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Ultradistributions and hyperfunctions

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In the last conference of March, 1971, the speaker announced the following theorem and applied it to the theory of ordinary differential equations with real analytic coefficients.

Theorem. Let $f = [F]$ be a hyperfunction on an interval (a, b) with a defining function F. Then f is an ultradistribution of Gevrey class of order s of Roumieu type (of Beurling type) if and only if for every compact interval $K \subset (a, b)$ and every $L > 0$ there is a constant C (there are constants L and C) such that

$$\sup_{x\in K} |\varphi(x+iy)| \leq C \exp\left\{\left(\frac{L}{|y|}\right)^{\frac{1}{s-1}}\right\}$$

In this lecture we develop the theory of ultradistributions and give a proof of the theorem in a generalized form.

1. Ultradifferentiable functions.

Let M_p, $p = 0, 1, 2, \cdots,$ be a sequence of positive numbers. An infinitely differentiable function f on an open set Ω in $\mathbb{R}^n$ will be called an ultradifferentiable function of class M_p of Roumieu type (of Beurling type) if for every compact set K in Ω there are constants h and C (and for every $h > 0$ there is a constant C) such that

$$\|D^\alpha f\|_{C(K)} \leq C h^p M_p, \quad |\alpha| = p = 0, 1, 2, \cdots . \tag{1}$$

We will impose the following conditions on M_p:

(M.1) (Logarithmic convexity)

(2) $$M_p^2 \leqq M_{p-1} M_{p+1}, \quad p = 1, 2, \cdots$$

(M.2) (Stability under convolution) There are constants A and H such that

(3) $$M_p \leqq A H^p \min_{0 \leqq q \leqq p} M_{p-q} M_q, \quad p = 0, 1, 2, \cdots .$$

(M.3) (Strong non-quasi-analyticity) There is a constant A such that

(4) $$\sum_{j=p}^{\infty} \frac{M_j}{M_{j+1}} \leqq A \frac{p M_p}{M_{p+1}}, \quad p = 1, 2, 3, \cdots .$$

In some problems (M.2) and (M.3) may be replaced by the following weaker conditions:

(M.2)' (Stability under differentiation)

(5) $$M_{p+1} \leqq A H^p M_p, \quad p = 0, 1, 2, \cdots .$$

(M.3)' (Non-quasi-analyticity)

(6) $$\sum_{j=0}^{\infty} \frac{M_j}{M_{j+1}} < \infty .$$

It is easy to check that the Gevrey sequences

(7) $$M_p = (p!)^s, \quad p^{sp} \quad \text{and} \quad \Gamma(1+sp),$$

where $s > 1$, satisfy these conditions. These sequences determine the same class of ultradifferentiable functions called the <u>Gevrey class of order</u> s.

It is convenient to relate the above conditions with the behavior of the <u>associated function</u>

(8) $$M(\rho) = \log \sup_p \frac{\rho^p M_0}{M_p} .$$

(M.1) is equivalent to

(9) $$\frac{M_p}{M_0} = \sup_{\rho > 0} \frac{\rho^p}{\exp M(\rho)} .$$

Under this condition

(10) $$M(\rho) = \int_0^{\rho} \frac{m(\lambda)}{\lambda} d\lambda ,$$

where $m(\lambda)$ is the number of ratios $m_j = M_j/M_{j-1}$ which does not exceed λ .

(M.2) is equivalent to

(11) $$2M(\rho) \leqq M(H\rho) + \log(A M_0) .$$

(M.3) implies

(12) $$\rho \int_{\rho}^{\infty} \frac{m(\lambda)}{\lambda^2} d\lambda \leqq A M(\rho).$$

On the other hand, (M.2)' is equivalent to

(13) $$m(\lambda) \geqq \frac{\log(\lambda/A')}{\log H} .$$

(M.3)' is equivalent to

(14) $$\int_0^{\infty} \frac{M(\rho)}{\rho^2} d\rho = \int_0^{\infty} \frac{m(\lambda)}{\lambda^2} d\lambda < \infty .$$

Definition 1. Let K be a regular compact set in $\mathbb{R}^n$ and $h > 0$. We denote by $\mathcal{E}^{\{M_p\},h}(K)$ the Banach space of all functions $f \in C^{\infty}(K)$ in the sense of Whitney such that

(15) $$\|f\|_{\mathcal{E}^{\{M_p\},h}(K)} = \sup_{\alpha,x} \frac{|D^{\alpha}f(x)|}{h^{|\alpha|}M_{|\alpha|}} < \infty ,$$

and by $\mathcal{D}_K^{\{M_p\},h}$ the Banach space of all functions $f \in C^{\infty}(\mathbb{R}^n)$ with support in K which satisfies (15).

