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Oscillatory Property for Second Order Differential Equations

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There are many results on oscillatory property of solutions of differential equations. In this article, we shall discuss oscillatory property of solutions and the existence of a bounded nonoscillatory solution of second order differential equations by applying Liapunov second method. Consider an equation

$$(1) \qquad (r(t)x')' + f(t, x, x') = 0 \qquad (' = \frac{d}{dt}),$$

where $r(t) > 0$ is continuous on $I = [0, \infty)$ and $f(t, x, u)$ is continuous on $I \times R \times R$, $R = (-\infty, \infty)$. To discuss oscillatory property of solutions of (1), we consider an equivalent system

$$(2) \qquad x' = \frac{y}{r(t)}, \qquad y' = -f(t, x, \frac{y}{r(t)}).$$

A solution $x(t)$ of (1) which exists in the future is said to be oscillatory if for every $T > 0$ there is a $t_0 > T$ such that $x(t_0) = 0$. Moreover, the equation (1) is said to be oscillatory if every solution of (1) which exists in the future is oscillatory.

Theorem 1. Assume that there exist two continuous scalar function $V(t, x, y)$ and $W(t, x, y)$ defined on $t \geq T$, $0 < x < K$, $|y| < \infty$ and on $t \geq T$, $-K < x < 0$, $|y| < \infty$, respectively, where T can be large and $K > 0$ or $K = \infty$, and assume that $V(t, x, y)$ and $W(t, x, y)$ satisfy the following conditions;

(i) $V(t, x, y) \to \infty$ uniformly for $0 < x < K$ and $-\infty < y < \infty$ as $t \to \infty$, and $W(t, x, y) \to \infty$ uniformly for $-K < x < 0$ and $-\infty < y < \infty$ as $t \to \infty$,

(ii) $\dot{V}_{(2)}(t, x(t), y(t)) \leqslant 0$ for all sufficiently large t, where $\{x(t), y(t)\}$ is a solution of (2) such that $0 < x(t) < K$ for all large t and

$$\dot{V}_{(2)}(t, x(t), y(t)) = \overline{\lim_{h\to 0^+}} \frac{1}{h} \{V(t+h, x(t+h), y(t+h)) - V(t, x(t), y(t))\},$$

(iii) $\dot{W}_{(2)}(t, x(t), y(t)) \leqslant 0$ for all sufficiently large t, where $\{x(t), y(t)\}$ is a solution of (2) such that $-K < x(t) < 0$ for all large t and

$$\dot{W}_{(2)}(t, x(t), y(t)) = \overline{\lim_{h\to 0^+}} \frac{1}{h} \{W(t+h, x(t+h), y(t+h)) - W(t, x(t), y(t))\}.$$

Then the solution $x(t)$ of (1) such that $|x(t)| < K$ for all large t is oscillatory. Moreover, if $K = \infty$, the equation (1) is oscillatory.

Proof. Let $x(t)$ be a solution of (1) which is defined on $[t_0, \infty)$ and bounded by K for all large t, and suppose that $x(t)$ is not oscillatory. Then $x(t)$ is either positive or negative for all large t. Now assume that $0 < x(t) < K$ for all $t \geqslant \sigma$ where $\sigma \geqslant T$. By the condition (i), if t is sufficiently large, say $t \geqslant t_1$, we have

$$V(\sigma, x(\sigma), y(\sigma)) < V(t, x, y)$$

for all $0 < x < K$, $|y| < \infty$. However, by the condition (ii), we have

$$V(t, x(t), y(t)) \leqslant V(\sigma, x(\sigma), y(\sigma))$$

for all $t \geqslant \sigma$, if necessary, choosing a large σ. This contradicts $V(t_1, x(t_1), y(t_1)) > V(\sigma, x(\sigma), y(\sigma))$. When we assume that $-K < x(t) < 0$ for all large t, we have also a contradiction by using $W(t, x(t), y(t))$. Thus we see that $x(t)$ is oscillatory.

