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等質確率場に関する2，3の話題

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先に行われた，"多重マルコフ性と予測理論への応用"に於て，筆者はE. WONG(1969)の紹介を行った。この論文にては，E. WONG(1969)の紹介及び問題点の指摘，(2) U. D. POPOV(1968)による線形外そう問題の結果の拡張を述べ，簡単な文献表を記す。(2)に関しては，英文原稿"On a Linear Extrapolation Problem Of Some Homogeneous Random Fields"を参照されたい。

§1. E. WONG(1969)の紹介

E. WONG(1969), "HOMOGENEOUS GAUSS-MARKOV RANDOM FIELDS", Ann. Math. Stat., 40, pp. 1625-1634 を紹介する。

この論文の構成は，2．で Second-order properties を扱い，3．で Homogeneous Gauss-Markov Fields を特徴づけ（定理1），4．で Generalized Homogeneous Gauss-Markov Fields を扱っている（定理2）．以下，順次，紹介する。

$(\Omega, \mathcal{A}, P)$: a fixed probability space

$\{x(\omega; z), \omega \in \Omega, z \in V_n\}$: n 次元パラメータ空間 V_n をもつ Gaussian random field

以下，V_n は (a) Euclid 空間 E^n，あるいは rank 1 なる定曲率対称空間 (b) 球面 S^n 及び (c) 実双曲空間 H^n である。

$GL(V_n)$: 距離を保つ V_n の運動の full linear group

Def： 任意の集合 $A = \{z_i\} \subset V_n$ について，
$\{x(z_i), z_i \in A\}$ と $\{x(gz_i), z_i \in A\}$ とが $g \in GL(V_n)$ のとき，同一分布をするとき，確率場 $\{x(z), z \in V_n\}$ は等質（homogeneous）であるという。

この論文で扱うマルコフ性は次の様に理解する。すなわち，∂D をなめらかな次元 $n-1$ なる V_n の閉曲面とし，V_n を有界領域 D^- と，領域 D^+ とに分つものとする。$\{x(z), z \in \partial D\}$ が与えられたとき，$\{x(z), z \in D^-\}$ と $\{x(z), z \in D^+\}$ とが

独立ならば，確率場 $\{x(t), t\in V_n\}$ はマルコフ性をもつという。 以下，確率場 $\{x(t), t\in V_n\}$ を2乗平均連続な Gaussian field で homogeneous とする。平均値関数を0とする。また，相関関数を $R(d(t,t'))$, $d(t,t')$ は $t, t'\in V_n$ の距離，と記す。 相関関数 $R(d(t,t'))$ は正値定符号である。

$\Delta(V_n)$: the Laplace-Beltrami operator

metric を

$$ds^2 = dr^2 + g^2(r)\sum_{i=1}^{n-1}\left(\prod_{k=i+1}^{n-1}\sin^2\varphi_k\right)d\varphi_i^2$$

と表わすとき，

$$\Delta(V_n) = \{g^{n-1}(r)\}^{-1}\frac{\partial}{\partial r}\left[g^{n-1}(r)\frac{\partial}{\partial r}\right] + (g^2(r))^{-1}\Delta(S^{n-1}),$$

$$\Delta_0(V_n) \equiv \{g^{n-1}(r)\}^{-1}\frac{\partial}{\partial r}\left[g^{n-1}(r)\frac{\partial}{\partial r}\right].$$

$\psi(0)=1$， $\Delta_0(V_n)\psi = \lambda\psi$ なる関数 ψ は，V_n 上の (zonal) spherical function である。（正値定符号）

L : $\int g^{n-1}(r)|f(r)|dr < +\infty$ なる複素数値関数の集合

$\mathfrak{M}$: すべての spherical function の集合

$\mathfrak{M}$ に次のような最弱位相を与える；

すなわち，$f \in L$ の Fourier 変換

$$\hat{f}(\varphi) = \int \varphi(r) f(r) g^{n-1}(r)\,dr, \quad \varphi \in \mathfrak{M}$$

が連続となる最弱位相を与える。このとき，Naimark, Normed Rings, p. 426 により，

$$R(r) = \int_{\mathfrak{M}} \varphi(r)\,\sigma(d\varphi),$$

$\sigma(d\varphi)$ は $\mathfrak{M}$ 上の finite Borel 測度.