$\mathcal{D}_K^{\{M_p\},h}$ may be looked upon as a closed subspace of $\mathcal{E}^{\{M_p\},h}(K)$.

Proposition 2. If $h < k$, the injections

(16) $$\mathcal{E}^{\{M_p\},h}(K) \subset \mathcal{E}^{\{M_p\},k}(K)$$

(17) $$\mathcal{D}_K^{\{M_p\},h} \subset \mathcal{D}_K^{\{M_p\},k}$$

are compact. If M_p satisfies (M.2)' in addition and if k/h is sufficiently large, then the injections are nuclear.

<u>Definition 3</u>. Let K be a regular compact set and Ω an open set in $\mathbb{R}^n$. We define the spaces of ultradifferentiable functions of Roumieu type $\mathcal{E}^{\{M_p\}}(K)$, $\mathcal{E}^{\{M_p\}}(\Omega)$ and those of Beurling type $\mathcal{E}^{(M_p)}(K)$ and $\mathcal{E}^{(M_p)}(\Omega)$ by

(18) $$\mathcal{E}^{\{M_p\}}(K) = \varinjlim_{h\to\infty} \mathcal{E}^{\{M_p\},h}(K),$$

(19) $$\mathcal{E}^{\{M_p\}}(\Omega) = \varprojlim_{K\Subset\Omega} \mathcal{E}^{\{M_p\}}(K) ,$$

(20) $$\mathcal{E}^{(M_p)}(K) = \varprojlim_{h\to 0} \mathcal{E}^{\{M_p\},h}(K) ,$$

(21) $$\mathcal{E}^{(M_p)}(\Omega) = \varprojlim_{K\Subset\Omega} \mathcal{E}^{(M_p)}(K) .$$

It follows from Proposition 2 that $\mathcal{E}^{\{M_p\}}(K)$ is a (DFS)-space and $\mathcal{E}^{(M_p)}(K)$ and $\mathcal{E}^{(M_p)}(\Omega)$ are (FS)-spaces. If M_p satisfies (M.2)', these spaces are all nuclear.

Similarly the spaces of ultra-differentiable functions with compact support are defined in the following way:

(22) $$\mathcal{D}_K^{\{M_p\}} = \varinjlim_{h\to\infty} \mathcal{D}_K^{\{M_p\},h} ,$$

(23) $$\mathcal{D}^{\{M_p\}}(\Omega) = \varinjlim_{K\Subset\Omega} \mathcal{D}_K^{\{M_p\}} ,$$

(24) $$\mathcal{D}_K^{(M_p)} = \varprojlim_{h\to 0} \mathcal{D}_K^{\{M_p\},h} ,$$

(25) $$\mathcal{D}^{(M_p)}(\Omega) = \varinjlim_{K\Subset\Omega} \mathcal{D}_K^{(M_p)} .$$

$\mathcal{D}_K^{\{M_p\}}$ and $\mathcal{D}^{\{M_p\}}(\Omega)$ are (DFS)-spaces, $\mathcal{D}_K^{(M_p)}$ is an (FS)-space and $\mathcal{D}^{(M_p)}(\Omega)$ is an (LF)-space as the strict inductive limit of a sequence of (FS)-spaces. Hence all spaces are Hausdorff, complete, reflexive and bornologic. If M_p satisfies (M.2)', then all spaces are nuclear.

A subset B of $\mathcal{D}_K^{\{M_p\}}$ or $\mathcal{D}^{\{M_p\}}(\Omega)$ is bounded if and only if it is contained in a $\mathcal{D}_K^{\{M_p\},h}$ and bounded there, while a subset B of $\mathcal{D}_K^{(M_p)}$ or $\mathcal{D}^{(M_p)}(\Omega)$ is bounded if and only if it is contained in a $\mathcal{D}_K^{(M_p)}$ for a K and bounded in all $\mathcal{D}_K^{\{M_p\},h}$.