To apply this theorem, the following lemmas play an important role. In the following, a scalar function $v(t, x, y)$ will be called a Liapunov function for (2), if $v(t, x, y)$ is continuous in (t, x, y) in the domain of definition and is locally Lipschitzian in (x, y). Moreover, we define $\dot{v}_{(2)}(t, x, y)$ by

$$\dot{v}_{(2)}(t, x, y) = \overline{\lim_{h\to 0^+}} \frac{1}{h} \{ v(t+h,\ x + h\frac{y}{r(t)},\ y - hf(t, x, \frac{y}{r(t)})) - v(t, x, y) \} .$$

<u>Lemma 1</u>. For $t \geqslant T^*$, $x > 0$, $|y| < \infty$, where T^* can be large, we assume that there exists a Liapunov function $v(t, x, y)$ which satisfies the following conditions;

(i) $yv(t, x, y) > 0$ for $t \geqslant T^*$, $x > 0$, $y \neq 0$,

(ii) $\dot{v}_{(2)}(t, x, y) \leqslant -\lambda(t)$, where $\lambda(t)$ is a continuous function defined on $t \geqslant T^*$ and $\underline{\lim}_{t\to\infty} \int_T^t \lambda(s)ds \geqslant 0$ for all large T.

Moreover, we assume that there is a τ and a $w(t, x, y)$ for all large T such that $\tau \geqslant T$ and $w(t, x, y)$ is a Liapunov function defined on $t \geqslant \tau$, $x > 0$, $y < 0$, which satisfies the following conditions;

(iii) $y \leqslant w(t, x, y)$ and $w(\tau, x, y) \leqslant b(y)$, where $b(y)$ is continuous, $b(0) = 0$ and $b(y) < 0$ $(y \neq 0)$,

(iv) $\dot{w}_{(2)}(t, x, y) \leqslant -\rho(t)w(t, x, y)$, where $\rho(t) \geqslant 0$ is continuous and

$$\int_\tau^\infty \frac{1}{r(t)} \exp\{-\int_\tau^t \rho(s)ds\}dt = \infty .$$

Then, if $\{x(t), y(t)\}$ is a solution of (2) such that $x(t) > 0$ for all large t, we have $y(t) \geqslant 0$ for all large t.

We can obtain a similar lemma for a solution $\{x(t), y(t)\}$ of (2) such that $x(t) < 0$ for all large t. For the proof of Lemma 1 and the details, see [5].

Proposition 1. For the equation (1) we assume that

(i) $\int_0^\infty \frac{dt}{r(t)} = \infty$,

(ii) for $t \geqslant 0$ and $x \geqslant 0$, there exist continuous functions $a(t)$ and $\alpha(x)$ such that

(3) $$\lim_{t \to \infty} \int_T^t a(s)ds \geqslant 0 \qquad \text{for all large } T$$

and that $x\alpha(x) > 0$ $(x \neq 0)$, $\alpha'(x) \geqslant 0$ and for all large t, $x \geqslant 0$, $|u| < \infty$

$$a(t)\alpha(x) \leqslant f(t, x, u),$$

(iii) for $t \geqslant 0$ and $x \leqslant 0$, there exist continuous functions $b(t)$ and $\beta(x)$ such that

$$\lim_{t \to \infty} \int_T^t b(s)ds \geqslant 0 \qquad \text{for all large } T$$

and that $x\beta(x) > 0$ $(x \neq 0)$, $\beta'(x) \geqslant 0$ and for all large t, $x \leqslant 0$, $|u| < \infty$

$$f(t, x, u) \leqslant b(t)\beta(x).$$

Then, if $\int_0^\infty a(t)dt = \infty$ and $\int_0^\infty b(t)dt = \infty$, the equation (1) is oscillatory. Moreover, if we have

(4) $$\int_0^\infty a(t)dt < \infty, \qquad \int_0^\infty \left(\frac{1}{r(s)} \int_s^\infty a(u)du\right)ds = \infty$$

and

(5) $$\int_0^\infty b(t)dt < \infty, \qquad \int_0^\infty \left(\frac{1}{r(s)} \int_s^\infty b(u)du\right)ds = \infty,$$

then all bounded solutions of (1) are oscillatory. In addition to the conditions above, if

(6) $$\int_{\epsilon}^{\infty} \frac{du}{\alpha(u)} < \infty, \quad \int_{-\epsilon}^{-\infty} \frac{du}{\beta(u)} < \infty \quad \text{for some } \epsilon > 0,$$

the equation (1) is oscillatory.