Λ：$\Delta_0(V_r)$ の固有値の集合

$\Lambda \ni \lambda$ に対応する spherical function を $\varphi(\lambda, r)$ と書くと，

$$R(r) = \int_{\Lambda} \varphi(\lambda, r)\,F(d\lambda),$$

ここに $F(d\lambda)$ は Λ 上の finite Borel 測度.

spherical function φ は，具体的には次のように与えられる。

(1) E^n のとき；

$$\varphi(\lambda, r) = \Gamma\left(\frac{n}{2}\right) \frac{J_{\frac{n-2}{2}}(\nu r)}{\left(\frac{\nu r}{2}\right)^{\frac{n-2}{2}}}, \quad \lambda = -\nu^2,\ 0 \leq \nu < +\infty$$

(2) S^n のとき；

$$\varphi(\lambda, r) = \frac{\Gamma(n-1)\,k!}{\Gamma(n+k-1)} C_k^{\frac{n-1}{2}}(\cos(r)),$$

$$\lambda = -k(k+n-1),\ k = 0, 1, 2, \ldots$$

(3) H^n のとき;

$$\psi(\lambda, r) = \frac{2^{\frac{n-2}{2}}\Gamma(\frac{n}{2})}{sh^{\frac{n-2}{2}}(r)} P^{\frac{2-n}{2}}_{-\sigma-\frac{n}{2}}(ch(r)),$$

$\sigma = -\frac{n-1}{2} + \sqrt{-1}\rho$ $(\rho \neq 0)$ (fundamental series) 及び σ: noninteger $\in (-n+1, 0)$ (complementary series) で,

$$\lambda = \sigma(\sigma + n - 1).$$

注意1: E. Wong, A.M. Yaglom (1961) では, H^n の場合の complementary series が含まれていない。

注意2: ここに扱う空間の場合には, Laplace-Beltrami 作用素が一意的に定まり, 従って spherical function も上述の要請により完全に定まる。

$L^2(S^{n-1})$: S^{n-1} 上の一様な測度に関して2乗可積分な関数の全体

$m = 0, 1, 2, \ldots$ に対して,

$$\Delta(S^{n-1})h = -m(m+n-2)h$$

の解 h の張る空間 ($GL(S^{n-1})$ 不変) を Z_m とし, $\dim(Z_m)$ を d_m とおく。Z_m の底を, 実数値関数で正規直交なるものにとり $\{h_{ml}, 1 \le l \le d_m\}$ とおく。

これらを用いて，我々は次の展開式をうる；

$$(1)\quad \psi(\lambda, d(z,z')) = \sum_{m=0}^{\infty}\sum_{l=1}^{d_m} h_{ml}(\varrho)h_{ml}(\varrho')\psi_m(\lambda,r)\psi_m(\lambda,r'),$$

ここに，

$$[\Delta_0(V_n) - \{g^2(r)\}^{-1}\{m(m+n-2)\}]\psi_m = \lambda\psi_m .$$

以上第2節の内容である。

(1) を用いて，

$$(2)\quad R(d(z,z')) = \sum_{m=0}^{\infty}\sum_{l=1}^{d_m} h_{ml}(\varrho)h_{ml}(\varrho')\int_{\Lambda}\psi_m(\lambda,r)\psi_m(\lambda,r')\,dF(\lambda).$$

