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渦動ソリトン

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§ 序

流体中の集中した渦運動は、大局的に見れば、完全流体中の渦糸の運動とみなすことができるが、それはHelmholtz-Lagrange-Kelvinの定理によって、個性を持った実体とみることができる。こうして特に縮まない流体の2次元運動については、古くからの研究が行なわれ、渦対、渦列、さらに進んで、Onsagerによる渦糸郡の統計力学に至るまで、かなりの研究が行なわれて来た。これは渦糸を点とみた抽象化が可能であるためである。これに対して3次元の渦糸は一本の渦糸をとつて見ても、その変形が複雑であって、渦輪やらせん渦を除いては、ほとんどしらべられていないのが現状である。

最近Hama (1962) はArmsによって示唆されたいわゆる局所誘導の考えのもとに、非常に細い渦糸(といってもそのまわりには、渦によって誘導された流体の運動が行なわれ

、たとえば渦輪に見るように数十倍の半径の流体の環をそのまわりに従えて運動することができる。）の運動を支配する漸近的な方程式(2.1)－(2.3)を導いて、いろいろな形の渦糸の運動を数値的に議論している。この近似の要点は自由な細い渦糸の運動が *Biot-Savart* の法則によれば局所的な曲率と循環に比例し、他の部分よりのものにくらべて大きな局所誘導速度によって支配されることに着眼し（たとえば *Batchelor* (1967)）更にその比例係数 G に含まれる対数関数の変化をその引数である、曲率の変化にくらべて無視する点にある。それが少くとも定性的にはよい近似を与えることは数値的には *Hama* (1962) によって放物線状渦に対し、渦輪の変形については *Kambe, Takao* の煙による実験(1971)によって確められている。ところで、この近似方程式は本質的にはかなり複雑な非線型のベクトル方程式であるが、2.3のばあいに厳密解を持つことがわかる。たとえば *Appendix* 2に見るように、平面上にあって剛体回転（角速度 $-\Omega$）する渦糸の有限変形解が存在し、その形が *flexual rigidity* B の一様な細い弾性糸が力 F に対してうける変形 (*Elastica*) と $F/B = \Omega/G$ の関係があれば同一であることがわかる。これは本質的には曲率が軸からの距離に等しいというばあいである。

しかしながら一般的な複雑な運動を議論するには渦糸の曲

率κと捩率τを支配する方程式（自然方程式）の方が便利である。実際Betchov（1965）はこのような方程式を導き、それが負の圧力を持ち、複雑な非線型の分散性ストレスの項を含むものであることを示しているが、その複雑さのために若干の性質の議論と、剛体回転する渦糸解で近似の性格からは受け入れられない自分自身に交わるloopを持つ解を導いたに過ぎなかった。

こゝでは微分幾何学の簡単な公式（Frenet Seret）の公式から出発して、曲率を振巾、捩れ角を位相とする複素変数 を支配する方程式を導く。これは非線型光学やプラズマ物理にあらわれる非線型のShrödinger方程式の1種であってその方面で得られた知識がたゞちに我々の問題を応用できることになる。

この方程式は簡単な孤立伝播波解を持つが、これは捩率の小さい、大きいに従って異なるが直線渦糸に生じたループあるいはらせん運動のこぶが前後に渦糸のらせん変形を引きおこして渦糸に沿って一定速度2τで伝播する状況をあらわすことが示される。最後にBetchovの方程式との関係と、κ^2を密度、2τ を速度と解釈することによって導かれる気体力学との類推とその保存形を示す。

Soliton on a vortex filament

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Intrinsic equation governing the curvature κ and the torsion τ of a very thin vortex filament in an incompressible inviscid fluid is reduced to a non-linear Schrödinger equation

$$\frac{1}{i}\frac{\partial\psi}{\partial t} = \frac{\partial^2\psi}{\partial s^2} + \frac{1}{2}(|\psi|^2 + A)\psi$$

where t is the time, s the length measured along the filament and ψ the complex variable

$$\psi = \kappa \exp\left(i\int_0^s \tau\, ds\right)$$

and A is a function of t .

It is found that this equation yields a solution describing the propagation of a loop or a hump of helical motion along a line vortex, with a constant velocity 2τ

The relation to the system of intrinsic equations derived by Betchov (1965) is discussed.

1. Introduction

Vortex filaments in a perfect fluid are known to preserve their identity and extensive investigations have been made on the two-dimensional motion of a system of vortices. In the three-dimensional case, however, few examples are known even for a single filament owing to its complicated behaviour.

Recently the so-called localized induction equation which describes asymptotically the motion of a very thin vortex filament has been derived by Arms (1963 ; from privated communication to Hama) and has been used by Hama(1962, 1963) in order to describe the motion of curved filaments of several shapes. As regards its derivation, Batchelor's book (1967) may be consulted.

The essential point of this approximation is to approximate the local motion of the filament by that of a thin circular vortex with the same curvature and to neglect slow variation of its coefficient. As long as the interaction between far distant portions along the filament is neglected, this approximation seems to be valid at least qualitatively as shown numerically by Hama (1962) for a parabola and experimentally by Kambe and Takao (1971) for a distorted vortex ring. On the basis of this approximation, Hasimoto (1971) has shown that the shape of a simply rotating plane filament is that of a plane elastic filament, *i.e.* the elastica.

