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Topological immersions and PL immersions

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1. Introduction, definitions and notation

In this paper we compute the homotopy groups $\Pi_i(V^{TOP}_{q,m}, V^{PL}_{q,m})$ where $V^{TOP}_{q,m}$(resp. $V^{PL}_{q,m}$) is the topological analogue (resp. the PL analogue) of the stiefel manifold, using the result of Kirby-Siebenmann [7]. And we consider the relation between topological immersions and PL immersions of PL manifolds.

Theorem 1.

1) If $q-m > 3$, $\Pi_k(V^{TOP}_{q,m}, V^{PL}_{q,m}) = 0$ for $0 \leqq k \leqq m$.

2) If $q-m = 1$ $q \geqq 1$ or $q-m = 2$ $q \geqq 6$,

$$\Pi_k(TOP_q, PL_q) \longrightarrow \Pi_k(V^{TOP}_{q,m}, V^{PL}_{q,m})$$

is isomorphic for $0 \leqq k \leqq m$, and

$$\Pi_{m+1}(TOP_q, PL_q) \longrightarrow \Pi_{m+1}(V^{TOP}_{q,m}, V^{PL}_{q,m})$$

is surjective.

Corollary 2.

If $q-m = 1$ $q \geqq 5$ or $q-m = 2$ $q \geqq 6$,

$$\Pi_k(V^{TOP}_{q,m}, V^{PL}_{q,m}) = \begin{cases} 0 & k \neq 3 \\ Z_2 & k = 3 \end{cases}$$

for $k \leqq m+1$.

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Theorem 3.

1) Let Q, M be PL manifolds with dim. q and m respectively, and $f : M \longrightarrow Q$ be a topological immersion. Then, if $q-m \geqq 3$, there is a topological regular homotopy which takes f to a PL immersion. If $q-m = 1$ $q \geqq 5$ or $q-m = 2$ $q \geqq 6$, there is a topological regular homotopy which takes f to a PL immersion if and only if the obstruction $c(f) \in H^3(M, Z_2)$ vanishes.

2) Suppose that $f_0, f_1 : M \longrightarrow Q$ are PL immersions, there is a topological regular homotopy which takes f_0 to f_1, and $q-m = 1$, $q \geqq 5$ or $q-m = 2$ $q \geqq 6$. Then there is a PL regular homotopy which takes f_0 to f_1 if the obstruction $d(f_0, f_1) \in H^2(M, Z_2)$ vanishes.

In this paper, immersions and embeddings are assumed to be locally flat.

Definition.

Topological (resp. PL) stiefel $V^{TOP}_{q,m}$ (resp. $V^{PL}_{q,m}$) is a Kan complex whose k-simplex f is a topological (resp. PL) embedding

$$f : \Delta^k \times R^m \longrightarrow \Delta^k \times R^q$$

such that $f|(\Delta^k \times 0) = \mathrm{id}$, $pr_1 \circ f = pr_1$ and f is locally flat uniformly with respect to Δ^k (i.e. for any $x \in \Delta^k$, there is a neighbourhood $U \subset \Delta^k$ of x, and homeomorphisms $g_1 : U \times R^m \longrightarrow U \times R^m$, $g_2 : U \times R^q \longrightarrow U \times R^q$, where $pr_1\, g_\alpha = pr_1$ $(\alpha = 1, 2)$, such that the diagram

$$\begin{array}{ccc} U \times R^m & \xrightarrow{f|U \times R^m} & U \times R^q \\ {\scriptstyle g_1}\uparrow & & \uparrow{\scriptstyle g_2} \\ U \times R^m & \hookrightarrow & U \times R^q \end{array}$$

commutes.)

Definition.

Let M', Q be topological (resp. PL) manifolds, $N \subset M \subset M'$ be locally flat proper submanifolds, $Q : N \longrightarrow Q$ be a topological (resp. PL) immersion. $I^{TOP}_{M',\theta}(M, Q)$ (resp. $I^{PL}_{M',\theta}(M, Q)$) is a Kan complex of topological (resp. PL) M'-immersions of M in Q whose restrictions on N in θ. $R^{TOP}_{M',\theta}(M, Q)$ (resp. $R^{PL}_{M',\theta}(M, Q)$) is a complex of its representations. When $N = \phi$, we omit the subscript θ.