いま，$\{x_{ml}(r)\}$ を独立な1次元 Gauss 過程とする。そこに，

$$E\,x_{ml}(r)x_{pk}(r') = \delta_{mp}\delta_{lk}\int_{\Lambda}\psi_m(\lambda,r)\psi_m(\lambda,r')\,dF(\lambda).$$

$$(3)\quad x(\varrho, r) \overset{\text{def}}{=} \sum_{m=0}^{\infty}\sum_{l=1}^{d_m} h_{ml}(\varrho)x_{ml}(r).$$

$$x_{ml}(r) = \int_{S^{n-1}} x(\varrho,r)h_{ml}(\varrho)\,dv(\varrho),$$

ここに $dv(\varrho)$ は S^{n-1} 上の normalized Haar measure.

Lemma 1 :

$x(z)$, $z=(r,\omega)$ を等質な Gauss-Markov field とする。

$$x_{ml}(r) \equiv \int_{S^{n-1}} x(z)\, h_{ml}(\omega)\, dS(\omega)$$

⟹ $\{x_{ml}(r)\}$ は、独立な1次元 Gauss-Markov 過程であり、

$$E\, x_{ml}(r)\, x_{ml}(r') = f_m(\max(r,r'))\, g_m(\min(r,r'))$$

をみたす関数 f_m, g_m が存在する。

Theorem 1 :

$\{x(z), z\in V_n\}$ を2乗平均連続な homogeneous Gaussian field とする。

⟹ $\{x(z), z\in V_n\}$ が Markovian であるための必十条件は、

(4) $\quad R(d(z,z')) = C\,\psi(d(z,z'))$,

ここに $C=R(0)$ であり、ψ は V_n 上の spherical function,

が成立することである。

注意3： (4) をみたす nontrivial な Markovian

field の存在の検討が残っている。事実，$V_n = \mathbb{E}^n$，∂D を半径 r，中心は原点なる球面ととると，∂D 上の値により，すべての $z \in V_n$ での値が確率 1 で定まってしまう。

以上が 3 節の内容である。

以下に紹介する 4 節の内容は検討していない。

$\mathcal{S} = \mathcal{S}(R^n)$ ：急減少な C^∞- 実数値関数の Schwartz 空間

$\mathcal{H}$ ：平均 0 なる実 gaussian 確率変数のヒルベルト空間

$X : \mathcal{S} \to \mathcal{H}$ cont. linear map を real zero-mean Gaussian generalized Random field という.

$g : \underset{(\text{isometry})}{R^n \to R^n}$ に対して，

$$(T_g f)(z) \overset{\text{def}}{=} f(g^{-1} z), \quad f \in \mathcal{S}.$$

Def. : $\forall g \in GL(R^n)$ に対して，

$$B(f_1, f_2) \overset{\text{def}}{=} \mathbb{E}\, X(f_1) X(f_2) = \mathbb{E}\, X(T_g f_1)\, X(T_g f_2),$$
$$f_1, f_2 \in \mathcal{S}$$

のとき，X を homogeneous という。B を X の covariance bilinear f'nal という。

- bilinear f'nal B on $\mathcal{S} \times \mathcal{S}^*$, homogeneous Gaussian generalized random field X の covariance f'nal であるための必十条件は,

(5) $$B(f_1, f_2) = \int_0^\infty \sum_{m,l} \hat{f}^{(1)}_{ml}(\lambda) \hat{f}^{(2)}_{ml}(\lambda)\, dF(\lambda),$$

F は $[0, \infty)$ 上のゆるやかな非減少関数であり

(6) $$\hat{f}_{ml}(\lambda) = \int_{S^{n-1}} d\nu(q) \int_0^\infty r^{n-1} dr\, f(r, q)\, \psi_m(-\lambda, r)\, h_{ml}(q)$$

$$, \quad m = 0, 1, 2, \ldots, \quad 1 \le l \le d_m$$

をみたすことである。

∂D: a smooth $n-1$ closed surface in R^n

$d\sigma$: the differential surface area on ∂D

$f \in \mathcal{L}^2(\partial D, d\sigma)$ に対して,

$$\tilde{f}_{ml}(\lambda) \overset{\text{def.}}{=} \int_{\partial D} f(x)\, \psi_m(-\lambda, r(x))\, h_{ml}(q(x))\, d\sigma,$$

$$\tilde{f}(q, \lambda) \overset{\text{def.}}{=} \sum_{m,l} h_{ml}(q) \tilde{f}_{ml}(\lambda).$$