Proof. Under our assumptions, if we consider a function $v(t, x, y) = \dfrac{y}{\alpha(x)}$ for large t, this function satisfies the conditions in Lemma 1 with $\lambda(t) = a(t)$. Since the condition (3) implies that for all large T, there is a τ such that $\tau \geqslant T$ and $\int_{\tau}^{t} a(s)ds \geqslant 0$ for all $t \geqslant \tau$, a function $w(t, x, y) = y + \alpha(x) \int_{\tau}^{t} a(s)ds$ defined on $t \geqslant \tau$, $x > 0$, $y < 0$ satisfies the conditions in Lemma 1 with $\rho(t) \equiv 0$. Thus we can see that if $\{x(t), y(t)\}$ is a solution of (2) such that $x(t) > 0$ for all large t, then $y(t) \geqslant 0$ for all large t. We can also see that for a solution such that $x(t) < 0$ for all large t, $y(t) \leqslant 0$ for all large t.

In the case where we assume that $\int_{0}^{\infty} a(t)dt = \infty$ and $\int_{0}^{\infty} b(t)dt = \infty$, for large t, if we define $V(t, x, y)$ and $W(t, x, y)$ by

$$V(t,x,y) = \begin{cases} \dfrac{y}{\alpha(x)} + \displaystyle\int_{0}^{t} a(s)ds & (y \geqslant 0) \\ \displaystyle\int_{0}^{t} a(s)ds & (y < 0) \end{cases}$$

and

$$W(t,x,y) = \begin{cases} \displaystyle\int_{0}^{t} b(s)ds & (y > 0) \\ \dfrac{y}{\beta(x)} + \displaystyle\int_{0}^{t} b(s)ds & (y \leqslant 0), \end{cases}$$

we can see that these functions satisfy the conditions in Theorem 1 for $K = \infty$, and hence the equation (1) is oscillatory.

In the case where we assume (4) and (5), letting $K > 0$ be a constant, se.

$$V(t, x, y) = \int_x^K \frac{du}{\alpha(u)} + \int_0^t \left(\frac{1}{r(s)} \int_s^\infty a(u)du\right)ds$$

for $t \geqslant 0$, $0 < x < K$ and $|y| < \infty$. For a solution $x(t)$ of (2) which satisfies $0 < x(t) < K$ for all large t, there is a $\sigma > 0$ such that $0 < x(t) < K$ and $y(t) \geqslant 0$ for $t \geqslant \sigma$, and hence

$$\dot{V}_{(2)}(t, x(t), y(t)) = \frac{1}{r(t)} \left\{ -\frac{y(t)}{\alpha(x(t))} + \int_t^\infty a(u)du \right\}.$$

If we set $V^*(t, x, y) = -\frac{y}{\alpha(x)} + \int_t^\infty a(u)du$, we have $\overline{\lim_{t\to\infty}}\, V^*(t, x(t), y(t)) \leqslant 0$. On the other hand, we have $\dot{V}^*_{(2)}(t, x, y) \geqslant 0$, and hence $V^*(t, x(t), y(t)) \leqslant 0$, which implies that $\dot{V}_{(2)}(t, x(t), y(t)) \leqslant 0$ for $t \geqslant \sigma$. For $t \geqslant 0$, $-K < x < 0$ and $|y| < \infty$, define $W(t, x, y)$ by

$$W(t, x, y) = \int_x^{-K} \frac{du}{\beta(u)} + \int_0^t \left(\frac{1}{r(s)} \int_s^\infty b(u)du\right)ds .$$

Then the conclusion follows from Theorem 1, because K is arbitrary.

In addition, when we assume (6), we can set $K = \infty$ in $V(t, x, y)$ and $W(t, x, y)$ above, and hence the equation (1) is oscillatory.

The result above contains Coles' result [2] and Macki and Wong's result [3].

Remark. It is clear that we can combine the conditions on $a(t)$ and $b(t)$. The Liapunov's method is also applicable to obtain Bobisud's [1] and Opial's [4] results, see [5].

Now we shall discuss the existence of a bounded nonoscillatory solution of (1). The following theorems will be applied. Consider an equation of the second order

$$x'' = F(t, x, x'), \tag{7}$$

where $F(t, x, y)$ is continuous on $I \times R \times R$. Let $\underline{\omega}(t)$ and $\overline{\omega}(t)$ be two functions defined on I, twice differentiable and bounded on I with their derivatives. We assume that $\underline{\omega}(t) \leqslant \overline{\omega}(t)$,

$$(8) \qquad \overline{\omega}''(t) \leqslant F(t, \overline{\omega}(t), \overline{\omega}'(t))$$

and

$$(9) \qquad \underline{\omega}''(t) \geqslant F(t, \underline{\omega}(t), \underline{\omega}'(t))$$

for all $t \geqslant 0$.