In order to consider the complicated behaviour of the filament,however, intrinsic equations for the curvature and the torsion of the filament seem to be useful. Betchov (1965) has derived such system of equations, which may be reduced to those for a fictitious gas with negative pressure, accompanied with complicated non-linear dispersive stresses.

In this paper, a simple intrinsic equation for a complex variable with the curvature as its amplitude and the torsion angle as its phase is derived by a simple procedure starting from the fundamental equations of differential geometay.

This equation is found to be a kind of non-linear Shorödinger equation which appears in the theories of non-linear optics and plasma physics (Karpman and Krushkal (1969), Taniuti and Yajima (1969), Asano, Taniuti and Yajima (1969)).

It is shown that this equation admits a solution describing a solitary wave propagating along a line vortex filament, which induces various types of motion of the filament according to the value of the torsion.

In Appendix, deduction of Betchov's intrinsic equation from our equation is made.

2. Fundamental Equations

The motion of a very thin vortex filament $\boldsymbol{X} = \boldsymbol{X}(s,t)$ of radius ε in an incompressible unbounded fluid is described asymptotically by

$$\frac{\partial \boldsymbol{X}}{\partial t} = G \varkappa \boldsymbol{b} \tag{2.1}$$

where s is the length measured along the filament, t the time, $\varkappa$ the curvature, $\boldsymbol{b}$ the unit vector in the direction of the binormal and G is the coefficient of local induction

$$G = \frac{\Gamma}{4\pi}\left[\log\left(\frac{1}{\varkappa\varepsilon}\right) + O(1)\right] \tag{2.2}$$

which is proportional to the circulation Γ of the filament and may be regarded as constant if we neglect the slow variation of the logarithm compared with that of its argument. It should be noted that the interaction between far distant portions of the filament is neglected in this approximation and the local motion is approximated by that of a thin circular ring with the same curvature.

Then a suitable choice of the units of time and length reduces (2.1) to the non-dimensional form

$$\dot{\boldsymbol{x}} = \varkappa \boldsymbol{b}, \tag{2.3}$$

where and hereafter a dot (.) and a prime (') denote respectively $\partial/\partial t$ and $\partial/\partial s$.

The equation (2.3) should be supplemented by the equations of differential gemetory(Frenet Seret Formulae)

$$\boldsymbol{X}' = \boldsymbol{t}, \tag{2.4}$$

$$\boldsymbol{t}' = \varkappa \boldsymbol{n}, \tag{2.5}$$

$$\boldsymbol{n}' = \tau \boldsymbol{b} - \varkappa \boldsymbol{t}, \tag{2.6}$$

$$\boldsymbol{b}' = -\tau \boldsymbol{n}, \tag{2.7}$$

where τ is the torsion and $\boldsymbol{t}$, $\boldsymbol{n}$ and $\boldsymbol{b}$ are a right-handed system of mutually perpendicular unit vectors parallel to the tangent, the principal normal and the binormal repectively.

Combining (2.6) and (2.7) we have

$$(\boldsymbol{n}+i\boldsymbol{b})' = -i\tau(\boldsymbol{n}+i\boldsymbol{b}) - \kappa\boldsymbol{t}, \tag{2.8}$$

which suggests the introduction of new variables

$$\boldsymbol{N} = (\boldsymbol{n}+i\,\boldsymbol{b})\exp\left(i\int_0^s \tau\, ds\right) \tag{2.9}$$

and

$$\psi = \kappa \exp\left(i\int_0^s \tau\, ds\right). \tag{2.10}$$

Then, we have from (2.8) and (2.5)

$$\boldsymbol{N}' = -\psi\boldsymbol{t} \tag{2.11}$$

and

$$\boldsymbol{t}' = Re\,[\psi\bar{\boldsymbol{N}}] = \frac{1}{2}(\bar{\psi}\boldsymbol{N} + \psi\bar{\boldsymbol{N}}), \tag{2.12}$$

where the bar (-) denotes the complex conjugate and Re the real part.

On the other hand, from (2.4), (2.3) and (2.7), we have,

$$\dot{\boldsymbol{t}} = \dot{\boldsymbol{X}}' = (\kappa\boldsymbol{b})' = \kappa'\boldsymbol{b} - \kappa\tau\boldsymbol{n} = \kappa\, Re\left[\left(\frac{\kappa'}{\kappa}+i\tau\right)(\boldsymbol{b}+i\boldsymbol{n})\right] \tag{2.13}$$

i.e.

$$\dot{\boldsymbol{t}} = Re\,[i\psi'\bar{\boldsymbol{N}}] = \frac{i}{2}(\psi'\bar{\boldsymbol{N}} - \bar{\psi}'\boldsymbol{N}) \tag{2.14}$$

where we have made use of (2.9) and (2.10).

