$\sigma_!(\rho^{-1}\mathcal{C}_N^{(n')}) \to \mathcal{C}_M^{(n)} ,$$

where $\mathcal{C}_N^{(n')}$ (resp. $\mathcal{C}_M^{(n)}$) means the sheaf $\mathcal{C}$ of n'-forms (resp. n-forms). ([3], §6)

Combining the method of F. John [5] with the theory of $\mathcal{C}$, we can easily derive the following

Theorem ([1]; [2]; [3] §8). <u>Let</u> P(x,D) <u>be a differential operator on</u> M <u>with the principal symbol</u> P_m , <u>and let</u> $F = \{(x,\bar{\eta}) \in S^*M \mid P_m(x,\eta) = 0 \}$. <u>Then the sheaf endomorphism of</u> $\mathcal{C}$ <u>induced by</u> P(x,D) <u>is bijective on</u> S*M-F.

More specifically, P_m is <u>invertible</u> on S*M-F in the sheaf of rings $\mathcal{P}$ over S*M consisting of 'pseudo-differential operators' operating on $\mathcal{C}$. (The sheaf $\mathcal{P}$ is defined to be $\text{Dist}^0(S^*M, \mathcal{C}_{M\times M}^{(0,n)})$. Here $\mathcal{C}_{M\times M}^{(0,n)}$ stands for the sheaf $\mathcal{C}$ over S*(M×M) that behaves as <u>n</u>-forms on the 2nd copy of M. S*M is regarded to be a submanifold of S*(M×M) by 'anti-diagonal' embedding. See [3] §6.) I note that, besides generalities about $\mathcal{C}$, the only fact we need to prove this theorem is the theorem of Cauchy-Kowalewski.

Corollary. <u>If</u> P(x,D) <u>is elliptic then every hyperfunction solution of the equation</u> Pu = 0 <u>is analytic</u>.

Proof. Since F is empty in this case, we have the isomorphism P: $\mathcal{C} \xrightarrow{\sim} \mathcal{C}$ valid on the whole S*M, and hence the isomorphism P: $\pi_*\mathcal{C} \xrightarrow{\sim} \pi_*\mathcal{C}$ on M. On the other hand, P : $\mathcal{A} \to \mathcal{A}$ is surjective (Cauchy-Kowalewski theorem); i.e. we have an exact sequence

$$0 \to \mathcal{A}^P \to \mathcal{A} \xrightarrow{P} \mathcal{A} \to 0 ,$$

where $\mathcal{A}^P$ denotes the sheaf of analytic solutions of $Pu = 0$. Now we observe the following diagram of exact sequences

$$\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \downarrow & & & & & & \\
 & & \mathcal{A}^P & & & & & & \\
 & & \downarrow & & & & & & \\
0 & \longrightarrow & \mathcal{A} & \longrightarrow & \mathcal{B} & \longrightarrow & \pi_*\mathcal{C} & \longrightarrow & 0 \\
 & & P\downarrow & & P\downarrow & & P\downarrow\wr & & \\
0 & \longrightarrow & \mathcal{A} & \longrightarrow & \mathcal{B} & \longrightarrow & \pi_*\mathcal{C} & \longrightarrow & 0 \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

and conclude that $0 \longrightarrow \mathcal{A}^P \longrightarrow \mathcal{B} \xrightarrow{P} \mathcal{B} \longrightarrow 0$ is exact. (q.e.d.)

M. Morimoto mentioned that a theorem of Bargman-Hall-Wightman on Jost points in the quantum field theory is also an easy corollary of the above theorem.