Theorem 2. Suppose that there exist two Liapunov functions $V(t, x, y)$ and $W(t, x, y)$ defined on $0 \leqslant t < \infty$, $\underline{\omega}(t) \leqslant x \leqslant \overline{\omega}(t)$, $y \geqslant K$ and on $0 \leqslant t < \infty$, $\underline{\omega}(t) \leqslant x \leqslant \overline{\omega}(t)$, $y \leqslant -K$, respectively, where $K > 0$ can be large, and assume that $V(t, x, y)$ and $W(t, x, y)$ satisfy the following conditions;

(i) $V(t, x, y) \leqslant b(y)$ and $W(t, x, y) \leqslant b(|y|)$, where $b(r) > 0$ is continuous,

(ii) $V(t, x, y) \to \infty$ as $y \to \infty$, $W(t, x, y) \to \infty$ as $y \to -\infty$, uniformly for t, x,

(iii) in the interior of their domains of definition

$$\dot{V}(t, x, y) = \overline{\lim_{h \to 0^+}} \frac{1}{h} \{ V(t+h, x+hy, y+hF(t, x, y)) - V(t, x, y) \} \geqslant 0$$

and

$$\dot{W}(t, x, y) = \overline{\lim_{h \to 0^+}} \frac{1}{h} \{ W(t+h, x+hy, y+hF(t, x, y)) - W(t, x, y) \} \leqslant 0$$

or

(iii)′ in the interior of their domains of definition

$$\dot{V}(t, x, y) \leqslant 0 \qquad \text{and} \qquad \dot{W}(t, x, y) \geqslant 0 .$$

Then the equation (7) has a solution $x(t)$ such that $\underline{\omega}(t) \leqslant x(t) \leqslant \overline{\omega}(t)$ and $x'(t)$ is bounded for all $t \geqslant 0$.

Theorem 3. Under the assumptions in Theorem 2, if $\underline{\omega}(0) = \overline{\omega}(0)$ and

$$\dot{V}(t, x, y) \leqslant 0 \quad \text{and} \quad \dot{W}(t, x, y) \leqslant 0$$

in the interiors of their domains of definition, then the equation (7) has a solution $x(t)$ such that $\underline{\omega}(t) \leqslant x \leqslant \overline{\omega}(t)$ and $x'(t)$ is bounded for all $t \geqslant 0$.

For the proofs, see [6].

In discussing the existence of a bounded nonoscillatory solution of (1), we assume that the derivative of $r(t)$ is continuous, and consequently the equation (1) can be written as

$$x'' + \frac{r'(t)}{r(t)} x' + \frac{1}{r(t)} f(t, x, x') = 0. \tag{10}$$

Proposition 2. Suppose that there exist functions $b(t)$ and $\beta(x)$ which satisfy the following conditions;

(i) $b(t)$ is continuous on I and $b(t) \geqslant 0$ for $t \geqslant T$, where T can be large,

(ii) $\beta(x)$ is continuous on $x \geqslant 0$,

(iii) for $t \geqslant T$, $x > 0$ and all y,

$$f(t, x, y) \leqslant b(t)\beta(x). \tag{11}$$

Moreover, we assume that there is a $c > 0$ such that $f(t, c, 0) \geqslant 0$ for $t \geqslant T$. Then, if

$$0 < \epsilon \leqslant r(t) \leqslant \rho \qquad \text{for some } \epsilon, \rho \text{ and all } t \geqslant 0 \tag{12}$$

and

$$\int_0^\infty t b(t) dt < \infty \tag{13}$$

or if, there is an $A > 0$ such that

(14) $$\left|\frac{r'(t)}{r(t)}\right| < A \qquad \text{for } t \geq 0$$

and we have

(15) $$\int_0^\infty \left(\frac{1}{r(s)} \int_s^\infty b(u)du\right)ds < \infty,$$

the equation (10) has a bounded nonoscillatory solution.