B^n denotes n-dim. ball. $\mathring{B}^n$ and ∂B^n denote the interior of B^n and boundary of B^n respectively.

Definition.

$\widetilde{I}_{B^k \times R^{m-k}}(B^k, R^q)$ is a complex of topological $B^k \times R^{m-k}$-immersions of B^k in R^q which is PL on a neighbourhood of $\partial B^k \times R^{m-k}$ in $B^k \times R^{m-k}$. $\widetilde{R}_{B^k \times R^{m-k}}(B^k \times R^q)$ is a complex of its representations.

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§2. Prelimilarly results

Lemma 5.

$$1)\quad \Pi_0(\tilde{I}_{B^k\times R^{m-k}}(B^k,\ R^q)) \cong \Pi_0(\tilde{R}_{B^k\times R^{m-k}}(B^k,\ R^q))$$
$$\cong \Pi_k(V^{TOP}{}_{q,m},\ V^{PL}{}_{q,m})$$

for $0 \leqq k \leqq m$.

$$2)\quad \Pi_i(\tilde{I}_{B^k\times R^{m-k}}(B^k,\ R^q),\ I^{PL}{}_{B^k\times R^{m-k}}(B^k,\ R^q))$$
$$\cong \Pi_i(\tilde{R}_{B^k\times R^{m-k}}(B^k,\ R^q),\ R^{PL}{}_{B^k\times R^{m-k}}(B^k,\ R^q))$$
$$\cong \Pi_{i+k}(V^{TOP}{}_{q,m},\ V^{PL}{}_{q,m})$$

for $0 \leqq k \leqq m$, $1 \leqq i$, $m < q$.

Proof.

1) is obtained from Kirby [7] and Kurata [10].

Proof of 2) is as follows. An isomorphism $\Pi_i(\tilde{R}_{B^k\times R^{m-k}}(B^k, R^q),$ $R^{PL}{}_{B^k\times R^{m-k}}(B^k\ R^q)) \longrightarrow \Pi_{i+k}(V^{TOP}{}_{q,m},\ V^{PL}{}_{q,m})$ is obtained similarly with the proof of 1).

The differential

$$d : I^{PL}{}_{B^k\times R^{m-k}}(B^k,\ R^q) \longrightarrow R^{PL}{}_{B^k\times R^{m-k}}(B^k,\ R^q)$$

is homotopy equivalent, therefore it is sufficient to prove the homotopy equivalence of

$$d : \tilde{I}_{B^k\times R^{m-k}}(B^k,\ R^q) \longrightarrow \tilde{R}_{B^k\times R^{m-k}}(B^k,\ R^q).$$

Consider the homotopy equivalence between fibrations

$$\begin{array}{ccc}
I^{TOP}_{2B^k\times R^{m-k},i}(B^k,\ R^q) & \longrightarrow & R^{TOP}_{2B^k\times R^{m-k},i}(B^k,\ R^q) \\
\downarrow & & \downarrow \\
I^{TOP}_{2B^k\times R^{m-k}}(B^k,\ R^q) & \longrightarrow & R^{TOP}_{2B^k\ R^{m-k}}(B^k,\ R^q) \\
\downarrow i_* & & \downarrow i_* \\
I^{TOP}_{2B^k\times R^{m-k}}(2B^k-\frac{1}{2}\mathring{B}^k,\ R^q) & \longrightarrow & R^{TOP}_{2B^k\times R^{m-k}}(2B^k-\frac{1}{2}\mathring{B}^k,\ R^q),
\end{array}$$

where i_* is induced by the inclusion $i : (2B^k-\frac{1}{2}\mathring{B}^k) \times R^{m-k} \longrightarrow 2B^k \times R^{m-k} \subset R^q$. Note that $I^{PL}_{2B^k\times R^{m-k}}(2B^k-\frac{1}{2}\mathring{B}^k,\ R^q)$ and $R^{PL}_{2B^k\times R^{m-k}}(2B^k-\frac{1}{2}\mathring{B}^k,\ R^q)$ are subcomplexes of $I^{TOP}_{2B^k\times R^{m-k}}(2B^k-\frac{1}{2}\mathring{B}^k,\ R^q)$ and $R^{TOP}_{2B^k\times R^{m-k}}(2B^k-\frac{1}{2}\mathring{B}^k,\ R^q)$ respectively. The restrictions