It should be noted that there are orthogonality relations among $\boldsymbol{t}$, $\boldsymbol{N}$ and $\bar{\boldsymbol{N}}$:

$$\boldsymbol{t}\cdot\boldsymbol{t} = 1, \quad \boldsymbol{N}\cdot\bar{\boldsymbol{N}} = 2, \quad \boldsymbol{N}\cdot\boldsymbol{N} = 0, \quad \boldsymbol{N}\cdot\boldsymbol{t} = 0 \quad etc. \tag{2.15}$$

The equation governing the evolution of $\boldsymbol{N}$ is obtained as follows. Letting

$$\dot{N} = \alpha N + \beta \bar{N} + \gamma t . \tag{2.16}$$

and by noting the orthogonality (2.15) and its time derivative we determine the coefficients α , β and γ as follows:

$$\alpha + \bar{\alpha} = \frac{1}{2} (\dot{N} \cdot \bar{N} + \dot{\bar{N}} \cdot N) = \frac{1}{2} (N \cdot \bar{N})^{\cdot} = 0 , \quad i.e. \ \alpha = i R$$

$$\beta = \frac{1}{2} \dot{N} \cdot N = \frac{1}{4} (N \cdot N)^{\cdot} = 0 ,$$

$$\gamma = - N \cdot \dot{t} = - i \psi' , \tag{2.17}$$

where R is a real unknown function and we have made use of (2.14) and (2.15). Thus we have

$$\dot{N} = i (R N - \psi' t) . \tag{2.18}$$

The time derivative of (2.11) and the s derivative of (2.18) yield respectively

$$\dot{N}' = - \dot{\psi} t - \psi \dot{t} = - \dot{\psi} t - \frac{i}{2} \psi (\psi' \bar{N} - \bar{\psi}' N) \tag{2.19}$$

and

$$\dot{N}' = i [R' N - R \psi t - \psi'' t - \frac{1}{2} \psi' (\bar{\psi} N + \psi \bar{N})] , \tag{2.20}$$

where we have made use of (2.14), (2.11) and (2.12).

Equating the coefficients of t and $i N$ respectively (those of $\bar{N}$ are identical) we have

$$- \dot{\psi} = - i (\psi'' + R \psi) \tag{2.21}$$

and

$$\frac{1}{2} \psi \bar{\psi}' = R' - \frac{1}{2} \psi' \bar{\psi} . \tag{2.22}$$

The comparison of $\dot{t}'$ from (2.12) and (2.14) leads only to (2.21). Solving (2.22) we have

$$R = \frac{1}{2}(\psi\bar{\psi} + A), \tag{2.23}$$

which reduces (2.21) to

$$\frac{1}{i}\frac{\partial\psi}{\partial t} = \frac{\partial^2\psi}{\partial s^2} + \frac{1}{2}(|\psi|^2 + A)\psi, \tag{2.24}$$

where A is a real function of t which can be eliminated by the introduction of the new variable

$$\Psi = \psi \exp\left[-\frac{i}{2}\int_0^s A(t)\,dt\right]. \tag{2.25}$$

This transformation is nothing but the shift of the origin of integration in (2.9) and (2.10); therefore we may take A in (2.24) to be zero without loss of generality.

Equation (2.24) is a non-linear Schrödinger equation which appears in the theories of non-linear optics and plasma physics. Hence the results in these cases can be easily transferrable to our problem.

3. Solitary Wave

As a special case, let us look for the solution of (2.24) which describes a solitary wave (soliton) which propagates steadily with a constant velocity c along the filament which is straight at infinity $i.e.$

$$\kappa = 0, \qquad \text{as} \qquad s \to \infty. \tag{3.1}$$

In the wave frame of reference in which κ and τ are functions of

$$\xi = s - ct \tag{3.2}$$

i.e.

$$\psi = \kappa(\xi) \exp\left[i\int_0^s \tau(\xi)\, ds\right], \tag{3.3}$$

the real and imaginary parts of (2.24) yield respectively

$$-c\kappa[\cancel{\tau(\xi)}-\tau(-ct)] = \kappa'' - \kappa\tau^2 + \tfrac{1}{2}(\kappa^2+A)\kappa \tag{3.4}$$

and

$$c\kappa' = 2\kappa'\tau + \kappa\tau'. \tag{3.5}$$

Equation (3.5) can be integrated to give

$$(c - 2\tau)\kappa^2 = 0, \tag{3.6}$$

where we have made use of (3.1) to determine the integration constant. According to (3.6) we have

$$\tau = \tau_0 = \tfrac{1}{2}c = const, \tag{3.7}$$

if κ is not identically zero; *i.e.* the torsion is constant along the filament and the velocity of propagation along it is twice the torsion.

Then by use of (3.1), (3.4) is integrated to give

$$\kappa = 2\nu \operatorname{sech} \nu\xi, \tag{3.8}$$

provided that A is a constant determined by

$$A = 2(\tau_0^2 - \nu^2). \tag{3.9}$$

The actual shape of the filament is determined by introducing (3.7) and (3.8) into (2.4) - (2.7). For this purpose, it is convenient to solve the equation for $\boldsymbol{b}$ obtained by the substitution of $\boldsymbol{n}$ and $\boldsymbol{t}$ from (2.7) and (2.6) into (2.5):

$$\tau_0(\mathbf{t}' - \kappa \mathbf{n}) = [\tfrac{1}{\kappa}(\mathbf{b}'' + \tau_0^2 \mathbf{b})]' + \kappa \mathbf{B}' = 0 \tag{3.10}$$

i.e.