また,

$$\hat{f}(q, \lambda) \overset{\text{def.}}{=} \sum_{m,l} \hat{f}_{ml}(\lambda)\, h_{ml}(q).$$

いま，$f \in \mathcal{L}^2(\partial D, d\sigma)$ のとき $\tilde{f} \in \mathcal{L}^2(dF, d\sigma)$ なるXに対して，

$$X_{\partial D}(f) \equiv \sum_{m,l} \int_0^\infty \tilde{f}_{ml}(\lambda)\, \hat{x}_{ml}(d\lambda),$$

ここに $\hat{x}_{ml}(d\lambda)$ は，

$$X(f) = \sum_{m,l} \int_0^\infty \hat{f}_{ml}(\lambda)\, \hat{x}_{ml}(d\lambda)$$

で定まる。

$\mathcal{M}(\partial D)$: the closed linear manifold generated by $\{X_{\partial D}(f),\ f \in \mathcal{L}^2(\partial D, d\sigma)\}$.

$P_{\partial D}\mathcal{M}(\partial D')$: $\mathcal{M}(\partial D)$ への射影 $P_{\partial D}$ による $\mathcal{M}(\partial D')$ の像

Def. : 任意の $\partial D_1 \subset \partial D \subset \partial D_2$ に対して，

$\mathcal{M}(\partial D_2) - P_{\partial D}\mathcal{M}(\partial D_2)$ と $\mathcal{M}(\partial D_1)$ とが直交するとき，X は Markovian であるという。

Theorem 2 :

$\{X(f),\ f \in \mathcal{S}\}$: homogeneous Gaussian generalized random field on R^n, with spectral distribution F

⤵ X が Markovian であるための必十条件は，

(7) $$\int_0^\infty \psi_0(-\lambda, r)\, F(d\lambda) = R(r), \quad r > 0$$

が，

(8)　$\{r^{n-1}\}^{-1}\frac{d}{dr}[r^{n-1}\frac{d}{dr}]R = \alpha R$，$\alpha$：正定数

をみたす $(0,\infty)$ 上の2回微分可能な関数を定義することである。

方程式(8)の解は，

(a)　$R(r) = C_1 J_{\frac{n-2}{2}}(\nu_0 r)/(\nu_0 r)^{\frac{n-2}{2}}$

及び

(b)　$R(r) = C_2 K_{\frac{n-2}{2}}(\nu_0 r)/(\nu_0 r)^{\frac{n-2}{2}}$

であり，(a)は $\lambda = \nu_0^2$ でのみ jump するスペクトル測度Fに対応し，(b)は

$$F(d\nu^2) = \frac{C_2}{\nu_0^n}\frac{\nu^{n-1}d\nu}{1+(\frac{\nu}{\nu_0})^2}$$

に対応する。(b)のとき，$n=1$ ならば，

$$R(\tau) = \frac{1}{2}\pi A e^{-\nu_0|\tau|}$$

となり，これは Ornstein - Uhlenbeck process の相関関数

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注意4：Wongは，S^n を含む Th. 2に相当する結果を得ているが，H^n の場合は未解決と述べている。

注意5：確率場の多重 Markov 性に関して残っている

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注意6：対称空間の rank > 1 の場合は，この論文の論法は適用困難と思われる。

On a Linear Extrapolation Problem Of Some Homogeneous Random Fields

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§0. Introduction and Summary

