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Borsuk-Ulam theorem and formal group

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In this lecture I shall give a generalization of the classical Borsuk-Ulam theorem in connection with the formal group for a generalized cohomology theory.

This work was done in cooperation with Prof. Munkholm.

§1. Formal group law for multiplicative cohomology

We recall first some facts on multiplicative cohomology theory (see Dold [1]).

We fix once and for all a multiplicative reduced cohomology theory h defined on the category of finite CW complexes with base point. There is the corresponding multiplicative cohomology theory h defined on the category of finite CW pairs.

Let ξ be a real n-dimensional vector bundle over a finite CW complex B, and denote by $M(\xi)$ the Thom space for ξ. For each $b \in B$ let ξ_b denote the restriction of ξ over b. Then $\tilde{h}(M(\xi_b))$ is a free h(pt)-module on one generator. ξ is said to be <u>h-orientable</u> if there exists $t(\xi) \in \tilde{h}^n(M(\xi))$ such that $t(\xi)|M(\xi_b)$ is a generator of $\tilde{h}(M(\xi_b))$ for each $b \in B$. Such $t(\xi)$ is called an <u>h-orientation</u> or a <u>Thom class</u> of ξ. By an <u>h-oriented</u> vector bundle we mean a vector bundle in which an h-orientation is given.

Let $D(\xi)$ (or $S(\xi)$) denote the total space of the disc bundle (or the sphere bundle) associated to ξ, and consider the homomorphism

$$\tilde{h}^n(M(\xi)) = h^n(D(\xi), S(\xi)) \xrightarrow{j^*} h^n(D(\xi)) \underset{\cong}{\xrightarrow{p^{*-1}}} h^n(B),$$

where j is the inclusion and p is the projection. The image of $t(\xi)$ under this homomorphism is called the <u>Euler class</u> of the h-oriented bundle ξ, and is denoted by $e(\xi)$.

The following facts are easily proved:

(1.1) If there is a bundle map $f : \xi \longrightarrow \xi'$ and ξ' is h-oriented, then ξ is h-oriented so that $f^* : h(B') \longrightarrow h(B)$ preserves the Euler classes.

(1.2) If ξ_1 and ξ_2 are h-oriented, then the Whitney sum $\xi_1 + \xi_2$ is h-oriented so that $e(\xi_1 + \xi_2) = e(\xi_1)e(\xi_2)$.

(1.3) If ξ has a non-zero cross section, then $e(\xi) = 0$.

The classical Leray-Hirsch theorem on fibering can be generalized to the multiplicative theory h, and so we have Thom isomorphism

$$\Phi : h(B) \cong \tilde{h}(M(\xi))$$

given by $\Phi(\alpha) = \alpha \cdot t(\xi)$. As a consequence, the Gysin exact sequence

$$\cdots \to h^{i-1}(S(\xi)) \to h^{i-n}(B) \xrightarrow{\cdot e(\xi)} h^{i}(B) \xrightarrow{p^*} h^{i}(S(\xi)) \to \cdots$$

holds.

A complex vector bundle ξ is called h-orientable if the real form ξ_R is h-orientable. Let η_n denote the canonical complex line bundle over the complex n-dimensional projective space CP^n. Throughout this section the following will be assumed :

(1.4) For each n, η_n is h-oriented so that the homomorphism $h(CP^{n+1}) \to h(CP^n)$ preserves the Euler classes.

It follows from this assumption that every complex line bundle ξ over a finite CW complex is h-oriented so that every homomorphism $f^* : h(B) \to h(B')$ induced by every bundle map $f : \xi \to \xi'$ preserves the Euler classes.

We can prove

(1.5) The algebra $h(CP^n)$ is a truncated polynomial algebra over $h(pt)$: $h(CP^n) = h(pt)[e(\eta_n)]/(e(\eta_n)^{n+1})$.