$$\frac{d^3}{d\eta^3}b + \tanh\eta \frac{d^2}{d\eta^2}b + (T^2 + \operatorname{sech}^2\eta)\frac{d}{d\eta}b + T^2 \tanh\eta\, b = 0, \tag{3.11}$$

where

$$\eta = \nu\xi \qquad \text{and} \qquad T = \tau/\nu\,. \tag{3.12}$$

The solution of this equation can be easily obtained by noting that

$$B = \frac{d}{d\eta}b + \tanh\eta\, b \tag{3.13}$$

is the solution of the equation

$$\frac{d^2}{d\eta^2}B + (T^2 + 2\operatorname{sech}^2\eta)B = 0, \tag{3.14}$$

which is satisfied by

$$(\tanh\eta \mp iT)\, e^{\pm iT\eta} \tag{3.15}$$

As particular solutions, we have

$$b = \operatorname{sech}\eta\,, \qquad (1 - T^2 \pm 2iT\tanh\eta)\, e^{\pm iT\eta}\,. \tag{3.16}$$

Let us introduce (3.16) into (2.7), (2.6) and (2.4), and determine the coefficients so as to satisfy without loss of generality the conditions that the flilament is parallel to the x-axis of the Cartesian coordinates (x , y , z) at infinity; i.e.

$$t_x \to 1 \qquad \text{as} \qquad \eta \to \infty \tag{3.17}$$

and

$$n_y + i n_z = i(b_y + i b_z) = e^{i(\tau_0 \xi + \sigma)} \quad \text{as} \quad \eta \to \infty, \tag{3.18}$$

the latter being suggested by the asymptotic behaviours of the solution of (2.5) - (2.7) and the orthogonalities among $\boldsymbol{t}$, $\boldsymbol{n}$ and $\boldsymbol{b}$. Here σ is a real constant and the suffices x, y, z denote the x-, y- and z components respectively.

After straightforward calculations we have

$$\begin{aligned}
&\boldsymbol{X}: \ x = \frac{1}{\nu}(\eta - 2\mu \tanh\eta), \quad y + iz = r e^{iT\eta}, \\
&\boldsymbol{t}: \ t_x = 1 - 2\mu \operatorname{sech}^2\eta, \quad t_y + i t_z = -\nu r(\tanh\eta - iT)e^{iT\eta}, \\
&\boldsymbol{n}: \ n_x = 2\mu \operatorname{sech}^2\eta \sinh\eta, \quad n_y + i n_z = -[1 - 2\mu(\tanh\eta - iT)\tanh\eta]\, e^{iT\eta}, \\
&\boldsymbol{b}: \ b_x = 2\mu T \operatorname{sech}\eta, \quad b_y + i b_z = i\mu(1 - T^2 - 2iT\tanh\eta)e^{iT\eta},
\end{aligned} \tag{3.19}$$

where

$$\begin{aligned}
&\mu = \frac{1}{1+T^2} = \frac{\nu^2}{\nu^2 + \tau_0^2}, \quad r = \frac{2\mu}{\nu} \operatorname{sech}\eta, \\
&\eta = \nu\xi = \nu(s - 2\tau_0 t), \quad T\eta = \tau_0 \xi
\end{aligned} \tag{3.20}$$

4. Numerical results and discussions

Figures 1-3a show the projections of the filament to the xy-, xz- and zy planes at an instant of time (say, $t = 0$).

It is seen that the filament is confined on an envelope of radius r (except near the centre $x \sim 0$ if $T < 1$) which decreases from its maximum $2\mu/\nu$ at $x = 0$ to $2\mu/[\nu \cosh(\nu x \pm 2\mu)] \sim 0$ as $x \to \pm\infty$. Our filament is a spiral surrounding this envelope, being approximated by

$$y + iz = r \exp(iT\eta) \to r \exp[i\tau_0(x \pm 2\mu/\nu)] \quad \text{as} \quad x \to \pm\infty.$$

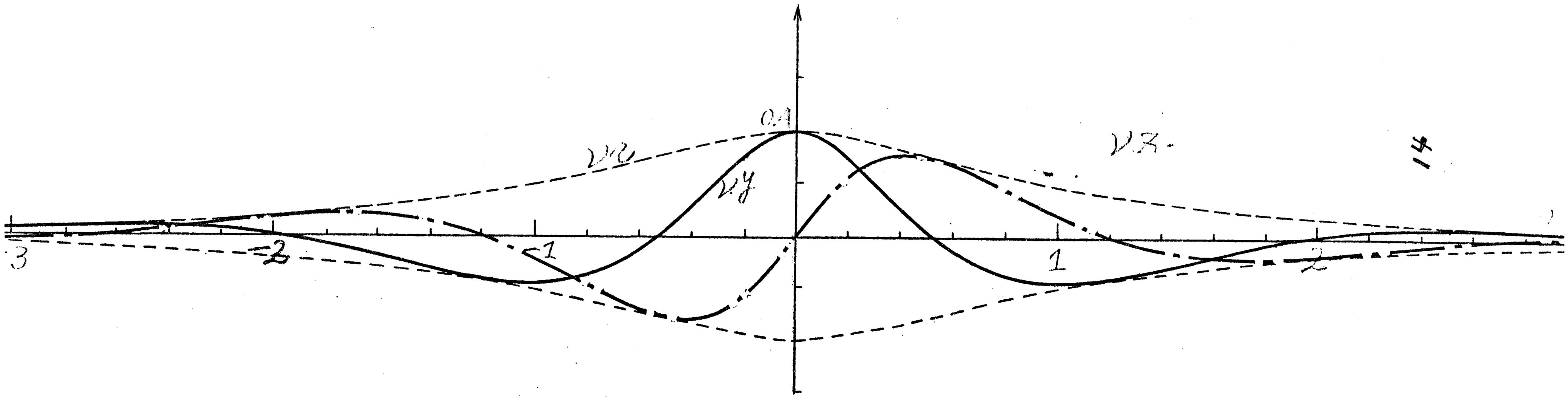


Figure 1. T = 2.0 ; ——— Projection to xy plane, — - — - — Projection to xz plane, --------envelope.