Proof. Under the conditions (12) and (13), the condition (13) implies that $\int_0^\infty b(t)dt < \infty$, and consequently $\int_s^\infty b(u)du$ exists and is small if s is sufficiently large, because $b(t) \geq 0$ eventually. Since $\epsilon \leq r(t) \leq \rho$, we have

$$\int_0^\infty \left(\frac{1}{r(s)} \int_s^\infty b(u)du\right)ds < \infty \quad \text{and} \quad \frac{1}{r(t)} \int_t^\infty b(u)du < \infty.$$

There is an $L > 0$ such that $\beta(c) \leq \frac{L}{2}$ and there is a $\delta > 0$ such that $\beta(x) \leq L$ if $|x-c| \leq \delta$. Choose $t_0 \geq T$ so large that

$$0 \leq L \int_{t_0}^t \left(\frac{1}{r(s)} \int_s^\infty b(u)du\right)ds \leq \delta \qquad \text{for all } t \geq t_0.$$

For $t_0 \leq t < \infty$, define $\underline{\omega}(t)$ and $\overline{\omega}(t)$ by

$$\underline{\omega}(t) = c \quad \text{and} \quad \overline{\omega}(t) = c + L \int_{t_0}^t \left(\frac{1}{r(s)} \int_s^\infty b(u)du\right)ds.$$

Then $0 < \underline{\omega}(t) \leq \overline{\omega}(t) \leq c + \delta$ for all $t \geq t_0$, and $\underline{\omega}(t)$, $\overline{\omega}(t)$ are bounded with their derivatives.

Clearly we have $\underline{\omega}''(t) \geq -\frac{r'(t)}{r(t)} \underline{\omega}'(t) - \frac{1}{r(t)} f(t, \underline{\omega}(t), \underline{\omega}'(t))$. On the other hand, $\overline{\omega}'(t) = \frac{L}{r(t)} \int_t^\infty b(u)du$ and $\overline{\omega}''(t) = -\frac{r'(t)}{r^2(t)} L \int_t^\infty b(u)du - \frac{L}{r(t)} b(t)$. Thus, using (11), we have

$$-\frac{r'(t)}{r(t)}\,\overline{\omega}\,'(t)-\frac{1}{r(t)}f(t,\overline{\omega}(t),\overline{\omega}\,'(t)) \geqslant -\frac{r'(t)}{r^2(t)}\,L\int_t^\infty b(u)du-\frac{1}{r(t)}\,b(t)\beta(\overline{\omega}(t))$$

$$\geqslant -\frac{r'(t)}{r^2(t)}\,L\int_t^\infty b(u)du-\frac{L}{r(t)}\,b(t)$$

$$\geqslant \overline{\omega}\,''(t)\,,$$

since $c \leqslant \overline{\omega}(t) \leqslant c+\delta$ for all $t \geqslant t_0$ and hence $\beta(\overline{\omega}(t)) \leqslant L$ for $t \geqslant t_0$.

For $t \geqslant t_0$, $\underline{\omega}(t) \leqslant x \leqslant \overline{\omega}(t)$ and $y \geqslant K$, define $V(t, x, y)$ by

$$V(t,x,y) = L\int_{t_0}^t b(s)ds + r(t)y$$

and for $t \geqslant t_0$, $\underline{\omega}(t) \leqslant x \leqslant \overline{\omega}(t)$ and $y \leqslant -K$, define $W(t, x, y)$ by

$$W(t,x,y) = -L\int_{t_0}^t b(s)ds - r(t)y\,.$$

Then it is clear that $V(t, x, y)$ and $W(t, x, y)$ satisfy the conditions (i) and (ii) in Theorem 2. Since $\beta(x) \leqslant L$ for $\underline{\omega}(t) \leqslant x \leqslant \overline{\omega}(t)$, we have

$$\dot{V}(t,x,y) = Lb(t) + r'(t)y + r(t)\{-\frac{r'(t)}{r(t)}y - \frac{1}{r(t)}f(t,x,y)\}$$

$$= Lb(t) - f(t,x,y)$$

$$\geqslant Lb(t) - b(t)\beta(x) \geqslant Lb(t) - Lb(t) = 0$$

and we have also $\dot{W}(t, x, y) \leqslant 0$. Therefore, it follows from Theorem 2 that the equation (10) has a solution $x(t)$ such that

$$0 < c \leqslant x(t) \leqslant c+\delta \qquad \text{for all} \quad t \geqslant t_0$$

and that $x'(t)$ is bounded for all $t \geqslant t_0$.