The improvement of the above theorem is now being worked out by T. Kawai, M. Kashiwara, and the present speaker along the lines of Lewy-Hörmander-Egorov-Nirenberg-Treves. (See also [6], [7].) The problem is to determine supports of the kernel and cokernel sheaves of P: $\mathcal{C} \to \mathcal{C}$. (These may be substantially smaller than F.) Or rather, to determine the kernel and cokernel sheaves themselves. For example, if $P = \frac{1}{2}(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}) - i(x_1+ix_2)\frac{\partial}{\partial x_3}$ is the operator of H. Lewy, and if Q (resp. $\overline{Q}$) is a pseudo-differential operator (which induces a well-defined sheaf homomorphism $\mathcal{C} \to \mathcal{C}$ on S^*M) defined by means of the kernel function $K(x,x')dx' = (x_3-x_3'+2(x_2x_1'-x_1x_2')+i((x_1-x_1')^2+(x_2-x_2')^2))^{-1}dx_1'dx_2'dx_3'$ (resp. the complex conjugate of $K(x,x')dx'$), then it is shown that

$$\mathcal{C} \xrightarrow{Q} \mathcal{C} \xrightarrow{P} \mathcal{C} \xrightarrow{\overline{Q}} \mathcal{C}$$

is an exact sequence. This implies that $Pu = f$ is solvable if and only if $\overline{Q}f = 0$ in $\mathcal{C}$, and that the supports of $\mathrm{Ker}_{\mathcal{C}}\, P$ and $\mathrm{Coker}_{\mathcal{C}}\, P$ are given by the supports of the operators Q and $\overline{Q}$, which are quite easy to determine.

Propagation of singularities. Let $P(x,D)$ be such that the principal symbol $P_m(x,\eta)$ is real and of principal type on M. Kawai and Kashiwara proved that a closed set $F \subset S^*M$ can be a $S.S._{\mathcal{C}}\, u$ for some $u \in \mathcal{B}(M)$ such that $Pu \in \mathcal{A}(M)$ or even $Pu = 0$, if and only if F is a union of bicharacteristic strips. An easy corollary of this is that a closed set of M can be a $S.S._M\, u$ for some hyperfunction solution of $Pu=0$ if and only if it is a union of bicharacteristic curves. For example, there exists a solution of $((\partial/\partial x_1)^2-(\partial/\partial x_2)^2-(\partial/\partial x_3)^2)u=0$ for which $S.S._M\, u = \{x \in \mathbb{R}^3 \mid x_2^2+x_3^2 \geqq 1\}$. (F. John).

I-hyperbolicity. Kawai introduced the notion of I-hyperbolic operators as an interesting generalization of hyperbolic operators.

§4. Construction of the sheaf C ([2], [3])

4.1. Relative cohomology groups in generalized sense. Let $\mathcal{F}$ denote a sheaf over a space X and let $f: Y \to X$ be a continuous map. Then the sheaf $f^{-1}\mathcal{F}$ over X called the inverse image of $\mathcal{F}$ is defined to be the fiber product over X of Y and $\mathcal{F}$. This functor $\mathcal{F} \mapsto f^{-1}\mathcal{F}$ is an exact one. Now the relative cohomology groups in the generalized sense, $H^p(X \leftarrow Y, \mathcal{F})$, are defined in a natural way ([2],[3]) so that we have an exact sequence

$$\cdots \to H^{p-1}(X,\mathcal{F}) \to H^{p-1}(Y,f^{-1}\mathcal{F}) \to H^p(X \leftarrow Y,\mathcal{F}) \to H^p(X,\mathcal{F}) \to H^p(Y,f^{-1}\mathcal{F}) \to \cdots$$

If $f: Y \hookrightarrow X$ is the natural embedding of an open subset our $H^p(X \leftarrow Y, \mathcal{F})$ reduces to the (ordinary) relative cohomology group $H^p(X \bmod Y, \mathcal{F})$, and if $Y = \phi$, the empty set, this reduces further to $H^p(X, \mathcal{F})$.

If we have still another space Z and another continuous map $Z \to Y$, we have the following exact sequence (which reduces to the above one when $Z = \phi$)

$$\cdots \to H^{p-1}(X \leftarrow Z,\mathcal{F}) \to H^{p-1}(Y \leftarrow Z,f^{-1}\mathcal{F}) \to H^p(X \leftarrow Y,\mathcal{F}) \to H^p(X \leftarrow Z,\mathcal{F}) \to H^p(Y \leftarrow Z,f^{-1}\mathcal{F}) \to \cdots .$$