Let V be the d-dimensional Euclidean space or d-dimensional sphere. Our random field ($X(\omega;z), z\in V$) has the mean function 0 and the covariance function $R(z,z')$, and it belongs to the class $L^2(\Omega)$. We call it the homogeneous random field whenever $R(gz,gz') = R(z,z')$ for any $g \in GL(V)$. In this case, the covariance function is of the form $R(\rho(z,z'))$, where $\rho(z,z')$ is a distance between z and z'. Moreover, we assume that the random field $X(z)$ is continuous in quadratic mean, then the covariance function $R(\rho(z,z'))$ is a positive definite function. As well known, the positive definite function has the spectral representation with spherical functions. In our case, it takes the following forms;

for $V = E^d$,

$$R(\rho(z,z')) = \Gamma\left(\frac{d}{2}\right)\int_0^{\infty}\frac{J_{\frac{d-2}{2}}(\nu\rho)}{\left(\frac{\nu\rho}{2}\right)^{\frac{d-2}{2}}}dF(\nu),$$

$$\text{, where } R(0) = F(+\infty) < +\infty, \tag{1}$$

for $V = S^d$,

$$R(\rho(z,z')) = \Gamma(d-1)\sum_{\nu=0}^{\infty}\frac{\Gamma(\nu+1)}{\Gamma(\nu+d-1)}C_\nu^{\frac{d-1}{2}}(\cos\rho)\,w(\nu),$$

$$\text{, where } w(\nu) \geqq 0 \text{ and } R(0) = \sum_{\nu=0}^{\infty} w(\nu)$$

We consider the following extrapolation problem; our data is observations on n concentric spheres with center at coordinate origin, and using these observations, we will extrapolate any one point outside of these regions. Let an extrapolated value be $\hat{X}(z)$; our extrapolation error is measured by

$$\delta^2(z) = E(X(z) - \hat{X}(z))^2. \tag{2}$$

Our extrapolator $\hat{X}(z)$ is of the form, when we denote a point z by coordinate (φ, θ), $\varphi \in S^{d-1}$ and $0 \leqq \theta < +\infty$ $(V = E^d)$, $0 \leqq \theta \leqq \pi$ $(V = S^d)$,

$$\hat{X}(z) = \sum_{i=1}^{n} \int_{S^{d-1}} c^{(i)}(z;\varphi) X(\varphi;\theta_i) dv(\varphi), \qquad (3)$$

,where $dv(\varphi)$ is the normalized Haar measure on S^{d-1}.

We will determine coefficients $c^{(i)}(z;\varphi)(1 \leq i \leq n)$ to minimize the error $\delta^2(z)$ and evaluate the minimum value of $\delta^2(z)$. We assume that these coefficients belong to the class $L^2(S^{d-1})$. In the following, we treat these problems and in particular, for the case of $n = 1$ and origin, we will give an explicit evaluation. U.D.Popov(1968) treated the same problem for $V = E^2$.

§1. Euclidean space

In this case, our coefficients $c^{(i)}(z,\varphi)$ have the Fourier expansion

$$c^{(i)}(z;\varphi) = \sum_{\ell=0}^{\infty} \sum_{K} \hat{c}^{(i)}(z; \overset{\ell}{\underset{K}{V}}) \Xi_K^{\ell}(\varphi), \qquad (4)$$

, where $\varphi = (\varphi^{(d-1)}, \ldots\ldots, \varphi^{(2)}, \varphi^{(1)})$, $0 \leq \varphi^{(1)} < 2\pi$, $0 \leq \varphi^{(j)} \leq \pi$ $(j \neq 1)$ and $\Xi_K^{\ell}(\varphi)$ $(\ell = 0,1,2,\ldots\ldots;K = (k_1,\ldots,k_{d-2})$ such that $k_0 = \ell \geq k_1 \geq k_2 \ \ldots\ldots k_{d-2} \geq 0)$ are the complete orthonormal system of $L^2(S^{d-1})$.