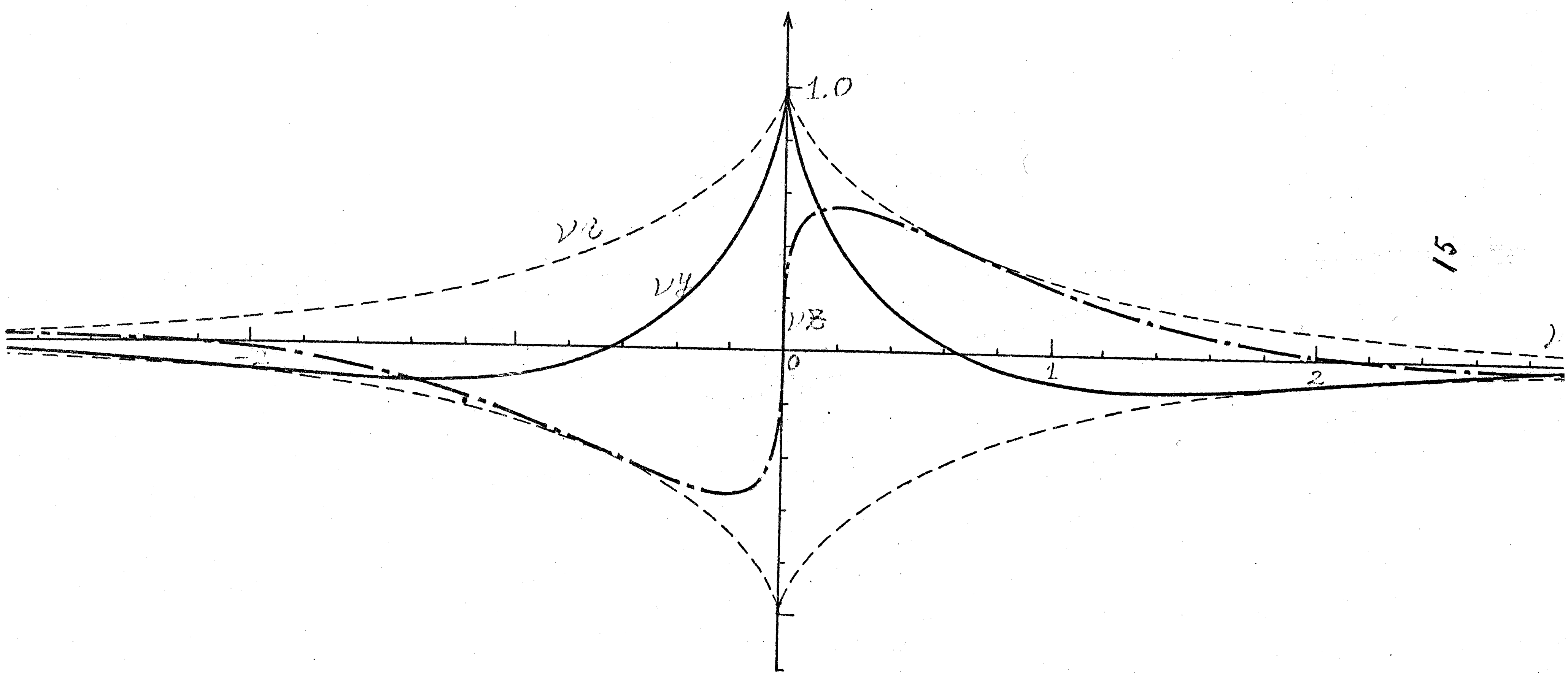


Figure 2. T = 1.0 ; ——— Projection to xy plane, — - — - — Projection to xz plane, --------envelope.