4.2. Real monoidal transform. Monoidal transforms, which are the most essential in desingularizing analytic spaces, are described as follows in the case of a (non-singular) complex analytic manifold X and a submanifold Y of arbitrary codimension in X . One removes Y from X and instead inserts the <u>normal projective bundle</u> $P_Y X$ over Y defined by $P_Y X = (T_Y X - \text{(zero-section)}) / \text{(non-zero complex numbers)}$. Here $T_Y X$ denotes the (tangential) normal vector bundle over Y defined as follows by means of tangent vector bundles TX and TY:

$$0 \to TY \to Y \times_X TX \to T_Y X \to 0.$$

This replacement of Y by $P_Y X$ or the blowing up is a natural one so that the transform $\tilde{X} = (X - Y) \sqcup P_Y X$ acquires a natural complex analytic structure and the natural projection map $\tau : \tilde{X} \to X$ becomes a proper morphism of analytic manifolds. The inverse image of Y by τ coincides with $P_Y X$ and $P_Y X$ lies in $\tilde{X}$ as a hypersurface or a submanifold of codimension 1.

What we need in construction of $\mathcal{C}$ is the real analytic version of the monoidal transform. Take a real analytic manifold M and a submanifold N of arbitrary codimension in M. Then the real monoidal transform of M at N is by definition

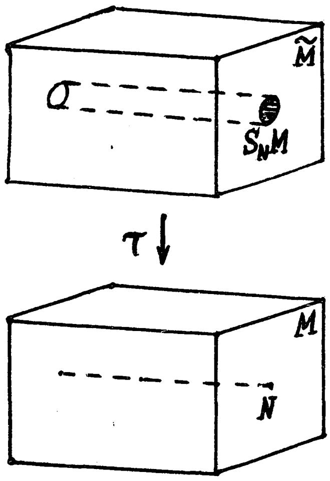
$$\tilde{M} = (M - N) \sqcup S_N M$$

where the normal sphere bundle $S_N M$ over N is defined by

$$S_N M = (T_N M - \text{(zero-section)}) / \text{(positive real numbers)}.$$

$\tilde{X}$ naturally acquires the structure[1] of real analytic manifold with boundary at $S_N M$, and the natural projection $\tau : \tilde{M} \to M$ becomes a proper morphism (Fig. 2). The inverse image by τ of N coincides with $S_N M$ and $S_N M$ is of codimension 1 in $\tilde{M}$.

Fig. 2



4.3. The sheaf $\mathcal{Q}$ over SM. Now return to the n-dimensional oriented real analytic manifold M and its complexification X. Since X has a structure of real analytic manifold of dimension $2n$, we can talk about the real monoidal transform $\tilde{X}$ of X at M. Here, furthermore, we can naturally identify the normal

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1) If antipodal pairs of points on each fiber of $S_N M$ are identified, $S_N M$ shrinks to the normal projective bundle $P_N M$ and $\tilde{M}$ shrinks to a real analytic manifold without boundary in which $P_N M$ lies as a hypersurface. We mention however that what we need in what follows is the topological structure of $\tilde{M}$ rather than the analytic structure thereof.

bundle $T_M X$ (resp. $S_M X$) with the tangent bundle TM (resp. SM) —— or rather with $\sqrt{-1}$ times TM(resp. SM) ——, because we have

$$0 \to TM \to M \times_X TX \to T_M X \to 0 \qquad \text{(by the definition)}$$

and

$$M \times_X TX = TM \otimes_{\mathbb{R}} \mathbb{C} = TM \oplus \sqrt{-1}\, TM .$$

Hence we can write $\widetilde{X} = (X - M) \sqcup SM$. The natural map $\tau : \widetilde{X} \to X$ is proper. On $\tau^{-1}M = SM$ the map τ gives the fiber structure $\tau : SM \to M$ while it reduces to a homeomorphism outside SM , τ: $\widetilde{X} - SM \approx X - M$.

