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On the fields of moduli for FM-structures.

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We shall give a rough description of our subject because the problem discussed here will be considered more thoroughly in a separate paper.

Throughout the paper we fix a universal domain $\mathbb{K}$ and are concerned with algebraic geometry over $\mathbb{K}$.

DEFINITION 1.1. $(\mathcal{V}, F)$ is a pair of a field F and a collection $\mathcal{V}$ of geometric objects $S, S', \cdots$, together with the following three laws:

i) Given $S \in \mathcal{V}$, the notion of fields of rationality for S is defined. (We consider only a field k containing F as a field of rationality for S.)

ii) Given S & $S' \in \mathscr{V}$, $S \cong S'$ or $S \not\cong S'$.

iii) Given $S \in \mathscr{V}$ and $\sigma \in \mathrm{Aut}(\mathbb{K}/F)$ = the group of automorphisms of $\mathbb{K}$ over F, S^σ is defined and belongs to $\mathscr{V}$.

The pair $(\mathscr{V}, F)$ is called an FM-system and an object S in $\mathscr{V}$ is called an FM-structure (in $(\mathscr{V}, F)$) if the following conditions are satisfied:

fm i) S is rational over k and $k' \supset k$ $\Longrightarrow$ S is rational over k'.

fm ii) $\cong$ is an equivalence relation.

fm iii) S & $S' \in \mathscr{V}$ and σ & $\tau \in \mathrm{Aut}(\mathbb{K}/F)$

1) $\sigma | k$ = identity, where k is a field of rationality for S. $\Longrightarrow$ $S^\sigma = S$

2) S is rational over k $\Longrightarrow$ S^σ is rational over k^σ.

3) $S \cong S'$ $\Longrightarrow$ $S^\sigma = S'^\sigma$.

4) $S^{\sigma\tau} = (S^\sigma)^\tau$.

DEFINITION 1.2. For an FM-system $(\mathfrak{T}, F)$ and $S \in \mathfrak{T}$, a field k_S containing F is called the "field of moduli for S" if the following two conditions are satisfied:

FM 1) For $\sigma \in \mathrm{Aut}(\mathbb{K}/F)$,

$$S^\sigma \cong S \iff \sigma \mid k_S = \text{identity}.$$

FM 2) $$k_S = \bigcap K,$$

where K runs over the set of all fields of rationality for all $S' \cong S$.

EXAMPLE 2.1. Let V be a complete variety rational over F, non-singular in codimension 1, containing an F-rational simple point; let $\mathfrak{D}_a(V)$ be the group of V-divisors algebraically equivalent to zero. The relation $\cong$ in $\mathfrak{D}_a(V)$ is defined by the linear equivalence. Then $(\mathfrak{D}_a(V), F)$ becomes an FM-system in a natural way and the field of moduli $k_{cl(X)}$ for $X \in \mathfrak{D}_a(V)$ exists and is generated by the point of the Picard variety $\hat{V}$ of V over F, which corresponds to the linear class

$\mathcal{C}l(X)$ determined by X, (if we choose a suitable model of the Picard variety of V).

EXAMPLE 2.2. Let V be a projective non-singular variety rational over F having an F-rational point. If we replace $\mathfrak{D}_a(V)$ and the linear equivalence in Ex. 2.1. respectively by the group $\mathfrak{Z}_a(V)$ of zero-cycles on V of degree zero and the regular equivalence in $\mathfrak{Z}_a(V)$, we get the same kind of results for $(\mathfrak{Z}_a(V), F)$ as those for $(\mathfrak{D}_a(V), F)$ in Ex 2.1., and the Albanese variety of V works as the Picard variety of V did in the previous case.

EXAMPLE 2.3. For a polarized abelian variety $\mathcal{P} = (A, \mathfrak{p}(X))$, the field of moduli $k_{\mathcal{P}}$ exists.

EXAMPLE 2.4. Let V be a complete non-singular curve. For the biregular isomorphy class determined by V, the field of moduli k_V exists.

REMARK. In the case of positive characteristic, if we consider a birational class determined by a curve (may be having singularities), the field of moduli for the class never exists. This is a consequence of the following easy lemma.

LEMMA 2.5. Let k' be a purely inseparable extension of a field k. If $k'(x)$ is a regular extension of k', there exists a subfield K of $k'(x)$ which is a regular extension of k such that $k'(x)$ is the composite of K and k'.

(Proof) Let $(t_1, \cdots, t_n) = (t)$ be a set of independent variables in $k'(x)$ over k' such that $k'(x)$ is separable algebraic over $k'(t)$. If we define a subfield K of $k'(x)$ by

$$K = \{ y \in k'(x) \mid y \text{ ... separable algeb. over } k(t) \},$$

K satisfies all the conditions we want.