$i_* : \tilde{I}_{2B^k\times R^{m-k}}(B^k,\ R^q) \longrightarrow I^{PL}_{2B^k\times R^{m-k}}(2B^k-\frac{1}{2}\mathring{B}^k,\ R^q)$, and

$i_* : \tilde{R}_{2B^k\times R^{m-k}}(B^k,\ R^q) \longrightarrow R^{PL}_{2B^k\times R^{m-k}}(2B^k-\frac{1}{2}\mathring{B}^k,\ R^q)$ are fibrations with fibres being $I^{TOP}_{2B^k\times R^{m-k},i}(B^k,\ R^q)$ and $R^{TOP}_{2B^k\times R^{m-k},i}(B^k,\ R^q)$ respectively, because

$i_*^{-1}(I^{PL}_{2B^k\times R^{m-k}}(2B^k-\frac{1}{2}\mathring{B}^k,\ R^q)) = \tilde{I}_{2B^k\times R^{m-k}}(B^k,\ R^q)$ and

$i_*^{-1}(R^{PL}_{2B^k\times R^{m-k}}(2B^k-\frac{1}{2}\mathring{B}^k,\ R^q)) = \tilde{R}_{2B^k\times R^{m-k}}(B^k,\ R^q)$. By homotopy equivalences of $d : I^{TOP}_{2B^k\times R^{m-k},i}(B^k,\ R^q) \longrightarrow R^{TOP}_{2B^k\times R^{m-k},i}(B^k,\ R^q)$ and $d : I^{PL}_{2B^k\times R^{m-k}}(2B^k-\frac{1}{2}\mathring{B}^k,\ R^q) \longrightarrow R^{PL}_{2B^k\times R^{m-k}}(2B^k-\frac{1}{2}\mathring{B}^k,\ R^q)$, it follows that $d : \tilde{I}_{B^k\times R^{m-k}}(B^k,\ R^q) \longrightarrow \tilde{R}_{B^k\times R^{m-k}}(B^k,\ R^q)$ is a homotopy equivalence.

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Lemma 6.

The natural inclusion $i : B^k \times R^{n-k} \hookrightarrow B^k \times R^{m-k}$ $(n<m)$ induces the following commutative diagrams.

$$\begin{array}{ccc}
\Pi_0(\tilde{I}_{2B^k\times R^{m-k}}(B^k,\ R^q)) & \xrightarrow{i^*} & \Pi_0(\tilde{I}_{2B^k\times R^{n-k}}(B^k,\ R^q)) \\
\downarrow \cong & & \downarrow \cong \\
\Pi_0(\tilde{R}_{2B^k\times R^{m-k}}(B^k,\ R^q)) & \xrightarrow{i^*} & \Pi_0(\tilde{R}_{2B^k\times R^{n-k}}(B^k,\ R^q)) \\
\downarrow \cong & & \downarrow \cong \\
\Pi_k(V^{TOP}_{q,m},\ V^{PL}_{q,m}) & \xrightarrow{i^*} & \Pi_k(V^{TOP}_{q,n},\ V^{PL}_{q,n})
\end{array},$$

$$\begin{array}{ccc}
\Pi_i(\tilde{I}_{2B^k\times R^{m-k}}(B^k,\ R^q),\ I^{PL}_{2B^k\times R^{m-k}}(B^k,\ R^q)) & & \\
& \searrow^{i^*} & \\
\downarrow \cong & & \Pi_i(\tilde{I}_{2B^k\times R^{n-k}}(B^k\times R^q),\ I^{PL}_{2B^k\times R^{n-k}}(B^k,\ R^q)) \\
& & \downarrow \cong \\
\Pi_i(\tilde{R}_{2B^k\times R^{m-k}}(B^k,\ R^q),\ R^{PL}_{2B^k\times R^{m-k}}(B^k,\ R^q)) & & \\
& \searrow^{i^*} & \\
\downarrow \cong & & \Pi_i(\tilde{R}_{2B^k\times R^{n-k}}(B^k\times R^q),\ R^{PL}_{2B^k\times R^{n-k}}(B^k,\ R^q)) \\
& & \downarrow \cong \\
\Pi_{i+k}(V^{TOP}_{q,m},\ V^{PL}_{q,m}) & \xrightarrow{i^*} & \Pi_i(V^{TOP}_{q,n},\ V^{PL}_{q,n})
\end{array}$$

Lemma 7.