Now we apply the exact sequence of 4.1. to the triple $\widetilde{X} - SM \hookrightarrow \widetilde{X} \overset{\tau}{\to} X$ with the structure sheaf $\mathcal{O} = \mathcal{O}_X$ as coefficients, and obtain

$$\cdots \to H^p(X \leftarrow \widetilde{X}, \mathcal{O}) \overset{\alpha}{\to} H^p(X \hookleftarrow \widetilde{X}\text{-}SM, \mathcal{O}) \overset{\beta}{\to} H^p(\widetilde{X} \hookleftarrow \widetilde{X}\text{-}SM, \tau^{-1}\mathcal{O}) \to \cdots \quad .$$

We have however $H^p(X \leftarrow \widetilde{X}, \mathcal{O}) = \mathcal{A}(M)$ (for p=n), = 0 (for $p \neq n$) and $H^p(X \hookleftarrow \widetilde{X} - SM, \mathcal{O}) = H^p(X \bmod \widetilde{X} - SM, \mathcal{O}) = \mathcal{B}(M)$ (for p = n), = 0 (for $p \neq n$). The latter is nothing but the fundamental fact in hyperfunction theory while the former is, as Kashiwara mentioned, quite an easy consequence of the elementary fact that $H^p(pt \leftarrow S^{n-1}, \mathbb{Z}) = \mathbb{Z}$ (for p = n), = 0 (for $p \neq n$). ('pt' denotes the space consisting of a single point and $\mathbb{Z}$ denotes rational integers.)

These vanishing theorems, together with the fact that $\alpha : \mathcal{A}(M) \to \mathcal{B}(M)$ is injective, immediately imply that the exact sequence mentioned above yields

$$0 \to \mathcal{A}(M) \to \mathcal{B}(M) \to H^n(\tilde{X} \bmod \tilde{X} - SM,\ \tau^{-1}\mathcal{O}) \to 0$$

and

$$H^p(\tilde{X} \bmod \tilde{X} - SM,\ \tau^{-1}\mathcal{O}) = 0 \qquad \text{if} \quad p \neq n.$$

According to the hyperfunction theory we know that M is purely $\underline{n}$-codimensional in X with respect to the structure sheaf $\mathcal{O}_X$: $\mathrm{Dist}^p(M, \mathcal{O}_X) = \mathcal{B}$ (for $p = n$), $= 0$ (for $p \neq n$). This includes at the same time the definition of the sheaf $\mathcal{B}$ of hyperfunctions. On the other hand, it is easy to show that the hypersurface SM in $\tilde{X}$ is purely 1-codimensional with respect to $\tau^{-1}\mathcal{O}_X$; i.e. we have $\mathrm{Dist}^p(SM, \tau^{-1}\mathcal{O}_X) = 0$ unless $p = 1$. Defining the sheaf $\mathcal{Q}$ over SM by $\mathcal{Q} = \mathrm{Dist}^1(SM, \tau^{-1}\mathcal{O}_X)$, we can deduce from this and the above facts that

$$H^p(SM, \mathcal{Q}) = \begin{cases} H^n(\tilde{X} \bmod \tilde{X} - SM,\ \tau^{-1}\mathcal{O}), & p = n-1, \\ 0, & p \neq n-1, \end{cases}$$

and

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{H}_\tau^{n-1}\mathcal{Q} \to 0 .$$

4.4. <u>The sheaf $\mathcal{C}$ over S^*M</u>. The next and final step to construction of the sheaf $\mathcal{C}$ is to transfer the sheaf $\mathcal{Q}$ from SM to the cosphere bundle S^*M. Define the fiber bundle DM over M by

$$DM = \{(x, \bar{\xi}, \bar{\eta}) \in SM \times {}_M S^*M \mid \xi \in T_xM - \{0\},\ \eta \in T^*_xM - \{0\},$$
$$\langle \xi, \eta \rangle \geqslant 0\} .$$

(We mention that each point in the fiber product $SM \times {}_M S^*M$ is a pair of tangent and cotangent vectors at the same point in M so that we can talk of inner product of them.) We have two chains of fiberings:

$$DM \overset{\pi'}{\to} SM \overset{\tau}{\to} M \quad \text{and} \quad DM \overset{\tau'}{\to} S^*M \overset{\pi}{\to} M,$$

consisting of proper maps. Also important is the fact that the fiberings π' and τ' have both contractible fibers of (closed) hemispheres $\frac{1}{2}S^{n-1}$, for this fact implies in particular that the map π' is purely 0-dimensional with respect to the inverse image sheaf $\pi'^{-1}\mathcal{Q}$ over DM, and the 0-th direct image coincides with the original sheaf $\mathcal{Q}$:

$$\mathcal{H}^p_{\pi'}(\pi'^{-1}\mathcal{Q}) = \begin{cases} \mathcal{Q}, & p = 0, \\ 0, & p \neq 0. \end{cases}$$