It is well known[1)] that $\mathcal{D}_K^{\{M_p\}} = \mathcal{D}_K^{\{M'_p\}}$, where M'_p is the greatest logarithmically convex sequence such that $M'_p \leqq M_p$ and that in case M_p is logarithmically convex, $\mathcal{D}_K^{\{M_p\}} \neq 0$ if and only if M_p satisfies (M.3)'. Conversely suppose that M_p satisfies (M.1) and (M.3)'. Then for any ball K of radius $\varepsilon > 0$ there is a function $\rho_\varepsilon \in \mathcal{D}_K^{\{M_p\}}$ such that $\rho_\varepsilon(x) \geqq 0$ and $\int \rho(x)\,dx = 1$. Hence it follows that $\mathcal{D}^{\{M_p\}}(\Omega)$ is dense in $\mathcal{D}(\Omega)$ and that there exists a partition of unity by functions in $\mathcal{D}^{\{M_p\}}(\Omega)$ subordinate to any open covering of Ω.

If M_p satisfies (M.1) and (M.3)', there is M'_p which satisfies (M.1), (M.3)' and

$$\lim_{p\to\infty} \frac{h^p M_p}{M'_p} = 0 \qquad \text{for any } h > 0. \tag{26}$$

Thus the same results as above hold for $\mathcal{D}^{(M_p)}(\Omega)$.

1) See Mandelbrojt [8], [9], Roumieu [10], [11] and Lions-Magenes [7] for the results up to the end of this section.

If M_p satisfies (M.1), the spaces $\mathcal{E}^{\{M_p\}}(K)$, $\mathcal{E}^{\{M_p\}}(\Omega)$, $\mathcal{E}^{(M_p)}(K)$ and $\mathcal{E}^{(M_p)}(\Omega)$ are stable under multiplication and the spaces $\mathcal{D}_K^{\{M_p\}}$, $\mathcal{D}^{\{M_p\}}(\Omega)$, $\mathcal{D}_K^{(M_p)}$ and $\mathcal{D}^{(M_p)}(\Omega)$ are stable under multiplication by functions in $\mathcal{E}^{\{M_p\}}(K)$, $\mathcal{E}^{\{M_p\}}(\Omega)$, $\mathcal{E}^{(M_p)}(K)$ and $\mathcal{E}^{(M_p)}(\Omega)$ respectively and the multiplications are hypo-continuous.

If (M.2)' holds, the above spaces are stable under differentiation and D^α is continuous for any α

The spaces of Roumieu type have been discussed by Roumieu [10] and [11]. However, it is not clear whether or not the topologies he employed coincide with the above natural topologies which have been introduced by Lions-Magenes [7].

The spaces of Beurling type have been discussed in Björck [1] from a little different point of view and in Lions-Magenes [7].

2. The Paley-Wiener theorem for ultra-differentiable functions.

Theorem 4. Suppose that M_p *satisfies* (M.1) *and* (M.2)' *and that* K *is a compact convex set in* $\mathbb{R}^n$. *Then a function* $\varphi(x)$ *belongs to* $\mathcal{D}_K^{\{M_p\}}$ $(\mathcal{D}_K^{(M_p)})$ *if and only if there are* h *and* C (*for any* $h > 0$ *there is* C) *such that the Fourier-Laplace transform*

$$\tilde{\varphi}(\zeta) = \mathcal{F}\varphi(\zeta) = \int_{\mathbb{R}^n} e^{ix\zeta}\varphi(x)\,dx \tag{27}$$

of φ satisfies

(28) $$|\tilde{\varphi}(\zeta)| \leqq C \exp(-M(|\zeta|/h) + H_K(\zeta)),$$

where

(29) $$H_K(\zeta) = \sup_{x \in K} \operatorname{Im} \langle x, \zeta \rangle .$$

A subset B of $\mathcal{D}_K^{\{M_p\}}$ $(\mathcal{D}_K^{(M_p)})$ is bounded if and only if we can choose constants h and C (for any $h > 0$ a constant C) uniformly for $\varphi \in B$.

A sequence of functions $\varphi_j \in \mathcal{D}_K^{\{M_p\}}$ $(\mathcal{D}_K^{(M_p)})$ converges if and only if for some $h > 0$ (for any $h > 0$) $\exp M(|\zeta|/h)\, \tilde{\varphi}_j(\zeta)$ converges uniformly on $\mathbb{R}^n$ or equivalently on a strip $|\operatorname{Im} \zeta| < a$, where $0 < a < \infty$.

Since $\mathcal{D}_K^{(M_p)}$ is a Fréchet space, this shows that the families of semi-norms

(30) $$\sup_{\zeta \in \mathbb{C}^n} \exp (M(k|\zeta|) - H_K(\zeta))\, \tilde{\varphi}(\zeta) , \quad k = 1, 2, \cdots$$

and

(31) $$\sup_{\xi \in \mathbb{R}^n} \exp M(k|\xi|)\, \tilde{\varphi}(\xi), \quad k = 1, 2, \cdots$$

determine the topology of $\mathcal{D}_K^{(M_p)}$.