Suppose $q-m = 1$, or $q-m = 2$ $q \geqq 6$. Then

1) $i^* : \Pi_0(\tilde{I}_{2B^k\times R^{q-k}}(B^k,\ R^q)) \longrightarrow \Pi_0(\tilde{I}_{2B^k\times R^{m-k}}(B^k,\ R^q))$

is an isomorphism for $0 \leqq k \leqq m$.

2) $i^* : \Pi_1(\tilde{I}_{2B^k\times R^{q-k}}(B^k,\ R^q),\ I^{PL}_{2B^k\times R^{q-k}}(B^k,\ R^q)) \longrightarrow$

$\Pi_1(\tilde{I}_{2B^k\times R^{m-k}}(B^k,\ R^q),\ I^{PL}_{2B^k\times R^{m-k}}(B^k,\ R^q))$

is a surjection for $0 \leqq k \leqq m$.

Proof of 1). Suppose that $f : 2B^k \times 2B^{m-k} \longrightarrow R^q$ represent a vertex of $\widetilde{I}_{2B^k \times R^{m-k}}(B^k, R^q)$. Let $(\widetilde{f}, V^q, \psi)$ be an induced neighbourhood of $2B^k \times 2B^{m-k}$ by f. (cf. [10]).

$$\begin{array}{ccc} 2B^k \times 2B^{m-k} & \xrightarrow{\widetilde{f}} & V^q \\ & {\scriptstyle f}\searrow \quad \circlearrowleft \quad \swarrow{\scriptstyle \psi} & \\ & R^q & \end{array}$$

We may assume that V^q is a q-dim. PL manifold with PL structure induced by ψ, ψ is PL immersion and $\widetilde{f}$ is an embedding.

We show that $\widetilde{f}$ can be extended to an embedding $\widetilde{F} : 2B^k \times 2B^{q-k} \longrightarrow V^q$ such that $\widetilde{F}|((2B^k - \frac{1}{2}\mathring{B}^k) \times 2B^{q-k})$ is PL, moreover such extension is unique up to isotopy. An extension $\widetilde{F}$ of $\widetilde{f}$ can be constructed as follows.

In the case where q-m = 1. By existance of codimension 1 topological normal bundle, there is a topological embedding $g : 2B^k \times 2B^{m-k} \times R \longrightarrow V^q$ such that $\widetilde{f} \circ pr_1 = g$, where $pr_1 : 2B^k \times 2B^{m-k} \times R \longrightarrow 2B^k \times 2B^{m-k}$ is the projection on $2B^k \times 2B^{m-k}$. By the existance of codimension 1 PL normal bundle, there is a PL bundle $p : E \longrightarrow (2B^k - \frac{1}{2}\mathring{B}^k) \times 2B^{m-k}$ with fibre being R, and PL embedding $g' : E \longrightarrow V^q$ such that $(\widetilde{f}|((2B^k - \frac{1}{2}\mathring{B}^k) \times 2B^{m-k}) \circ p = g'$. Because codimension 1 topological normal bundle is unique up to isotopy (cf. Brown [2]), g can be chosen such that $p \circ g'^{-1} \circ g = pr_1$. Therefore $p : E \longrightarrow (2B^k - \frac{1}{2}\mathring{B}^k) \times 2B^{m-k}$ is topological trivial bundle. The homotopy equivalence between TOP_1 and PL_1 implies that $p : E \longrightarrow (2B^k - \frac{1}{2}\mathring{B}^k) \times 2B^{m-k}$

is PL trivial bundle. Then we may assume that $E = (2B^k - \frac{1}{2}\mathring{B}^k) \times 2B^{m-k} \times R$. $p = pr_1$. By the argument of Brown [2], PL embedding $g' : (2B^k - \frac{1}{2}\mathring{B}^k) \times 2B^{m-k} \times R \longrightarrow V^q$ can be extended to a topological embedding $\tilde{F} : 2B^k \times 2B^{m-k} \times R \longrightarrow V^q$.