(1.6) Put $e(\eta_m)_1 = p_1^* e(\eta_m)$ and $e(\eta_n)_2 = p_2^* e(\eta_n)$ for the projections $p_1 : CP^m \times CP^n \to CP^m$ and $p_2 : CP^m \times CP^n \to CP^n$. Then the isomorphism

$$h(CP^m \times CP^n) = h(pt)[e(\eta_m)_1, e(\eta_n)_2]/(e(\eta_m)_1^{m+1}, e(\eta_n)_2^{n+1})$$

holds.

For a CW complex X with finite skelton, we define $h(X)$ as the inverse limit with respect to skelton:

$$h(X) = \varprojlim h(X^n).$$

Then, for the infinite dimensional projective space CP^∞, the following result is obtained from (1.5) and (1.6).

(1.7) $h(CP^\infty)$ and $h(CP^\infty \times CP^\infty)$ are rings of formal power series:

$$h(CP^\infty) = h[[x]], \quad h(CP^\infty \times CP^\infty) = h[[x_1, x_2]],$$

where x, x_1 x_2 are the elements defined by $e(\eta_n)$, $e(\eta_n)_1$, $e(\eta_n)_2$ respectively.

Let η denote the canonical bundle over CP^∞, and consider the external tensor product $\eta \hat{\otimes} \eta$ which is a complex line bundle over $CP^\infty \times CP^\infty$. Let $\mu : CP^\infty \times CP^\infty \to CP^\infty$ be a classifying map for $\eta \hat{\otimes} \eta$ (which is cellular), and put

$$\mu^*(x) = \sum_{i,j \geq 0} a_{ij}\, x_1^i\, x_2^j \qquad (a_{ij} \in h^{2(1-i-j)}(pt))$$

for $\mu^* : h(CP^\infty) \to h(CP^\infty \times CP^\infty)$. Then we obtain easily

(1.8) For the tensor product $\xi_1 \otimes \xi_2$ of any complex line bundles ξ_1 and ξ_2 over a finite CW complex,

$$e(\xi_1 \otimes \xi_2) = \sum_{i,j\geqq 0} a_{ij}\, e(\xi_1)^i\, e(\xi_2)^j$$

holds.

Consider now a power series $F(x,y)$ with coefficients in $h(pt)$, which is defined by

$$F(x, y) = \sum_{i,j\geqq 0} a_{ij}\, x^i\, y^j$$

with a_{ij} above. Then it follows that $F(x, y)$ is a formal group law over $h(pt)$, i.e. the identities

$$F(x, 0) = x = F(0, x)\ ,\ F(x, y) = F(y, x),$$

$$F(x, F(y, z)) = F(F(x, y), z)$$

hold. For each integer $i \geqq 1$, let $[i](x) \in h[[x]]$ denote the operation of "multiplication by i" for the formal group, i.e.

$$[1](x) = x, \qquad [i](x) = F([i-1](x), x).$$

Since the formula in (1.8) is rewritten as

$$e(\xi_1 \otimes \xi_2) = F(e(\xi_1), e(\xi_2)),$$

we have for the i-fold tensor product $\xi^i = \xi \otimes \cdots \otimes \xi$

$$e(\xi^i) = [i](e(\xi)).$$

Given a positive integer q, let G denote a cyclic group of order q. Define a G-action on the standard $(2n+1)$-sphere $S^{2n+1} = \{(z_0, z_1, \cdots, z_n) \in \mathbf{C}^{n+1} \mid \sum_i |z_i|^2 = 1\}$ by

$$(z_0, \cdots, z_n)g_0 = (z_0 \exp \frac{2\pi i}{q}, \cdots, z_n \exp \frac{2\pi i}{q}),$$

where g_0 is the generator of G. This yields a principal G-bundle $\rho'_n : S^{2n+1} \longrightarrow L^n(q)$ over the lens space $L^n(q)$. Let L denote a 1-dimension complex G-module given by $c \cdot g_0 = c \exp 2\pi i/q$, and consider the associated complex line bundle $\rho_n = \rho'_n \times_G L$. For the canonical projection $\pi : L^n(q) \longrightarrow CP^n$ we have $\rho_n = \pi^*(\eta_n)$, and hence $e(\rho_n)^{n+1} = 0$ holds.