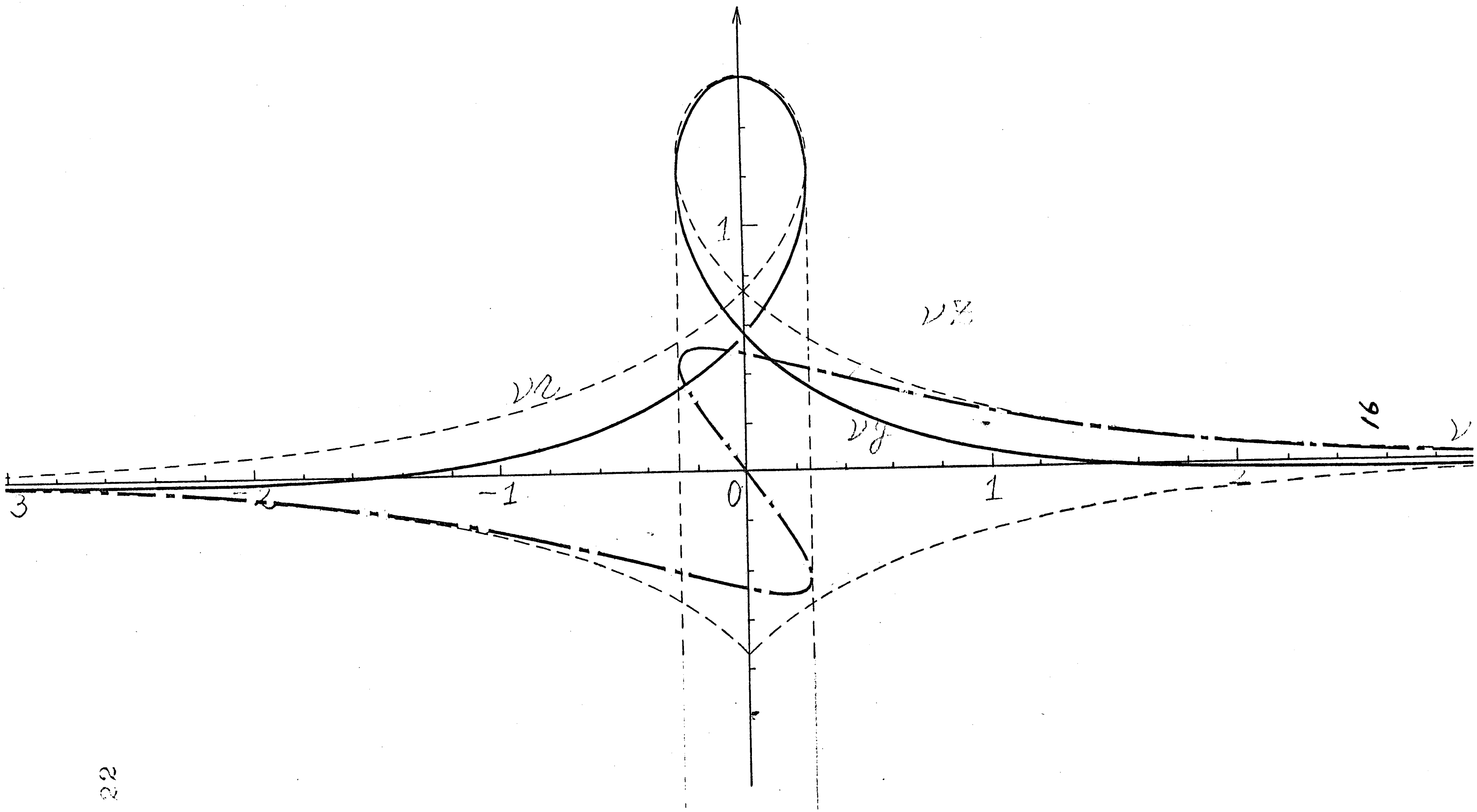


Figure 3. T = 0.5 ; ———— Projection to xy plane, — - — - — Projection to xz plane, --------envelope.

Figure 1a. T = 2.0 ; ——— Projection zy plane.

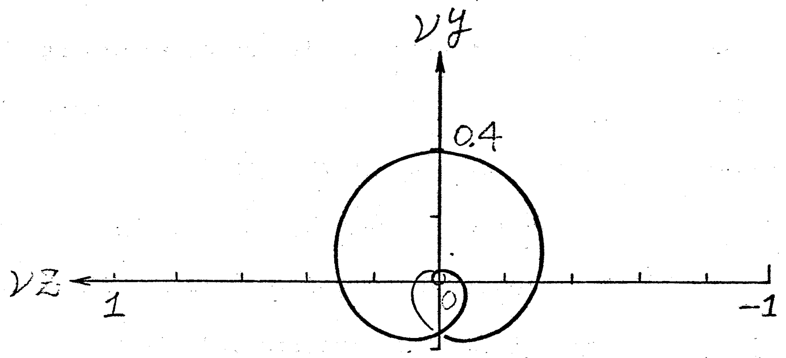


Figure 2a. T = 1.0 ; ——— Projection to zy plane.