Let $\tilde{F}_\alpha : 2B^k \times 2B^{m-k} \times R \longrightarrow V^q$ $(\alpha = 0,1)$ be extensions of $\tilde{f}$ such that $\tilde{F}_\alpha | ((2B^k - \frac{1}{2}\mathring{B}^k) \times 2B^{m-k} \times R)$ is PL. By the uniqueness up to isotopy of codimension 1 PL normal bundle and the isotopy extension theorem of topological manifolds (Edward-Kirby [3]), there is a topological isotopy $H_t : 2B^k \times 2B^{m-k} \times R \longrightarrow 2B^k \times 2B^{m-k} \times R$ $(0 \leq t \leq 1)$ fixing $2B^k \times 2B^{m-k} \times 0$, such that $H_0 = id$. $H_t | ((2B^k - \frac{1}{2}\mathring{B}^k) \times 2B^{m-k} \times R)$ is PL, and $H_1 \circ \tilde{F}_1^{-1} \circ \tilde{F}_0 | ((2B^k - \frac{1}{2}\mathring{B}^k) \times 2B^{m-k} \times R)$ commutes with the projection on $(2B^k - \frac{1}{2}\mathring{B}^k) \times 2B^{m-k}$. By the uniqueness up to isotopy of codimension 1 topological normal bundle, there is a topological isotopy $H_t (1 \leq t \leq 2)$ fixing $((2B^k - \frac{1}{2}\mathring{B}^k) \times 2B^{m-k} \times R) \cup (2B^k \times 2B^{m-k} \times 0)$ such that $H_2 \circ \tilde{F}_1^{-1} \circ \tilde{F}_0$ commutes with the projection on $2B^k \times 2B^{m-k}$. Because $\Pi_i(TOP_1, PL_1) = 0$ for $i \geq 0$, there is a topological isotopy $H_t (2 \leq t \leq 3)$ such that $H_t | ((2B^k - \frac{1}{2}\mathring{B}^k) \times 2B^{m-k} \times R)$ $(2 \leq t \leq 3)$ is PL, $H_t \circ \tilde{F}_1^{-1} \circ \tilde{F}_0$ $(2 \leq t \leq 3)$ commutes with the projection on $2B^k \times 2B^{m-k}$, $H_3 \circ \tilde{F}_1^{-1} \circ \tilde{F}_0$ is PL. The contractibility of $2B^k \times 2B^{m-k}$ implies that there is a PL isotopy $H_t (3 \leq t \leq 4)$ which commutes with the projection on the $2B^k \times 2B^{m-k}$ such that $H_4 \circ \tilde{F}_1^{-1} \circ \tilde{F}_0 = id$. $\tilde{F}_1 \circ H_t^{-1}$ $(0 \leq t \leq 4)$ is the required isotopy from $\tilde{F}_1$ to $\tilde{F}_0$.

In the case where $q-m = 2$, $q \geq 6$. There is a topological extension $F : 2B^k \times 2B^{m-k} \times R^2 \longrightarrow V^q$ os f. By Kirby-Siebenmann [8], $F | ((2B^k - \frac{1}{2}\mathring{B}^k) \times 2B^{m-k} \times R^2)$ is isotopic to a PL embedding.

Extending this isotopy by isotopy extension theorem ([3]), we obtain the required embedding $\check{F}$. Similar argument to the proof in the case where q-m = 1 implies the uniqueness of the extension of $\hat{f}$, by the existance and uniqueness of codimension 2 PL normal bundle ([12]) and the homotopy equivalence between TOP_2 and PL_2 (Akiba [1], Kneser [9]).