Under the conditions (14) and (15), we can use the same $\underline{\omega}(t)$ and $\overline{\omega}(t)$, since (14) and (15) imply that $\frac{1}{r(t)}\int_t^\infty b(u)du < \infty$. Moreover, (14) and (15) imply that $\int_0^\infty \frac{b(s)}{r(s)}ds < \infty$, and hence it is sufficient to consider

$$V(t, x, y) = y + Ax + L\int_{t_0}^{t} \frac{b(s)}{r(s)} ds$$

and

$$W(t, x, y) = -y + Ax - L\int_{t_0}^{t} \frac{b(s)}{r(s)} ds .$$

Remark. Assuming the existence of functions $a(t)$ and $\alpha(x)$ such that $a(t)\alpha(x) \leqslant f(t, x, y)$ for $t \geqslant T$, $x < 0$ and all y, we can obtain a result similar to Proposition 2.

By applying Theorem 3, we shall now prove the following proposition.

Proposition 3. Suppose that there exist two functions $b(t)$ and $\beta(x, y)$ which satisfy the following conditions;

(i) $b(t)$ is continuous on I and $b(t) \geqslant 0$ for $t \geqslant T$, where T can be large,

(ii) $\beta(x,y)$ is continuous on $x \geqslant 0$ and $y \geqslant 0$,

(iii) for $t \geqslant T$, $x > 0$ and $y \geqslant 0$,

$$f(t, x, y) \leqslant b(t)\beta(x, y). \tag{16}$$

Moreover, we assume that there is a $c > 0$ such that

$$f(t, c, 0) \geqslant 0 \qquad \text{for } t \geqslant T \tag{17}$$

and that for some $M > 0$ such that $c < M$

$$f(t, x, y) \geqslant 0 \quad \text{for} \quad t \geqslant T, \ M \geqslant x \geqslant c \ \text{and} \ y > K, \tag{18}$$

(19) $\qquad f(t,x,y) \leqslant 0 \quad$ for $\quad t \geqslant T, \quad M \geqslant x \geqslant c \quad$ and $\quad y < -K$,

where K can be large. If $0 < \epsilon \leqslant r(t) \leqslant \rho$ for some ϵ, ρ and all $t \geqslant 0$ and

(20) $$\int_0^\infty tb(t)dt < \infty .$$

or if there is an $A > 0$ such that

(21) $$-A < \frac{r'(t)}{r(t)} \qquad \text{for all} \quad t \geqslant 0$$

and we have

(22) $$\int_0^\infty \left(\frac{1}{r(s)} \int_s^\infty b(u)du\right)ds < \infty,$$

then the equation (10) has a bounded nonoscillatory solution.

Proof. In both cases, we have (22) and the fact that $\frac{1}{r(t)} \int_t^\infty b(u)du \to 0$ as $t \to \infty$. For the c, $\beta(c, 0) \leqslant \frac{L}{2}$ for some L. Since $\beta(x,y)$ is continuous, there is a $\delta > 0$ such that if $|x-c| \leqslant \delta$ and $0 \leqslant y \leqslant \delta$, we have $\beta(x,y) \leqslant L$. Choose $t_0 \geqslant T$ so large that

$$L \int_{t_0}^t \left(\frac{1}{r(s)} \int_s^\infty b(u)du\right)ds \leqslant \min(M-c, \delta) \qquad \text{for} \quad t \geqslant t_0$$

and

$$L \frac{1}{r(t)} \int_t^\infty b(u)du \leqslant \delta \qquad \text{for} \quad t \geqslant t_0 .$$

Then, in the same way as in the proof of Proposition 2, we can see that

$$\underline{\omega}(t) \equiv c \qquad \text{and} \qquad \overline{\omega}(t) = c + L \int_{t_0}^t \left(\frac{1}{r(s)} \int_s^\infty b(u)du\right)ds$$

satisfy the conditions (8) and (9) for $t \geqslant t_0$.

In the case where we have $\epsilon \leqslant r(t) \leqslant \rho$, for $t \geqslant t_0$, $\underline{\omega}(t) \leqslant x \leqslant \overline{\omega}(t)$ and $y > K$, define $V(t,x,y)$ by $V(t,x,y) = r(t)y$. Then we have

$$\begin{aligned}\dot{V}(t,x,y) &= r'(t)y + r(t)\{-\frac{r'(t)}{r(t)}y - \frac{1}{r(t)}f(t,x,y)\} \\ &= \quad - \; f(t,x,y) \\ &\leqslant 0.\end{aligned}$$

For $t \geqslant t_0$, $\underline{\omega}(t) \leqslant x \leqslant \overline{\omega}(t)$ and $y < -K$, $W(t,x,y) = -r(t)y$ satisfies $\dot{W}(t,x,y) \leqslant 0$. In the case where we have the condition (21), $V(t,x,y) = y - Ax$ and $W(t,x,y) = -y + Ax$ are Liapunov functions that we desire. Therefore, it follows from Theorem 3 that the equation (10) has a bounded nonoscillatory solution.