Proposition 1. <u>Let $P(x) \in h(pt)[[x]]$. Then the element $P(e(\rho_n))$ of $h(L^n(q))$ is zero if and only if $P(x)$ is in the ideal generated by x^{n+1} and $[q](x)$.</u>

Proof. Consider the q-fold tensor product $\eta_n^q = \eta_n \otimes \cdots \otimes \eta_n$. Then it is easily checked that the total space $S(\eta_n^q)$ of the sphere bundle associated to η_n^q is homeomorphic with $L^n(q)$. Therefore we have the Gysin sequence

$$\cdots \to h^{i-2}(CP^n) \xrightarrow{\cdot e(\eta_n{}^q)} h^i(CP^n) \xrightarrow{\pi^*} h^i(L^n(q)) \longrightarrow \cdots .$$

Since $e(\eta_n{}^q) = [q](e(\eta_n))$, the sequence and (1.5) derive the desired result (see [2]).

§2. The element $s^*(\theta)$

As in §1, let G denote a cyclic group of order q. We shall assume that q is <u>odd</u>.

For any space X, let XG denote the product of q copies of X. Writing its elements as $\sum_{g\in G} x_g g$, a G-action on XG is given by

$$(\sum_{g\in G} x_g g)\cdot h = \sum_{g\in G} x_{gh^{-1}} g \qquad (h \in G).$$

We denote by ΔX the diagonal in XG.

Let Σ be a homotopy $(2n+1)$-sphere (which is a differentiable manifold), and assume that there is given a free differentiable G-action on Σ. We denote by Σ_G the orbit space.

Let M be a differentiable manifold, and consider the diagonal action on $\Sigma \times MG$ whose orbit space is denoted by $\Sigma \underset{G}{\times} MG$. $\Sigma \times \Delta M$ is an invariant submanifold of the G-manifold $\Sigma \times MG$, and its orbit space is regarded as $\Sigma_G \times \Delta M$. We denote by ν the normal bundle of $\Sigma_G \times \Delta M$ in $\Sigma \underset{G}{\times} MG$.

This is a real m(q-1)-dimensional vector bundle.

Choose a point $y_0 \in M$, and we shall identify Σ_G with a subspace $\Sigma_G \times y_0 G$ of $\Sigma_G \times \Delta M$.

Let $\lambda' : \Sigma \to \Sigma_G$ denote the principal G-bundle defined by the G-action on Σ, and consider the associated complex line bundle $\lambda = \lambda' \times_G L$.

We have the following (see [4])

Proposition 2. <u>The normal bundle ν has a complex structure for which</u>

$$\underline{i^*(\nu) = m(\lambda \oplus \lambda^2 \oplus \cdots \oplus \lambda^{(q-1)/2})}$$

<u>holds, where $i : \Sigma_G \longrightarrow \Sigma_G \times \Delta M$ is the inclusion.</u>

Proof. If $\nu_1 : N_1 \longrightarrow \Delta M$ denote the normal G-vector bundle of ΔM in MG, then we have $\nu = 1 \times_G \nu_1 : \Sigma \times_G N_1 \longrightarrow \Sigma_G \times \Delta M$. Therefore it suffices to prove that there exists an G-equivariant complex structure on ν_1 with the fiber over $y_0 G$ being $m(L \oplus \cdots \oplus L^{(q-1)/2})$.