In order to find a family of semi-norms similar to (30) or (31) which determines the topology of $\mathcal{D}_K^{\{M_p\}}$, we imbed the Fourier-Laplace transform of $\mathcal{D}_K^{\{M_p\}}$ in a (DFS*)-space.

Let $1 < r < \infty$ be fixed and consider the sequence of Banach spaces

(32) $$Y_h = \{\psi \in L^r_{loc}(\mathbf{C}^n);\ \exp(M(|\zeta|/h) - H_K(\zeta))\psi(\zeta) \in L^r(\mathbf{C}^n)\},$$
$$h = 1, 2, \cdots$$

with the identity mappings $Y_h \longrightarrow Y_{h+1}$. Since Y_h are reflexive Banach spaces, this forms a weakly compact sequence and its limit $Y = \underrightarrow{\lim}\, Y_h$ is a (DFS*)-space.

A modified form of Morera's theorem shows that

(33) $$X_h = \{\psi \in Y_h;\ \psi \text{ is entire on } \mathbf{C}^n\}$$

is a closed subspace of Y_h. We can prove that

(34) $$\mathcal{F}\mathcal{D}_K^{\{M_p\}} = \lim_{\overrightarrow{h\to\infty}} X_h$$

including the topology. Morera's theorem proves also that the set $\mathcal{F}\mathcal{D}_K^{\{M_p\}}$ is closed in Y and that $X_h = Y_h \cap \mathcal{F}\mathcal{D}_K^{\{M_p\}}$.

Since $\mathcal{D}_K^{\{M_p\}}$ is a Montel space, it is proved that the original topology of $\mathcal{F}\mathcal{D}_K^{\{M_p\}}$ induced by that of $\mathcal{D}_K^{\{M_p\}}$ coincides with the relative topology induced by that of Y (cf. [5] Theorem 7).

<u>Theorem 5</u>. <u>Under the same assumptions as in Theorem 4 the topology of</u> $\mathcal{F}\mathcal{D}_K^{\{M_p\}}$ <u>is determined by the family of semi-norms</u>

(35) $$\sup_{\zeta\in\mathbf{C}^n} |\exp(M(\varepsilon(|\zeta|)) - H_K(\zeta))\tilde{\varphi}(\zeta)|$$

<u>when</u> $\varepsilon(\rho)$ <u>runs through the increasing functions on</u> $[0, \infty)$ <u>satisfying</u>

(36) $$\lim_{\rho\to\infty} \frac{\varepsilon(\rho)}{\rho} = 0.$$

From the Paley-Wiener theorem (Theorem 4) we get easily the following

Theorem 6. Suppose that M_p *satisfies* (M.1), (M.2) *and* (M.3)'. *Let*

$$J(\zeta) = \sum_{|\alpha|=0}^{\infty} a_\alpha \zeta^\alpha \tag{37}$$

be an entire function with the growth order that for any $L > 0$ *there is* C (*there are* L *and* C) *such that*

$$|J(\zeta)| \leqq C \exp M(L|\zeta|), \qquad \zeta \in \mathbb{C}^n. \tag{38}$$

Then, for any compact convex set K *in* $\mathbb{R}^n$ *the differential operator of infinite order*

$$J(D) = \sum_{|\alpha|=0}^{\infty} a_\alpha D^\alpha \tag{39}$$

maps $\mathcal{D}_K^{\{M_p\}}$ ($\mathcal{D}_K^{(M_p)}$) *continuously into itself. Moreover, the right hand side of*

$$J(D)\varphi(x) = \sum_{|\alpha|=0}^{\infty} a_\alpha D^\alpha \varphi(x) \tag{40}$$

converges absolutely in the topology of $\mathcal{D}_K^{\{M_p\}}$ ($\mathcal{D}_K^{(M_p)}$) *and* (40) *holds for any* $\varphi \in \mathcal{D}_K^{\{M_p\}}$ ($\mathcal{D}_K^{(M_p)}$). *More precisely if* φ *is contained in a bounded set of* $\mathcal{D}_K^{\{M_p\}}$ ($\mathcal{D}_K^{(M_p)}$), *the partial sums of* (40) *are contained in an absolutely convex bounded set* B *and the series converges absolutely in the normed space generated by* B.