Remark. Assuming the existence of functions $a(t)$ and $\alpha(x,y)$ such that $a(t)\alpha(x,y) \leqslant f(t,x,y)$ for $t \geqslant T$, $x < 0$ and $y \leqslant 0$, we can obtain a result similar to Proposition 3.

Now consider the equation (1). We assume that there exist continuous functions $a(t)$, $b(t)$, $\alpha(x)$ and $\beta(x)$ which satisfy the following conditions;

(i) $a(t)$ and $b(t)$ are nonnegative for $t \geqslant T$, where T can be large,

(ii) $x\alpha(x) > 0$ and $x\beta(x) > 0$ for $x \neq 0$, and $\alpha'(x) \geqslant 0$, $\beta'(x) \geqslant 0$,

(iii) $a(t)\alpha(x) \leqslant f(t,x,u) \leqslant b(t)\beta(x)$ for $t \geqslant T$, $|x| < \infty$ and $|u| < \infty$.

Moreover, we assume that the derivative of $r(t)$ is continuous and that $0 < \epsilon \leqslant r(t) \leqslant \rho$ for some ϵ, ρ and all $t \geqslant 0$.

Under the assumptions above, the following results follow immediately from Propositions 1 and 2 with the remark.

Proposition 4. A necessary and sufficient condition in order that all bounded solutions of (1) are oscillatory is that

$$\int_0^\infty ta(t)dt = \infty \qquad \text{and} \qquad \int_0^\infty tb(t)dt = \infty .$$

Proposition 5. A necessary and sufficient condition in order that the equation (1) has a bounded nonoscillatory solution is that

$$\int_0^\infty ta(t)dt < \infty \qquad \text{or} \qquad \int_0^\infty tb(t)dt < \infty .$$

Proposition 6. In addition to the assumptions above, if

$$\int_\epsilon^\infty \frac{du}{\alpha(u)} < \infty \quad \text{and} \quad \int_{-\epsilon}^{-\infty} \frac{du}{\beta(u)} < \infty \quad \text{for some} \quad \epsilon > 0 ,$$

a necessary and sufficient condition in order that the equation (1) is oscillatory is that

$$\int_0^\infty ta(t)dt = \infty \qquad \text{and} \qquad \int_0^\infty tb(t)dt = \infty .$$

For the equation (1), we assume that there exist continuous functions $a(t)$, $b(t)$, $\alpha(x)$ and $\beta(x)$ mentioned above. Moreover, we assume that the derivative of $r(t)$ is continuous, $\int_0^\infty \frac{dt}{r(t)} = \infty$ and there is an $A > 0$ such that $\left| \frac{r'(t)}{r(t)} \right| < A$ for $t \geqq 0$. Then, by Proposition 1 if we have the condition A that

$$(23) \qquad \begin{cases} \displaystyle\int_0^\infty a(s)ds = \infty \\ \text{or} \\ \displaystyle\int_0^\infty a(s)ds < \infty , \quad \int_0^\infty \Big(\frac{1}{r(s)} \int_s^\infty a(u)du \Big) ds = \infty \end{cases}$$

and

$$(24)\quad \begin{cases} \int_0^\infty b(s)ds = \infty \\ \text{or} \\ \int_0^\infty b(s)ds < \infty, \quad \int_0^\infty (\frac{1}{r(s)} \int_s^\infty b(u)du)ds = \infty, \end{cases}$$

all bounded solutions of (1) are oscillatory. On the other hand, if we have the condition B that

$$(25)\quad \int_0^\infty a(s)ds < \infty \quad \text{and} \quad \int_0^\infty (\frac{1}{r(s)} \int_s^\infty a(u)du)ds < \infty$$

or

$$(26)\quad \int_0^\infty b(s)ds < \infty \quad \text{and} \quad \int_0^\infty (\frac{1}{r(s)} \int_s^\infty b(u)du)ds < \infty,$$

the equation (1) has a bounded nonoscillatory solution. Therefore the condition A is a necessary and sufficient condition in order that all bounded solutions of (1) are oscillatory, and the condition B is a necessary and sufficient condition in order that the equation (1) has a bounded nonoscillatory solution.

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