To prove this, let IG be defined by the exact sequence of real G-modules

$$0 \to \Delta\mathbb{R} \to \mathbb{R}G \to IG \to 0.$$

View this as a sequence of real G-vector bundles over a point, and identify ΔM with $M \times pt = M$ in the obvious way. Then we have the exact sequence

$$0 \to \tau M \hat{\otimes} \Delta \mathbb{R} \to \tau M \hat{\otimes} \mathbb{R}G \to \tau M \hat{\otimes} IG \to 0$$

of real G-vector bundle over M, where τM denotes the tangent bundle over M. Since $\tau(MG) = (\tau M)G$, an equivariant isomorphism

$$\beta : \tau(MG)|\Delta M \to \tau M \hat{\otimes} \mathbb{R}G$$

can be given by

$$\beta(\sum_g v_g g) = \sum_g v_g \otimes g \qquad (v_g \in \tau_y(M),\ y \in M).$$

Since $\sum_g v_g g$ is in $\tau(\Delta M)$ if and only if all v_g are equal, β maps $\tau(\Delta M)$ onto $\tau M \hat{\otimes} \Delta\mathbb{R}$. Thus it holds that $\nu_1 \cong \tau M \hat{\otimes} IG$ as real G-vector bundles. From elementary representation theory of groups, it follows that IG is the real form of $L \oplus \cdots \oplus L^{(q-1)/2}$. This gives ν_1 its complex structure, and we get

$$(\nu_1)_{y_0} = \tau_{y_0} M \otimes (L \oplus \cdots \oplus L^{(q-1)/2})$$
$$= \mathbb{R}^m \otimes (L + \cdots + L^{(q-1)/2}) = m(L \oplus \cdots \oplus L^{(q-1)/2})$$

as desired. This completes the proof.

As in 1, let h be a given multiplicative cohomology theory. In the following we shall assume the following conditions:

(2.1) every complex vector bundle of any dimension is h-orientable.

(2.2) $h^{odd}(pt) = 0$.

Assuming that M is closed, consider the normal bundle ν. Then, by Proposition 2 and (2.1), we have a Thom class $t(\nu) \in \tilde{h}^{m(q-1)}(M(\nu))$ and the corresponding Euler class $e(\nu) \in h^{m(q-1)}(\Sigma_G \times_\Delta M)$ such that

$$
\begin{aligned}
i^*e(\nu) &= e(m(\lambda \oplus \lambda^2 \oplus \cdots \oplus \lambda^{(q-1)/2}) \\
(2.3) \qquad &= \left(\prod_{i=1}^{(q-1)/2} [i](e(\lambda))\right)^m.
\end{aligned}
$$

As usual we shall regard the total space N of ν as a tubular neighborhood of $\Sigma_G \times_\Delta M$ in $\Sigma \times_G MG$. Then we can identify $\tilde{h}(M(\nu))$ with $h(\Sigma \times_G MG, \Sigma \times_G MG - N)$ canonically. Define

$$\vartheta \in h^{m(q-1)}(\Sigma \times_G MG)$$

to be the image of the Thom class $t(\nu)$ under the homomorphism

ℓ^* : $h(\Sigma \times_G MG, \Sigma \times_G MG - N) \to h(\Sigma \times_G MG)$ induced by the inclusion. We have immediately

(2.4) For the homomorphism $j^* : h(\Sigma \times_G MG) \to h(\Sigma_G \times \Delta M)$ induced by the inclusion, $j^*(\theta) = e(\nu)$ holds.

Given a continuous map $f : \Sigma \to M$, define a continuous map $s : \Sigma_G \to \Sigma \times_G MG$ by

$$s(xG) = (x, \sum_g f(xg^{-1})g)G.$$

For the projection $p : \Sigma \times_G MG \to \Sigma_G$, $p \circ s$ is the identity.

Proposition 3. <u>For the homomorphism $s^* : h(\Sigma \times_G MG) \to h(\Sigma_G)$ and the homomorphism $i^* : h(\Sigma_G \times \Delta M) \to h(\Sigma_G)$, we have</u>

$$s^*(\theta) = i^*(e(\nu)).$$

Proof. It is easily seen that there exist a continuous map $f_1 : \Sigma \to M$ and an open set V of Σ satisfying the following conditions: i) f is homotopic to f_1, ii) V is homeomorphic to R^{2n+1}, iii) $f_1(\Sigma - V) = y_0$, iv) $xg \notin V$ for any $g \neq 1$ and any $x \in V$. Define $s_1 : \Sigma_G \to \Sigma \times_G MG$ from f_1 as in s, then s and s_1 are homotopic. Let $(MG)_1$