An entire function $J(\zeta)$ satisfying (38) will be called a *multiplier* for the class $\{M_p\}$ ((M_p)). It is easy to see that (37) is a multiplier for $\{M_p\}$ ((M_p)) if and only if for any $L > 0$ there is C (there are L and C) such that

$$|a_\alpha| \leqq C L^{|\alpha|} / M_{|\alpha|}, \qquad |\alpha| = 0, 1, 2, \cdots . \tag{41}$$

Proposition 7. Suppose that M_p satisfies (M.1), (M.2) and

(M.3). Then an entire function $J(\zeta)$ of one variable is a multiplier for $\{M_p\}$ ((M_p)) if and only if it has Hadamard's factorization ([2], p.22)

(42) $$J(\zeta) = a\zeta^{n_0} \prod_{j=1}^{\infty} (1 - \frac{\zeta}{c_j})$$

and for any $L > 0$ there is C (there are L and C) such that

(43) $$N(\rho) = \int_0^{\rho} \frac{n(\lambda) - n_0}{\lambda} d\lambda$$

$$\leqq M(L\rho) + \log C, \qquad 0 < \rho < \infty,$$

where $n(\lambda)$ is the number of c_j with $|c_j| \leqq \lambda$.

Finally we obtain a characterization of the Fourier-Laplace transforms of ultra-differentiable functions with compact support in a way similar to Ehrenpreis [3]. Since (40) converges absolutely in the original topology, ours may be said a better characterization.

Theorem 8. *Suppose that* M_p *satisfies* (M.1), (M.2) *and* (M.3) *and that* K *is a compact convex set in* $\mathbb{R}^n$. *Then a function* $\varphi(x)$ *belongs to* $\mathcal{D}_K^{\{M_p\}}$ ($\mathcal{D}_K^{(M_p)}$) *if and only if its Fourier-Laplace transform* $\tilde{\varphi}(\zeta)$ *satisfies*

(44) $$\sup_{\zeta} |\exp(-H_K(\zeta))J(\zeta)\tilde{\varphi}(\zeta)| < \infty$$

for any entire function $J(\zeta)$ *of the form*

(45) $$J(\zeta) = J_0(s_1\zeta_1) \cdots J_0(s_n\zeta_n),$$

(46) $$J_0(\zeta) = \prod_{j=1}^{\infty} (1 + \frac{\ell_j\zeta}{m_j}),$$

where s_i *is* $+1$ *or* -1 *and* ℓ_j *is a sequence of positive numbers converging to zero* (ℓ_j *is a positive constant*).

Moreover, the family of semi-norms (44) *determines the*

<u>topology of</u> $\mathcal{D}_K^{\{M_p\}}$ $(\mathcal{D}_K^{(M_p)})$.

3. Ultra-distributions.

<u>Definition 9</u>. Suppose that M_p satisfies (M.1) and (M.3)' and that Ω is an open set in $\mathbb{R}^n$. We denote by $\mathcal{D}^{\{M_p\}'}(\Omega)$ $(\mathcal{D}^{(M_p)'}(\Omega))$ the strong dual space of $\mathcal{D}^{\{M_p\}}(\Omega)$ $(\mathcal{D}^{(M_p)}(\Omega))$ and call its elements <u>ultra-distributions</u> on Ω of class M_p of Roumieu type (Beurling type) or of class $\{M_p\}$ $((M_p))$ for short.

Since $\mathcal{D}^{\{M_p\}}(\Omega)$ $(\mathcal{D}^{(M_p)}(\Omega))$ is a dense subspace of $\mathcal{D}(\Omega)$ and the injection is continuous, $\mathcal{D}^{\{M_p\}'}(\Omega)$ $(\mathcal{D}^{(M_p)'}(\Omega))$ contains the distributions $\mathcal{D}'(\Omega)$ as a dense subspace.

On the other hand, since the real analytic functions on Ω are continuously and densely contained in $\mathcal{E}^{\{M_p\}}(\Omega)$ $(\mathcal{E}^{(M_p)}(\Omega))$, it follows that every ultra-distribution is a hyperfunction.