For fixed $z_j \in E$, a set of observation points, from (3) we have

$$R(\rho(z,z_j)) = \sum_{i=1}^{n} \int_{S^{d-1}} c^{(i)}(z;\varphi) R(\rho(z_i,z_j)) dv(\varphi). \qquad (5)$$

Putting $z_i = (\theta_i, \varphi)$, $z_j = (\theta_j, \varphi_j)$, we get

$$R(\rho(z_i,z_j)) = \Gamma^2(\tfrac{d}{2}) \sum_{\ell=0}^{\infty} \sum_{L} (-1)^{\ell} d-2 \, a_{\ell}^{(d)}(\theta_i,\theta_j) \overline{\Xi_L^{\ell}(\varphi)} \Xi_L^{\ell}(\varphi_j) \qquad (6)$$

,where $a_{\ell}^{(d)}(\theta_i,\theta_j) = \int_0^{\infty} \frac{J_{\ell+\frac{d-2}{2}}(\nu\theta_i)}{(\nu\theta_i)^{\frac{d-2}{2}}} \frac{J_{\ell+\frac{d-2}{2}}(\nu\theta_j)}{(\nu\theta_j)^{\frac{d-2}{2}}} dF(\nu)$.

Substituting (4) and (6) in (5), we get

$$R(\rho(z,z_j)) = \Gamma^2(\tfrac{d}{2}) \sum_{\ell=0}^{\infty} \sum_{L} (-1)^{\ell} d-2 \sum_{i=1}^{n} a_{\ell}^{(d)}(\theta_i,\theta_j) \hat{c}^{(i)}(z; \overset{\ell}{\underset{L}{V}}) \Xi_L^{\ell}(\varphi_j)$$

$$= \sum_{\ell=0}^{\infty} \sum_{L} \gamma_j^{(d)}(z;\ell,L) \mathcal{H}_L^{\ell}(\varphi_j), \tag{7}$$

, where $\gamma_j^{(d)}(z;\ell,L) = \Gamma^2(\tfrac{d}{2})(-1)^{\ell}{}_{d-2} \sum_{i=1}^{n} a_{\ell}^{(d)}(\theta_i, \theta_j) \cdot \hat{c}^{(i)}(z;\ell,L)$.

From the expression (7), we can determine coefficients $\gamma_j^{(d)}(z;\ell,L)$ uniquely and the matrix $A_{\ell}^{(d)} \doteq (a_{\ell}^{(d)}(\theta_i, \theta_j))$ is a nonnegative definite symmetric matrix, so that whenever $\det(A_{\ell}^{(d)}) \neq 0$, we can determine coefficients $c^{(i)}(z,\varphi)$ uniquely. To evaluate the error $\delta_n^2(z)$, we need an evaluation of $\| \hat{X}(z) \|^2 = E(\hat{X}(z))^2$. While from (3) and (5), using (7), we get

$$\| \hat{X}(z) \|^2 = \Gamma^2(\tfrac{d}{2}) \sum_{\ell=0}^{\infty} \sum_{L} (-1)^{\ell}{}_{d-2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{\ell}^{(d)}(\theta_i, \theta_j) \overline{\hat{c}^{(i)}(z;\ell,L)} \hat{c}^{(j)}(z;\ell,L)$$
$$= \sum_{\ell=0}^{\infty} \sum_{L} \sum_{i=1}^{n} \overline{\hat{c}^{(i)}(z;\ell,L)} \gamma_i^{(d)}(z;\ell,L) \tag{8}$$

In particular, when n = 1, we have

$$\delta_1^2(z) = F(+\infty) - \sum_{\ell=0}^{\infty} \sum_{L} (-1)^{\ell}{}_{d-2} \Gamma^2(\tfrac{d}{2}) \frac{[a_{\ell}^{(d)}(\theta_0,\theta_1)]^2}{a_{\ell}^{(d)}(\theta_1,\theta_1)} |\mathcal{H}_L^{\ell}(\varphi_0)|^2. \tag{9}$$

Thus we have a proposition

<u>Proposition 1:</u> In the Euclidean case, using a extrapolator of the form (3), when we have observations on one sphere with center at origin, the extrapolation error is given by the formula (9), and in particular, for d = 2, it depends only on distance from origin.