On the other hand, the map τ' is shown to have pure dimension $\underline{n}$-1 with respect to $\pi'^{-1}\mathcal{Q}: \mathcal{H}^p_{\tau'}(\pi'^{-1}\mathcal{Q}) = 0$, $(p \neq n - 1)$. This is a fact equivalent to a result of M. Morimoto [4] about the edge-of-the-wedge theorems. Now we define the sheaf $\mathcal{C}$ over S^*M by

$$\mathcal{C} =_{\text{def}} \mathcal{H}^{n-1}_{\tau'}(\pi'^{-1}\mathcal{Q}),$$

and obtain the formulae

$$\mathcal{H}^{p-1}_{\tau}\mathcal{Q} = \mathcal{H}^{p-1}_{\tilde{\omega}}(\pi'^{-1}\mathcal{Q}) = \mathcal{H}^{p-n}_{\pi}\mathcal{C} = \begin{cases} \mathcal{B}/\mathcal{A} & p = n, \\ 0, & p \neq n, \end{cases}$$

$$0 \to \mathcal{A} \to \mathcal{B} \to \pi_{*}\mathcal{C} \to 0 \ ,$$

where $\tilde{\omega}$ is the abbreviation of $\tau \circ \pi' = \pi \circ \tau' : DM \to M$. (We understand the cohomology group for negative dimension is always 0.)

We mention that a further consideration gives us two diagrams consisting of exact sequences of sheaves over SM and S*M respectively:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 \to & \tau^{-1}\mathcal{A} & \to \tilde{\mathcal{A}} & \to & \mathcal{Q} & \to & 0 \\
 & \| & \downarrow & & \downarrow & & \\
0 \to & \tau^{-1}\mathcal{A} & \to \tau^{-1}\mathcal{B} & \to & \tau^{-1}\pi_{*}\mathcal{C} & \to & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & \pi'_{*}\tau'^{-1}\mathcal{C}^{a} & = & \pi'_{*}\tau'^{-1}\mathcal{C}^{a} & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}
\qquad
\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 \to & \pi^{-1}\mathcal{A} & \to \tilde{\mathcal{A}}^{*a} & \to & \mathcal{Q}^{*a} & \to & 0 \\
 & \| & \downarrow & & \downarrow & & \\
0 \to & \pi^{-1}\mathcal{A} & \to \pi^{-1}\mathcal{B} & \to & \pi^{-1}\pi_{*}\mathcal{C} & \to & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & \mathcal{C} & = & \mathcal{C} & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

Here $\tilde{\mathcal{A}}$ is the sheaf of 'ideal boundary values' of holomorphic functions defined as follows: Define a sheaf $\tilde{\mathcal{O}}$ over $\tilde{X}$ to be the 0-th direct image of $\mathcal{O}_{\tilde{X}-SM} = \mathcal{O}_{X-M}$ by the natural embedding $\tilde{X} - SM \hookrightarrow \tilde{X}$. Then $\tilde{\mathcal{A}}$ is the restriction of $\tilde{\mathcal{O}}$ onto SM, i.e. the inverse image of $\tilde{\mathcal{O}}$ by the natural embedding SM $\hookrightarrow \tilde{X}$. (We omit the definitions of sheaves $\tilde{\mathcal{A}}^{*}$ and $\mathcal{Q}^{*}$. See[3].)

The symbol 'a' on the right shoulder stands for the direct (and at the same time inverse) image by the antipodal mapping on SM or S*M. The exact sequences $0 \to \tilde{\mathcal{A}} \to \tau^{-1}\mathcal{B} \to \cdots$ and $\cdots \to \pi^{-1}\mathcal{B} \to \mathcal{C} \to 0$ in the above diagrams are of great importance in further study and applications of hyperfunction theory.

For complete accounts and proofs the reader is referred to [3].

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