denote the subspace of MG consisting of points with at most one coordinate $\neq y_0$. Then $(MG)_1$ is an invariant subspace of the G-space MG, and the orbit space $\Sigma \times_G (MG)_1$ contains $s_1(\Sigma_G)$. Since $\Sigma - V$ is contractible, there exists a homotopy $\psi_t : (\bar{V}, \partial V) \to (\Sigma, \Sigma - V)$ such that ψ_0 is the inclusion and $\psi_1(\partial V) = x_0$ $(\in \partial V)$. Put $V_G = \pi(V)$ for the projection $\pi : \Sigma \to \Sigma_G$, and let $\bar{V}$, $\bar{V}_G$ denote the closure of V, V_G respectively. Consider now the following diagram:

$$
\begin{array}{ccccccc}
 & & \Sigma_G & \xrightarrow{s_1} & \Sigma \times_G (MG)_1 \\
 & & \downarrow j_2 & & \downarrow j_1 \\
(\bar{V}_G, \partial V_G) & \xrightarrow{k} & (\Sigma_G, \Sigma_G - V_G) & \xrightarrow{s_1} & (\Sigma \times_G (MG)_1, \Sigma_G \times y_0 G) \\
\uparrow \pi & & & & \uparrow \varphi \\
(\bar{V}, \partial V) & \xrightarrow{\psi_1} & (\Sigma, x_0) & \xrightarrow{(1, f_1)} & (\Sigma \times M, \Sigma \times y_0),
\end{array}
$$

where j_1, j_2, k are the inclusions, and φ is given by

$$\varphi(x, y) = (x, y \cdot 1 + \sum_{g \neq 1} y_0 g)G.$$

Since a homotopy $\bar{\psi}_t : (\bar{V}, \partial V) \to (\Sigma \times_G (MG)_1, \Sigma_G \times y_0 G)$ of $s_1 \circ k \circ \pi$ to $\varphi \circ (1, f_1) \circ \psi_1$ can be defined by

$$\bar{h}_t(v) = (h_t(v),\ f_1(h_t(v))\cdot 1 + \sum_{g\neq 1} y_0 g)G \qquad (v \in \bar{V}),$$

the above diagram is homotopy commutative. Therefore we have $\pi^* \circ k^* \circ s_1^* = \psi_1^* \circ (1, f_1)^* \circ \rho^* : h(\Sigma \times_G (MG)_1, \Sigma_G \times y_0 G) \to h(\bar{V}, \partial V)$.

We have

$$h^{m(q-1)}(\Sigma, x_0) = \tilde{h}^{m(q-1)}(S^{2n-1}) = h^{m(q-1)-(2n-1)}(pt) = 0$$

by (2.2), and k^* and π^* are isomorphisms. Therefore $s_1^* : h^{m(q-1)}(\Sigma \times_G (MG)_1, \Sigma_G \times y_0 G) \to h^{m(q-1)}(\Sigma_G, \Sigma_G - V_G)$ is trivial, and consequently

$$s_1^* \circ j_1^* ; h^{m(q-1)}(\Sigma \times_G (MG)_1, \Sigma_G \times y_0 G) \to h^{m(q-1)}(\Sigma_G)$$

is trivial.

Next consider the commutative diagram

$$\begin{array}{ccccc}
h(\Sigma \times_G MG) & & \xrightarrow{\quad j^* \quad} & & h(\Sigma_G \times \Delta M) \\
\big\downarrow s^* & \searrow i^* & & & \big\downarrow i^* \\
 & & h(\quad {}_G (MG)_1) & & \\
 & \swarrow s_1^* & \big\uparrow p^* & \searrow i_1^* & \\
h(\Sigma_G) & = & h(\Sigma_G) & = & h(\Sigma_G \quad y_0 G)
\end{array}$$