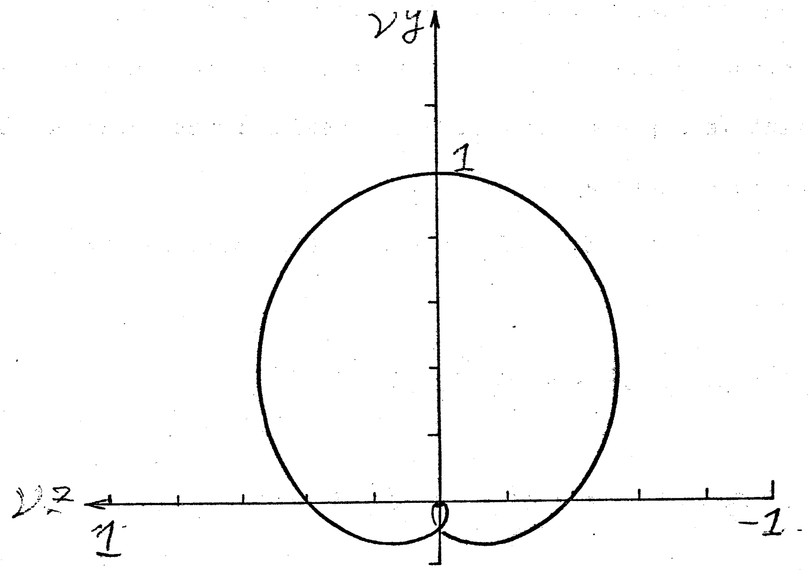
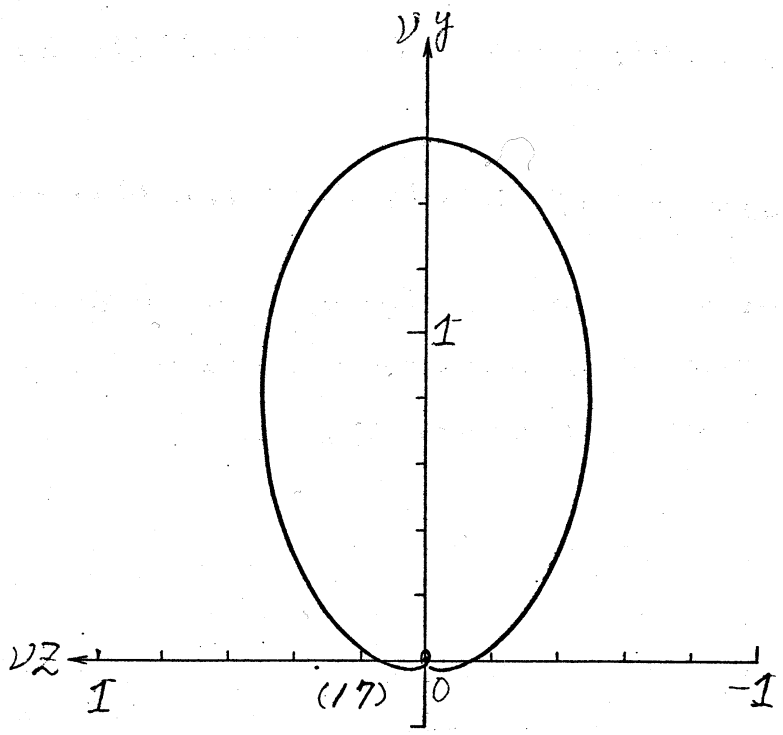


Figure 3a. T = 0.5 ; ——— Projection to zy plane.



However its behaviour at $x \sim 0$ is different according to the value of T. As long as $T \geqq 1$, $y + iz$ is a single-valued function of x, since x is a monotonic function of η. If $T = 1$, $dx/d\eta$ is zero at $x = \eta = 0$ and the cusp of the envelope appears at $x = 0$, $z = 2\mu/\nu$ though no singularity exists on the filament.

For $T < 1$, the filament is twisted and yields a loop in its side view, though no real crossing point exists. As the torsion is decreased the projection to the zy plane is flattened and in the limit of $T = 0$, the filament is a plane curve with a crossing point which has been noted by Betchov to be unacceptable.

The velocity $\mathbf{V} = \dot{\mathbf{X}}$ of the filament is obtained from (2.3), (3.8), (3.13) and (3.19) as

$$V_x = 4\mu T \operatorname{sech}^2\eta = \tau_0(\tau_0^2+\nu^2)\nu^2, \tag{4.1}$$

$$W = V_y + iV_z = W_{rot} + W_{rad}, \tag{4.2}$$

where

$$W_{rot} = i\nu(1-T^2)(y+iz) = i(\nu^2-\tau_0^2)(y+iz) \tag{4.3}$$

and

$$W_{rad} = 2\nu^2 T(y+iz)\tanh\eta = 2\nu\tau_0(y+iz)\tanh\eta. \tag{4.4}$$

It is seen that the motion can be decomposed into three parts : i) longitudinal motion V_x, ii) rotation about the x-axis W_{rot} and iii) radial contraction and expansion W_{rad}.

The rotation changes its sign according to that of T - 1 *i.e.* A in (3.9). If T < 1, the direction of rotation is that of the vorticity at $x = \pm\infty$ and if T > 1 it is opposite. It is intersting to notice that no actual rotation occures if T = 1 *i.e.* $A = 0$. This behaviour may be attributed to the appearence of a loop in the side view for T < 1 in contrast to the dominance of spiral for T > 1.

The magnitude of V_x depends on the curvature proportional to κ and the orientation of the looping to the x-axis *i.e.* b_x which is proportional to $\tau\kappa$. The faster motion of the larger looping seems to be coupled with the radial expansion by W_{rad} , leading to the propagation of our solitaray wave along the filament. Notice that we have radial expansion or contraction according as $\tau\kappa$ is positive or negative.

Some of these features are seen to be in accordance with Hama's numerical experiments (1963) on the filament which has Gaussian shape initially.

Though these behaviours of the vortex filament may be temporary judging from its approximate nature and possible instability, the author hopes that they might be observed in some vortex system such as highly sheared stream or rotating flow.

In this connection, it may be noted that Yajima and Outi (1971) made a numerical calculation on the stability of solitary waves for a nonlinear Schrödinger equation and have shown that they are fairly stable.