For the case of d = 2, the expression (9) becomes as

$$\delta_1^2(z) = F(+\infty) - \sum_{\ell=-\infty}^{\infty} (a_{\ell}^{(2)}(\theta_0, \theta_1))^2 / a_{\ell}^{(2)}(\theta_1, \theta_1).$$

Finally, we consider extrapolation of origin. In this case,

$$R(\rho(z,z_j)) = R(\theta_j) = \sum_{\ell=0}^{\infty} \sum_{L} \gamma_j^{(d)}(z;\ell,L) \mathcal{H}_L^{\ell}(\varphi_j),$$

so that $\gamma_j^{(d)}(z;\ell,L) = \delta(\ell)\delta(L)R(\theta_j)$ and

$$\delta_n^2(z) = F(+\infty) - \Gamma^2(\tfrac{d}{2}) \sum_{i=1}^{n} \sum_{j=1}^{n} a_0^{(d)}(\theta_i, \theta_j) \overline{\hat{c}^{(i)}(z;0)} \hat{c}^{(j)}(z;0) \tag{10}$$

,in particular when n = 1, we have

$$\delta_1^2(z) = F(+\infty) - \frac{1}{2^{d-2}} \frac{\left[\Gamma(\tfrac{d}{2}) \int_0^{\infty} \frac{J_{\frac{d-2}{2}}(\nu\theta_1)}{(\nu\theta_1)^{\frac{d-2}{2}}} dF(\nu)\right]^2}{\int_0^{\infty} \left[\Gamma(\tfrac{d}{2}) \frac{J_{\frac{d-2}{2}}(\nu\theta_1)}{(\nu\theta_1)^{\frac{d-2}{2}}}\right]^2 dF(\nu)}.$$

In the latter case, using Cauchy-Schwarz inequality, we get a lower bound of error, $\delta_1^2(z) \geqslant F(+\infty)(1 - 1/2^{d-2})$, and the lower bound is attained iff the spectral distribution function $F(\nu)$ jumps at ν-values such that $\Gamma(\frac{d}{2}) \frac{J_{\frac{d-2}{2}}(\nu\theta_1)}{(\frac{\nu\theta_1}{2})^{\frac{d-2}{2}}} = C,\ 0 < C \leqslant 1.$

§2. Spherical space

In this case, our coefficients $c^{(i)}(z;\varphi)$ have a Fourier expansion (4) and corresponding to (6) we have

$$R(\rho(z_i,z_j)) = \frac{2^{d-2}}{\pi}\Gamma^2(\tfrac{d}{2})\sum_{\ell=0}^{\infty}\sum_{L}(-1)^{\ell}\, d-2\, a_\ell^{(d)}(\theta_i,\theta_j)\,\overline{H_L^{\ell}(\varphi_i)}\, H_L^{\ell}(\varphi_j),$$

$$\text{where } a_\ell^{(d)} = \sum_{\nu=\ell}^{\infty} w(\nu)\left(\frac{\Gamma(\nu+1)}{\Gamma(\nu+d-1)}\,\frac{2^{2\ell+d-2}(\nu-\ell)!\,\Gamma^2(\frac{d-1}{2}+\ell)}{\Gamma(\nu+\ell+d-1)}\right.$$
$$\left. C_{\nu-\ell}^{(d-1)/2+\ell}(\cos\theta_i)\sin^{\ell}(\theta_i)\,C_{\nu-\ell}^{(d-1)/2+\ell}(\cos\theta_j)\sin^{\ell}(\theta_j)\right) \qquad (11)$$

and corresponding to (7) we have

$$R(\rho(z,z_j)) = \frac{2^{d-2}}{\pi}\Gamma^2(\tfrac{d}{2})\sum_{\ell=0}^{\infty}\sum_{L}(-1)^{\ell}\, d-2 \sum_{i=1}^{n} a_\ell^{(d)}(\theta_i,\theta_j)\,\hat{c}^{(i)}(z;{}^{\ell}_{L})\, H_L^{\ell}(\varphi_j) \qquad (12)$$