where i_1, i_2 are the inclusions. Putting $\theta' = p^* i_1^* i_2^*(\theta) - i_2^*(\theta)$ we have

$$s_1^*(\theta') = i^* j^*(\theta) - s^*(\theta) = i^*(e(\nu)) - s^*(\theta)$$

by (2.4), and $i_1^*(\theta') = 0$. Therefore θ' is in the image of $j_1^* : h^{m(q-1)}(\Sigma \underset{G}{\times} (MG)_1, \Sigma_G \times y_0 G) \longrightarrow h^{m(q-1)}(\Sigma_G \times (MG)_1)$, and hence $s_1^*(\theta) = 0$ by the fact proved above. Thus we have proved $i^*(e(\ )) = s^*(\ )$.

§ 3. Generalization of Borsuk-Ulam theorem

Let Σ be as in § 2, and let $f : \Sigma \longrightarrow M$ be a continuous map to a differentiable m-manifold. Put

$$A(f) = \{ x \in \Sigma \mid f(x) = f(xg) \text{ for any } g \in G \}.$$

In this section we shall consider the covering dimension of $A(f)$.

For the image $A(f)_G = \pi(A(f))$, it follows from dimension theory that $\dim A(f) = \dim A(f)_G$.

Proposition 4. <u>Assume that M is closed. Then $\dim A(f) < 2d$ implies $e(d\lambda)\, s^*(\theta) = 0$.</u>

Proof. Since $\dim A(f)_G \leqq 2d - 1$, it follows that $d\lambda$ has a non-zero cross section over $A(f)_G$ (see [3], Lemma 2). By standard facts on extension of cross section, this cross section extends to a non-zero cross section over the closure $\bar{W}$ of some neighborhood W of $A(f)_G$ in Σ_G. Here we may assume that $\bar{W}$ is a finite CW complex, and that $s(\Sigma_G - W) \subset \Sigma \underset{G}{\times} MG - N$ by taking N small. We have then $e(d\lambda \mid \bar{W}) = 0$, and so $e(d\lambda)$ is in the image of $\ell_1^* : h(\Sigma_G, \bar{W}) \longrightarrow h(\Sigma_G)$ induced by the inclusion.

On the other hand, it follows from the commutative diagram

$$\begin{array}{ccc} h(\Sigma \underset{G}{\times} MG,\ \Sigma \underset{G}{\times} MG - N) & \xrightarrow{\ell^*} & h(\Sigma \underset{G}{\times} MG) \\ \Big\downarrow s^* & & \Big\downarrow s^* \\ h(\Sigma_G,\ \Sigma_G - W) & \xrightarrow{\ell_2^*} & h(\Sigma_G) \end{array}$$

(ℓ, ℓ_2 : inclusions), that $s^*(\theta)$ is in the image of ℓ_2^*.

Therefore $e(d\lambda)\, s^*(\theta)$ is in the image of the homomorphism $h(\Sigma_G, \bar{W} \cup (\Sigma_G - W)) = h(\Sigma_G, \Sigma_G) \longrightarrow h(\Sigma_G)$, and hence we have the desired result.

We shall now prove the main theorem.

Theorem 1. <u>Let G be a cyclic group of odd order q,</u>

and Σ be a (2n+1)-sphere on which a free differentiable G-action is given. Let M be a differentiable m-manifold. Assume that there exists a continuous map $f : \Sigma \to M$ with $\dim A(f) < 2d$. Then, for any multiplicative cohomology theory h defined on the category of finite CW pairs and satisfying the conditions (2.1), (2.2),

$$x^d\left(\prod_{i=1}^{(q-1)/2} [i]\,(x)\right)^m \in h[[x]]$$

is contained in the ideal generated by x^{n+1} and $[q](x)$.

Proof. Recall that any differentiable m-manifold is regarded as an increasing union of compact differentiable m-manifold, and that any differentiable m-manifold with boundary is contained in a differentiable m-manifold without boundary. Since Σ is connected and compact, it follows from these facts that we may assume M to be closed without loss of generality.

