

Unitary representations of p-adic division algebras

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Let $p = a$ a prime, $k = a$ \mathfrak{F} -adic field or $F_q((t))$ such that \mathfrak{F} divides p or q is a power of p , $E = a$ central simple algebra over k , $E^\times = a$ the multiplicative group of E , $E^{(1)} = \{ \chi \in E^\times \mid \text{reduced norm of } \chi = 1 \}$ and $n^2 = [E : k]$. Then, by the natural topology from k , E^\times is a locally compact (totally disconnected) topological group, and its center is exactly k^\times .

Problem (I) : Find all the unitary representations of E^\times , $E^{(1)}$ and E^\times/k^\times .

Note that E^\times (resp. $E^{(1)}$; resp. E^\times/k^\times) is the group of k -rational points of a k -form of GL_n (resp. SL_n ; resp. PGL_n). The problem (I) is solved so far only when $n = 2$, p odd and $E = M_2(k)$ [by Kirillov]. Suppose E to be a division algebra, then E has a unique maximal compact subring $R = R(E)$, and for any subring L of E , $R(L) = R(E) \cap L$ is a unique maximal compact subring of L . Then since we have $E^{(1)} \subset R(E)$ (hence $E^{(1)} = R^\times \cap E^{(1)}$), and since $R^\times/R(k)^\times$ is of finite index in E^\times/k^\times , the solution of (I) is not very far from that of the following.

Problem (I') : Find all the unitary representations of $R(E)^\times/R(k)^\times$.

Again under the assumption that $n = 2$, (i.e. E is a quaternion) and p odd, (I') is solved by Gelfond-Graev. Now we have solved (I') and (I) under the following

Assumption : (1) E is a division algebra, (2) n is an odd prime different from p , (3) k contains a primitive n -th root of unity.

Our result is surprisingly simple, for instance simpler than the

Gelfond-Graev case of $n = 2$. Indeed every representation of $R^x/R(k)^x$ can be obtained as a monomial representation. We start with a maximal subfield L , take a character χ of $R(L)^x/R(k)^x$, extend χ to a linear character $\tilde{\chi}$ of a certain open subgroup A which we choose as big as possible, then induce $\tilde{\chi}$ from A to whole $R^x/R(k)^x$. Then we get all the representations.

Severe as our assumptions are, our result provides the first example of a solution of (I) for groups with (absolute) rank greater than 2.