Proof of 2). Let $f : 2B^k \times 2B^{m-k} \times I \longrightarrow R^q \times I$ be an 1-simplex of $\tilde{I}_{2B^k \times R^{m-k}}(B^k, R^q)$, such that $f|(2B^k \times 2B^{m-k} \times \dot{I})$ is a PL immersion. There is a topological immersion $g : 2B^k \times 2B^{m-k} \times R^{q-m} \times I \longrightarrow R^q \times I$ such that g is an extension of f, g commutes with the projection on I and $g|((2B^k - \frac{1}{2}\mathring{B}^k) \times 2B^{m-k} \times R^{q-m} \times I)$ is PL. Similarly to the proof of 1), there is regular homotopies rel. $2B^k \times 2B^{m-k} \times \{\alpha\}$ $g_t : 2B^k \times 2B^{m-k} \times R^{q-m} \times \{\alpha\} \longrightarrow R^q \times \{\alpha\}$ $(\alpha = 0, 1)$, keeping PL on $(2B^k - \frac{1}{2}\mathring{B}^k) \times 2B^{m-k} \times R^{q-m} \times \{\alpha\}$, which take $g|(2B^k \times 2B^{m-k} \times R^{q-m} \times \{\alpha\})$ to PL immersions. Extending g_t to a regular homotopy over $2B^k \times 2B^{m-k} \times R^{q-m} \times I$. We obtain an immersion $g' : 2B^k \times 2B^{m-k} \times R^{q-m} \times I \longrightarrow R^q \times I$, which satisfies the following. g' is an 1-simplix of $\tilde{I}_{2B^k \times R^{q-k}}(B^k, R^q)$, an extension of $\tilde{f}$ and $g'|(2B^k \times 2B^{m-k} \times R^{q-m} \times \dot{I})$ is PL immersion. This shows that

$$i^* : \Pi_1(\tilde{I}_{2B^k \times R^{q-k}}(B^k, R^q), I^{PL}_{2B^k \times R^{q-k}}(B^k, R^q)) \longrightarrow \Pi_1(\tilde{I}_{2B^k \times R^{m-k}}(B^k, R^q), I^{PL}_{2B^k \times R^{m-k}}(B^k, R^q))$$

is surjective. The proof is complete.

Lemma 8.

Suppose $q-m \geqq 3$, $f : 2B^k \times 2B^{m-k} \longrightarrow R^q$ be a vertex of

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$\tilde{I}_{2B^k \times 2B^{m-k}}(B^k, R^q)$. Then there is a regular homotopy g_t which takes f to a PL immersion with $g_t \in \tilde{I}_{2B^k \times 2B^{m-k}}(B^k, R^q)$.

Proof.

Let $(\hat{f}, V^q, \psi)$ be an induced neighbourhood by f of $2B^k \times 2B^{m-k}$, where ψ is PL immersion, V^q is PL manifold and $\hat{f}$ is embedding with $\hat{f}|((2B^k - \frac{1}{2}\mathring{B}^k) \times 2B^{m-k})$ being PL. By Kirby-Siebenmann [8] and Miller [11], there is an isotopy $\hat{f}_t$ which takes $\hat{f}$ to a PL embedding keeping PL on $(2B^k - \frac{1}{2}\mathring{B}^k) \times 2B^{m-k}$. $g_t = \psi \circ \hat{f}_t$ is a required regular homotopy.

§3. Proof of theorems.

Theorem 1 is obtained by lemma 5~8.

Corollary 2 follows immediately from Theorem 1 by Kirby-Siebenmann [7].

Proof of Theorem 3.

Let $f : M^m \longrightarrow Q^q$ be a topological immersion. $df : TM \longrightarrow TQ$ is its differential. There is a homotopy $f_t : M \longrightarrow Q$ which takes f to a PL map f_1. There is a homotopy of locally flat bundle monomorphism $\psi_t : TM \longrightarrow TQ$ such that $\psi_0 = df$ and ψ_t covers f_t. When $q-m > 3$, ψ_1 can deform to a PL bundle map and when $q-m = 1$ $q \geq 5$ or $q-m = 2$ $q \geq 6$ the abstruction to deform ψ_1 to a PL bundle map lies in $H^3(M, Z_2)$ by Corollary 2. By topological and PL immersion theorems ([5], [10]), it follows from theorem 3.1). Proof of 2) is similar to those of 1).

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