Then, in virtue of (2.3), Prop. 3 and Prop. 4, we have

$$e(\lambda)^d\left(\prod_{i=1}^{(q-1)/2} [i]\,(e(\lambda))\right)^m$$

$$= e(d\lambda)\cdot i^*e(\nu) = e(d\lambda)\cdot s^*(\theta) = 0.$$

Since ρ'_n is a principal G-bundle whose base space is

(2n+1)-dimensional CW complex, and since λ' is a (2n+1)-universal principal G-bundle, there is a bundle map of ρ_n to λ. Hence the last equation implies

$$e(\rho_n)^d \Big(\prod_{i=1}^{(q-1)/2} [i]\, (e(\rho_n)) \Big)^m = 0$$

From this and Prop. 1 we have the desired result.

As typical examples of the multiplicative cohomology theory satisfying the conditions in Theorem, we have the classical integral cohomology theory $H^*(\ ;\mathbb{Z})$, the Grothendieck-Atiyah·Hirzebruch periodic cohomology theory $K^*(\ )$ of K-theory, and the complex cobordism theory $U^*(\ )$ obtained from the Milnor spectrum MU (see [5]).

As is well known, $H^i(pt;\mathbb{Z}) = \mathbb{Z}$ $(i = 0)$, $= 0$ $(i \neq 0)$ and the formal group law for $H^*(\ ;\mathbb{Z})$ is given by $F(x, y) = x + y$. Hence the conclusion in Theorem 1 for $h(\ ) = H^*(\ ;\mathbb{Z})$ is stated that

$$\Big(\frac{q-1}{2}!\Big)^m x^{d+m(q-1)/2} \in \mathbb{Z}[x]$$

is contained in the ideal generated by x^{n+1} and qx. From this we obtain the following result.

(3.1) If q is an odd prime, for any continuous map $f : \Sigma \to M$ we have $\dim A(f) \geqq 2n - m(q-1)$.

Remark. The conclusion in (3.1) is strengthened to $\dim A(f) \geqq 2n + 1 - m(q-1)$ (see [6], [7]).

For $K^*(\ )$ it is known that $K^{even}(pt) = \mathbf{Z}$, $K^{odd}(pt) = 0$ and the formal group law is given by $F(x, y) = x + y + xy$. Therefore the conclusion in Theorem for $h(\ ) = K^*(\ )$ is stated that

$$x^d \left(\prod_{i=1}^{(q-1)/2} ((x + 1)^i - 1) \right)^m \in \mathbf{Z}[x]$$

is contained in the ideal generated by x^{n+1} and $(x+1)^q - 1$. Putting $y = x + 1$ this is restated that

$$(y - 1)^d \left(\prod_{i=1}^{(q-1)/2} (y^i - 1) \right)^m \in \mathbf{Z}[y]$$

is contained in the ideal generated by $(y - 1)^{n+1}$ and y^{q-1}. If q is an odd prime power p^a, it can be proved by making use of elementary algebraic number theory that the above statement is equivalent to

$$d \geqq n + p^{a-1} - \frac{1}{2} am(p^a - p^{a-1})$$

(see [3]). Thus theorem 1 implies the following theorem containing (3.1) and being a generalization of a result in [3].

Theorem 2. <u>If q is an odd prime power p^a,</u> <u>for any continuous map $f : \Sigma \to M$ we have</u>

$$\underline{\dim A(f) \geqq 2n + 2p^{a-1} - am(p^a - p^{a-1}) - 2}.$$

For $U^*(\ )$ it is known that $U^*(pt)$ is a polynomial ring over $\mathbb{Z}$ with one generator of degree $-2i$ for each positive integer i, However the formal group law for $U^*(\ )$ is rather complicated(see e.g. [8]) and I have no method to derive numerical condition equivalent to the conclusion in Theorem 1. Since the cobordism theory is stronger than K-theory in general, it is expected that sharper result than Theorem 2 will be obtained from Theorem 1 for complex cobordism.

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