Also corresponding to (8),

$$\|\hat{X}_n(z)\|^2 = \frac{2^{d-2}}{\pi}\Gamma^2(\tfrac{d}{2})\sum_{\ell=0}^{\infty}\sum_{L}(-1)^{\ell}\, d-2 \sum_{i=1}^{n}\sum_{j=1}^{n} a_\ell^{(d)}(\theta_i,\theta_j)\,\hat{c}^{(i)}(z;{}^{\ell}_{L})\,\overline{\hat{c}^{(j)}(z;{}^{\ell}_{L})}$$

and we have

$$\delta_1^2(z) = R(0) - \frac{2^{d-2}}{\pi}\Gamma^2(\tfrac{d}{2})\sum_{\ell=0}^{\infty}\sum_{L}(-1)^{\ell}\, d-2\, \frac{|a_\ell^{(d)}(\theta_0,\theta_1)|^2}{a_\ell^{(d)}(\theta_1,\theta_1)}\,|H_L^{\ell}(\varphi_0)|^2 \qquad (13)$$

Thus we get a proposition

Proposition 2: In the spherical case, using a extrapolator of the form (3), when we have observations on one sphere with center at origin(north pole), the extrapolation error is given by the formula (13), and in particular, for d = 2, it depends only on distance from origin.

For the case of d = 2, the formula (13) becomes as

$$\delta_1^2(z) = R(0) - \sum_{\nu=0}^{\infty}\sum_{\ell=-\nu}^{\nu} w(\nu)(-1)^{\ell}\frac{(\nu-\ell)!}{(\nu+\ell)!}\left|\frac{a_\ell^{(2)}(\theta_0,\theta_1)}{a_\ell^{(2)}(\theta_1,\theta_1)}P_\nu^{\ell}(\cos\theta_1)\right|^2.$$

We note that this proposition parallels proposition 1 in the Euclidean case. But as seen in the following, a lower bound of error of extrapo -lation of origin with $n = 1$, differs.

Let an extrapolated point be the origin; then we have results

$$\|\hat{X}_n(z)\|^2 = \frac{2^{d-2}}{\pi}\Gamma^2\left(\frac{d}{2}\right)\sum_{i,j=1}^{n} a_0^{(d)}(\theta_i, \theta_j)\hat{c}^{(i)}(z;0)\overline{\hat{c}^{(j)}(z;0)},$$

and

$$\delta_1^2(z) = R(0) - \frac{\left[\Gamma(d-1)\sum_{\nu=0}^{\infty}\frac{\Gamma(\nu+1)}{\Gamma(\nu+d-1)}C_\nu^{\frac{d-1}{2}}(\cos\theta_1)w(\nu)\right]^2}{\sum_{\nu=0}^{\infty}\left[\Gamma(d-1)\frac{\Gamma(\nu+1)}{\Gamma(\nu+d-1)}C_\nu^{\frac{d-1}{2}}(\cos\theta_1)\right]^2 w(\nu)}.$$

Thus in the spherical case, error function $\delta_1^2(z)$, z is the origin, has lower bound 0 and this lower bound is attained iff spectrum distributes on ν-values such that $\frac{\Gamma(d-1)\Gamma(\nu+1)}{\Gamma(\nu+d-1)}C_\nu^{\frac{d-2}{2}}(\cos\theta_1) = C$, $0 < C \leq 1$.

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