If $a \in \mathcal{E}^{\{M_p\}}(\Omega)$ $(\mathcal{E}^{(M_p)}(\Omega))$ and $f \in \mathcal{D}^{\{M_p\}'}(\Omega)$ $(\mathcal{D}^{(M_p)'}(\Omega))$, the product af is defined by

(47) $$\langle af, \varphi\rangle = \langle f, a\varphi\rangle, \quad \varphi \in \mathcal{D}^{\{M_p\}}(\Omega) \ (\mathcal{D}^{(M_p)}(\Omega)).$$

If M_p satisfies (M.2)', the derivative $D^\alpha f$ is defined by

(48) $$\langle D^\alpha f, \varphi\rangle = (-1)^{|\alpha|}\langle f, D^\alpha\varphi\rangle, \quad \varphi \in \mathcal{D}^{\{M_p\}}(\Omega) \ (\mathcal{D}^{(M_p)}(\Omega)).$$

Similarly if M_p satisfies (M.2) and $J(\zeta)$ is a multiplier for $\{M_p\}$ $((M_p))$, $J(D)f$ is defined by

(49) $$\langle J(D)f, \varphi\rangle = \langle f, J(-D)\varphi\rangle, \quad \varphi \in \mathcal{D}^{\{M_p\}}(\Omega) \ (\mathcal{D}^{(M_p)}(\Omega)).$$

As in the case of distributions, the existence of partition of unity implies that $\mathcal{D}^{\{M_p\}'}(\Omega)$ $(\mathcal{D}^{(M_p)'}(\Omega))$, $\Omega \subset \mathbb{R}^n$, with the natural restriction mappings forms a soft sheaf on $\mathbb{R}^n$. In particular, the notion of support is defined.

If S is a closed set in Ω, the subspace of all ultradistributions f in $\mathcal{D}^{\{M_p\}'}(\Omega)$ ($\mathcal{D}^{(M_p)'}(\Omega)$) with supp $f \subset S$ is closed.

Multiplications $a\cdot$, differentiations D^α and $J(D)$ are sheaf homomorphisms. Namely they do not enlarge the support.

The dual space $\mathcal{E}^{\{M_p\}'}(\Omega)$ ($\mathcal{E}^{(M_p)'}(\Omega)$) of $\mathcal{E}^{\{M_p\}}(\Omega)$ ($\mathcal{E}^{(M_p)}(\Omega)$) is identified with the subspace composed of all $f \in \mathcal{D}^{\{M_p\}'}(\Omega)$ ($\mathcal{D}^{(M_p)'}(\Omega)$) with compact support.

<u>Theorem 10</u>. <u>Suppose that</u> M_p <u>satisfies</u> (M.1), (M.2)' <u>and</u> (M.3)'. <u>Then, a hyperfunction</u> f <u>on an open set</u> Ω <u>in</u> $\mathbb{R}^n$ <u>belongs to</u> $\mathcal{D}^{\{M_p\}'}(\Omega)$ ($\mathcal{D}^{(M_p)'}(\Omega)$) <u>if and only if its restriction</u> $f\,|\,G$ <u>to any relatively compact open set</u> G <u>in</u> Ω <u>can be written</u>

$$(50) \qquad f\,|\,G = \sum_{|\alpha|=0}^{\infty} D^\alpha f_\alpha ,$$

<u>where</u> $f \in C(\bar{G})'$ <u>or</u> $L^r(G)$, $1 \leq r \leq \infty$, <u>and for any</u> $L > 0$ <u>there is</u> C (<u>there are</u> L <u>and</u> C) <u>such that</u>

$$(51) \qquad \|f_\alpha\| \leq C \frac{L^{|\alpha|}}{M_{|\alpha|}} .$$

(50) <u>converges strongly in</u> $\mathcal{D}^{\{M_p\}'}(G)$ ($\mathcal{D}^{(M_p)'}(G)$).

<u>A subset</u> B <u>of</u> $\mathcal{D}^{\{M_p\}'}(\Omega)$ ($\mathcal{D}^{(M_p)'}(\Omega)$) <u>is bounded if and only if constant(s)</u> C (<u>and</u> L) <u>in</u> (51) <u>can be chosen uniformly in</u> $f \in B$.

Roumieu [10], Chap.I, théorème 1 gives a stronger statement.

By the Phragmen-Lindelöf theorem we can show that the semi-norm (44) in Theorem 8 is equivalent to

$$\sup_{\xi\in\mathbb{R}^n} |J(\xi)\tilde{\varphi}(\xi)| .$$

Hence we obtain another structure theorem:

Theorem 11. Suppose that M_p *satisfies (M.1), (M.2) and (M.3). Then, a hyperfunction* f *on an open set* Ω *in* $\mathbb{R}^n$ *belongs to* $\mathcal{D}^{\{M_p\}'}(\Omega)$ $(\mathcal{D}^{(M_p)'}(\Omega))$ *if and only if for any relatively compact convex open set* G *there is a multiplier* $J(\zeta)$ *for the class* $\{M_p\}$ $((M_p))$ *and a finite measure* f *on* G *such that*

$$f\,|\,G = J(D)f . \tag{52}$$

A subset B *of* $\mathcal{D}^{\{M_p\}'}(\Omega)$ $(\mathcal{D}^{(M_p)'}(\Omega))$ *is bounded if and only if there is* $J(D)$ *independent of* $f \in B$ *and* $\|f\|$ *are bounded.*

4. Characterization of ultra-distributions. In this section we consider only the case where $n = 1$ for the sake of simplicity.