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Appendix. Relation to Betchov's intrinsic equation

Betchov (1965) has derived a system of intrinsic equations governing essentially two variables

$$\rho = \kappa^2 , \quad u = 2\tau \tag{A.1}$$

In order to derive his equation, it is convenient to introduce the potential

$$\Phi = \int_0^s u\,ds = \frac{1}{2}\int_0^s \tau\,ds \tag{A.2}$$

into (2.24) and differentate $\psi = \sqrt{\rho}\,\exp(2i\Phi)$ logarithmically ; comparing the real and imaginary parts of

$$\frac{1}{2i}\left(\frac{\dot{\rho}}{\rho} + i\dot{\Phi}\right) = \frac{1}{2}\left(\frac{\rho''}{\rho} - \frac{\rho'^2}{\rho^2} + i\Phi''\right) + \frac{1}{4}\left(\frac{\rho'}{\rho} + i\Phi'\right)^2 + \frac{1}{2}(\rho + A) \tag{A.3}$$

and differentiating the former with respect to s , we have

$$\dot{u} + u u' = \rho' + \left(-\frac{\rho'^2}{2\rho^2} + \frac{\rho''}{\rho}\right)', \tag{A.4}$$

$$\dot{\rho} + u\rho' = -\rho u' . \tag{A.5}$$

These are reduced to the same equations as given by Betchev((1965), (2.16) and (2.23)) if we put $\rho = K$ and $u = 2T$ there.

It may be noted that (A.5) and (A.4) supplemented by (A.5) yield the conservation forms :

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial s}(\rho u) = 0 \tag{A.6}$$

and

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial s}\left[\rho u^2 - \frac{1}{2}\rho^2 - \rho\frac{\partial^2}{\partial s^2}(\log \rho)\right] = 0 \tag{A.7}$$

respectively. By assuming the same dependence on ξ as that in (3.2) and (3.3) we can obtain the same results as in section 3.

Appendix 2. Motion of a Vortex Filament and its Relation to Elastica

The motion of a very thin vortex filament in an unbounded perfect fluid has been discussed by several authors (Hama (1962,1963), Betchov (1965), Batchelor (1967) on the basis of the so-called localized induction equation which is initially suggested by Arms (private communication to Hama(1962)) and is valid for very small values of the radius a of the filament compared with its radius of curvature $1/\varkappa$, where $\varkappa$ denotes the curvature.

In this note, it is shown that an analytic solution of this equation corresponding to the simple rotation of a plane filament exists and is perfectly equivalent to the finite deformation of a plane elastic filament, *i.e.* the elastica.

The localized-induction equation states that the velocity of the filament $\boldsymbol{v}$ is $G\varkappa\boldsymbol{b}$ in the direction of its binormal $\boldsymbol{b}$:

$$\boldsymbol{v} = G\varkappa\boldsymbol{b} , \qquad (1)$$

where G is the coeficient of local induction proportional to the circulation of the filament and may be assumed to be constant for very small values of $a\varkappa$.

Let y be the distance from the x-axis about which our filament rotates with uniform angular velocity $-\Omega$, s be the distance measured along the filament, and θ be the angle between the tangent to the filament and the x-axis. Then, by noting $\boldsymbol{v} = -\Omega y \boldsymbol{b}$ and $\varkappa = d\theta/ds$, eq. (1) reduces to $G\, d\theta/ds = -\Omega y$, or by differentiation with respect to s :

$$G\frac{d^2\theta}{ds^2} = -\Omega\frac{dy}{ds} = -\Omega \sin\theta . \qquad (2)$$

This equation is nothing but the equation for Euler's elastica (e.g. Love's Treatise ,1927, p401) *i.e.* the plane elastic filament of flexual rigidity B under the action of thrust F applied at its ends, provided that

$$G/\Omega = B/F. \tag{3}$$

The kinetic analogue to a simple pendulum in which s is replaced by the time t is also valid.

In this manner, the classical solution for elastica in terms of Jacobian elliptic function is easily transferable to our problem :

$$x = \left(\frac{G}{\Omega}\right)^{\frac{1}{2}}\left[-\xi + 2\int_0^{\xi} dn^2(\xi, k)\, d\xi\right], \quad y = A\, cn(\xi, k), \tag{4}$$

where A is the maximum distance from the axis and

$$\xi = \left(\frac{\Omega}{G}\right)^{\frac{1}{2}} s, \quad k = \frac{1}{2}\left(\frac{\Omega}{G}\right)^{\frac{1}{2}} A. \tag{5}$$

For small values of k eqs. (4)--(5) yield

$$\frac{y}{A} = \left(1 - \frac{k^2}{16}\right)\cos v + \frac{k^2}{16}\cos 3v + O(k^4), \quad v = \left(\frac{G}{\Omega}\right)^{\frac{1}{2}}\left(1 + \frac{3}{4}k^2\right)x, \tag{6}$$

which is in accordance with the form calculated by Kambe and Takao (1971) in relation to their experiment on the stability of a vortex ring and corresponds to Kelvin's sinsoidal filament for $k = 0$.

The distance Δx between nodal points $\xi = \pm K$, *i.e.*

$$\Delta x = 2\left(\frac{G}{\Omega}\right)^{\frac{1}{2}}(2E - K) \tag{7}$$

is shown in Fig. 1 as a function of k, where K and E are the complete elliptic integrals of the first and the second kinds repectively.

As k increases Δx decreases and the " bay " formed by the curve bulges, so that the filament cuts the axis perpendicularly at $k^2 = \frac{1}{2}$, and at $k \fallingdotseq 0.8551$ neighbouring bays start to cross ; thereafter our solution is not acceptable since the interaction between two widely separated values of s is neglected in our approximation.

The other solution represented by $y = A\,dn$ is unacceptable, since it yields a series of loops lying altogether on one side of the axis.

The general feature of our acceptable case is partly in accordance with that found by numerical experiments by Hama (1963) and Takaki and Yoshizawa (1971) on initially sinusoidal filaments, except for three-dimensional deformation for large k .