When M_p is a sequence satisfying (M.1), we write

$$M^*(\rho) = \log \sup_p \frac{\rho^p p!\, M_0}{M_p} , \tag{53}$$

$$M_p^* = M_0 \sup_{\rho>0} \frac{\rho^p}{\exp M^*(\rho)} . \tag{54}$$

If m_p/p is increasing, we have $M_p^* = M_p/p!$.

Theorem 12. Suppose that M_p *satisfies* (M.1), (M.2) *and* (M.3). *Then, a hyperfunction* $f = [F]$ *on an interval* (a, b) *belongs to* $\mathcal{D}^{\{M_p\}'}(a, b)$ $(\mathcal{D}^{(M_p)'}(a, b))$ *if and only if for any compact* interval K in (a, b) and for any $L > 0$ there is C (there are L and C) such that the defining function F satisfies

$$\sup_{x \in K} |F(x+iy)| \leq C \exp M^*\left(\frac{L}{|y|}\right) \tag{55}$$

for sufficiently small $|y|$.

A subset B of $\mathcal{D}^{\{M_p\}'}(a, b)$ ($\mathcal{D}^{(M_p)'}(a, b)$) is bounded if and only if the constant(s) C (and L) can be chosen uniformly in $f \in B$.

Sketch of Proof. Suppose that F satisfies (55) for $K = [c, d]$. We will find multipliers $J_+(\zeta)$ and $J_-(\zeta)$ and holomorphic functions G_+ and G_- which are bounded near (c, d) such that

$$F(x+iy) = \begin{cases} J_+(D)G_+(x+iy), & y > 0 \\ J_-(D)G_-(x+iy), & y < 0. \end{cases} \tag{56}$$

Then $f = J_+(D)G_+(x+i0) - J_-(D)G_-(x-i0)$ belongs to $\mathcal{D}^{\{M_p\}'}(c, d)$ ($\mathcal{D}^{(M_p)'}(c, d)$).

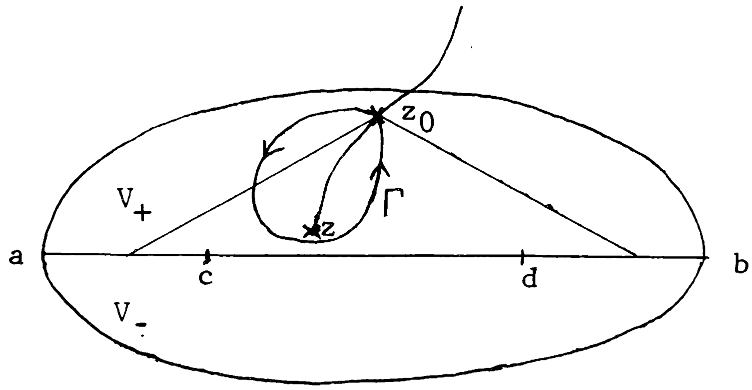
Let $y > 0$ and

$$J_+(\zeta) = (1+\zeta)^2 \prod_{j=1}^{\infty} \left(1+\frac{\ell_j \zeta}{m_j}\right), \tag{57}$$

where ℓ_j is a positive sequence converging to zero (a positive constant) Since $J_+(\zeta)^{-1}$ is infra-exponential except on the negative real axis,

$$G_+(z) = \frac{1}{2\pi} \int_0^{\infty e^{i\alpha}} J_+(\zeta)^{-1} e^{iz\zeta} d\zeta \tag{58}$$

defines a holomorphic function on the Riemann surface $-\pi < \arg z < 2\pi$.



Choose a point z_0 in the upper domain V_+ of F and define for z in the cone $-\pi+\varepsilon < \arg(z-z_0) < -\varepsilon \subset V_+$

(59) $$G_+F(z) = \int_\Gamma G_+(z-w)F(w)dw ,$$

where Γ is a simple closed curve starting z_0 and encircling the slit $[z, z_0]$ counterclockwise. Then we have

(60) $$J_+(D)G_+F(z) = F(z).$$

By deforming the contour Γ we have

(61) $$G_+F(z) = i\int_0^t g_+(-iv)F(z+iv)dv + \cdots ,$$

where

(62) $$g_+(-iy) = \frac{1}{2\pi i}\int_{\xi-i\infty}^{\xi+i\infty} J_+(\xi+i\eta)^{-1} e^{y(\xi+i\eta)}\, d\eta .$$

Taking it into account that

(63) $$|g(-iy)| \leqq \frac{1}{2\pi}\left|\int_{-i\infty}^{i\infty}\frac{d\zeta}{(1+\zeta)^2}\right| \inf_{\xi>0}\frac{e^{y\xi}}{\prod\left|1+\frac{\ell_j\xi}{m_j}\right|}$$
$$\leqq C\inf_{\xi>0}\frac{e^{y\xi}}{\exp\tilde{M}(\xi)} ,$$

where

(64) $$\tilde{M}(\xi) = \log\sup_p\frac{\ell_1\cdots\ell_p\xi^p M_0}{M_p} ,$$

we can choose a sequence ℓ_j so that the first term of (61) is bounded. The remainder is also bounded. Hence we have (56).

The proof shows that if the estimate (55) is uniform in $f \in B$, then $J_{\pm}(D)^{-1}$ constructed above map $F(x \pm i0)$ into a bounded set in $L^{\infty}(c, d)$, and hence B is bounded.

Conversely suppose that $f \in \mathcal{D}^{\{M_p\}'}(a, b)$ $(\mathcal{D}^{(M_p)'}(a, b))$. It follows from Theorem 10 that

$$f\,|\,(c, d) = \sum_{p=0}^{\infty} D^p f_p , \quad f_p \in C([c, d])' \text{ and}$$

$$\|f_p\|_{C([c,d])'} \leqq C \frac{L^p}{M_p} .$$

Let F_p be the standard defining function of f_p. Then we have the estimate

$$\sup_{x \in [c,d]} |D^p F_p(x+iy)| \leqq \frac{C}{2\pi} \frac{L^p p!}{|y|^{p+1} M_p}$$

$$\leqq \frac{1}{2^p} \frac{CA}{2\pi M_0} \sup_p \frac{(2HL)^{p+1}(p+1)!\, M_0}{|y|^{p+1} M_{p+1}} .$$

Therefore

$$F(x+iy) = \sum_{p=0}^{\infty} D^p F_p(x+iy)$$

is a defining function of f and it satisfies

$$\sup_{x \in [c,d]} |F(x+iy)| \leqq \frac{CA}{\pi M_0} \exp M^*\left(\frac{2HL}{|y|}\right) .$$

It is clear that if a defining function satisfies (55), any other defining function satisfies it also.

For the Gevrey sequence of order s, $M^*(\rho)$ is equivalent to $\rho^{\frac{1}{s-1}}$. Therefore the theorem in the introduction is a special case of Theorem 12.

References

[1] G. Björck, Linear partial differential operators and generalized distributions, Ark. f. Mat. 6 (1966), 351-407.

[2] R. P. Boas, Jr., Entire Functions, Acad. Press, New York, 1954.

[3] L. Ehrenpreis, Theory of infinite derivatives, Amer. J. Math. 81 (1959), 799-845.

[4] L. Ehrenpreis, Fourier Analysis in Several Complex Variables, Wiley-Interscience, New York·London·Sydney·Toront, 1970.

[5] H. Komatsu, Projective and injective limits of weakly compact sequences of convex spaces, J. Math. Soc. Japan 19 (1967), 366-383.

[6] H. Komatsu, Hyperfunctions and Partial Differential Equations with Constant Coefficients, Seminar Notes No.22, Dept. Math. Univ. Tokyo, 1968 (in Japanese).

[7] J.-L. Lions - E. Magenes, Problèmes aux Limites Non Homogènes et Applications, Vol.3, Dunod, Paris, 1970.

[8] S. Mandelbrojt, Série Adhérentes. Régularisation des Suites. Applications, Gauthier-Villars, Paris, 1952.

[9] S. Mandelbrojt, Fonctions Entieres et Transformées de Fourier, Applications, Math. Soc. Japan, Tokyo, 1967.

[10] C. Roumieu, Sur quelques extensions de la notion de distribution, Ann. Sci. Ecole Norm. Sup. 3 ser. 77 (1960), 41-121.

[11] C. Roumieu, Ultra-distributions definies sur $\mathbb{R}^n$ et sur certains classes de variétés différentiables, J. Analyse Math. 10 (1962/3), 153-192.