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Some applications of hyperfunctions to the abstract Cauchy problem and stationary random processes

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Recently the theory of hyperfunctions makes progress, and it becomes a powerful tool to investigate the theory of partial differential equations. But, except it, the theory of hyperfunctions is scarcely used.

In this note we will show a few applications of hyperfunctions. In §2 we report a few results on hyperfunction solutions of the abstract Cauchy problem

$$\begin{cases} \dfrac{du(t)}{dt} = Au(t) \\ u(0) = a \end{cases},$$

where A is a closed linear operator in a Banach space X and $a \in X$. We discuss conditions for existence, uniqueness and regularity. In §3 we define stationary random hyperfunctions which are more general than stationary random distributions studied by I.M. Gelfand and N.Y. Vilenkin [7] and K. Itô [8], and show their elementary properties. Appendix is concerned with positive definite hyperfunctions.

The author hopes that the methods and the notions in the theory of hyperfunctions are useful for various fields of analysis.

§1. Preliminaries. We denote by $\mathcal{B}$ the sheaves over R^n of germs of hyperfunctions and by $\mathcal{O}$ the sheaves over C^n of holomorphic functions. Let U be a set in R^n (or C^n). Then $\mathcal{B}(U)$ ($\mathcal{O}(U)$) denotes the set $\Gamma(U,\mathcal{B})$ ($\Gamma(U,\mathcal{O})$, respectively) of sections over U.

In the following we shall use vector valued hyperfunctions of one variable. Let E be a Banach space. Consider the space $\mathcal{O}(\Omega, E)$ of all E-valued holomorphic functions defined on Ω, where Ω is an open set in C^1. Let S be an open interval in R^1. We define an

E-valued hyperfunction to be an element of quotient space:

$$\mathcal{B}(S, E) = \frac{\mathcal{O}(D - S, E)}{\mathcal{O}(D, E)},$$

where D is a complex neibourhood of S , which contains S as a closed set. For E-valued hyperfunctions we can establish results similar to the case of scalar hyperfunctions. We refer the reader to H. Komatsu [9], M. Sato [12] and P. Schapira [13] for the theory of hyperfunctions.

§ 2. Hyperfunction solutions of the abstract Cauchy problem.

Let E and F be Banach spaces whose norms are $\|\cdot\|_E$ and $\|\cdot\|_F$ respectively. L(E,F) is a Banach space consisting of all bounded linear operators from E to F equipped with the operator norm denoted by $\|\cdot\|_{E\to F}$. The set L(E,E) is written L(E) for short.

Let X be a Banach space, and A a closed linear operator in X. The domain of A with the graph norm is a Banach space and is denoted by [D(A)]. $\rho(A)$ means the resolvent set of A. In the following I is the idetity mapping, in particular we shall use notaions I_X and $I_{[D(A)]}$ which are the identity on X and on [D(A)] respectively.

Definition 2.1. A closed operator A is said to be well-posed for the Cauchy problem at t = 0 in the sense of hyperfunction (well-posed, for short), if there exists $T \in \mathcal{B}(R^1, L(X, [D(A)]))$ satisfying the following conditions:

(2.1) support of $T \subset [0,\infty)$;

(2.2) $(\delta^{(1)}(t) \otimes I - \delta(t) \otimes A) * T = \delta(t) \otimes I_X$,

$T * (\delta^{(1)}(t) \otimes I - \delta(t) \otimes A) = \delta(t) \otimes I_{[D(A)]}$

where $\delta(t-\tau)$ is Dirac measure at $t = \tau$, $*$ means convolution, $\delta^{(k)}(t)$ k-th derivative of $\delta(t)$ and $\otimes$ tensor product.

We shall call T in the definition a fundamental solution.

Remark 2.1. If A is well-posed, then the fundamental solution T is unique in $\mathcal{B}(R^1; L(X, [D(A)]))$. This result easily follows from the facts that T is a two-sided fundamental solution and its support is contained in $[0,\infty)$.

Theorem 2.1. A closed linear operator A is well-posed if and only if the resolvent of A satisfies the condition:

(2.3) For any $\varepsilon > 0$ there exists K_ε such that $\Sigma_\varepsilon = \{\lambda ; \mathrm{Re}\lambda \geq \varepsilon|\mathrm{Im}\lambda| + K_\varepsilon\}$ is contained in $\rho(A)$ and in this set $\|(\lambda - A)^{-1}\|_{X\to X} \leq C_\varepsilon \exp(\varepsilon|\lambda|)$ holds.

Theorem 2.2. A closed operator A is well-posed and its fundamental solution is holomorphic in the sector $\Sigma = \{z ; |\arg z| < \alpha, 0 < \alpha < \frac{\pi}{2}\}$, if and only if A satisfies the following condition:

(2.4) For any $\varepsilon > 0$, there exists a real ω_ε, and for any λ in the sector $\Sigma_\varepsilon = \{\lambda ; |\arg(\lambda - \omega_\varepsilon)| < \theta, \theta = \frac{\pi}{2} + \alpha - \varepsilon\}$, we have $(\lambda - A)^{-1} \in L(X)$ with the estimate $\|(\lambda - A)^{-1}\|_{X\to X} \leq C_\varepsilon \exp(\varepsilon|\lambda|)$.

Theorem 2.3. A closed operator A is well-posed and its fundamental solution is real analytic on the positive real axis, if and only if A satisfies the condition:

(2.5) For any $\varepsilon > 0$ there exists K_ε and $0 < \delta_\varepsilon \leq \varepsilon$, and for any λ in the set $\Sigma_\varepsilon = \{\lambda ; \varepsilon \mathrm{Re}\lambda \geq -\delta_\varepsilon|\mathrm{Im}\lambda| + K_\varepsilon\}$, $(\lambda - A)^{-1} \in L(X)$ exists and the estimate $\|(\lambda - A)^{-1}\|_{X\to X} \leq C_\varepsilon \exp(\varepsilon|\mathrm{Re}\lambda| + \delta_\varepsilon|\mathrm{Im}\lambda|)$ holds.

Remark 2.2. Distribution solutions were investigated by J. Chazarain [3], G. Da Prato and U. Mosco [5], D. Fujiwara [6], J.L. Lions [10] and T. Ushijima [14]. J. Chazarain [3] characterized well-posedness of the abstract Cauchy problem in the sense of distribution. Compairing Theorem 2.1. with his result, we conclude that operators which are well-posed in the sense of hyperfunction contain those which are well-posed in the sense of distribution.

The criterion which correspond to Theorem 2.2 in the case of distribution solutions was given by G. Da Prato and U. Mosco [5] and Fujiwara [6].

The same results in this section were reported in S. Ōuchi [11] and in it outline of proof of Theorem 2.1 is shown.

§3. Stationary random hyperfunctions. Let Ω be a measurable space of elements ω with a σ- algebra $\mathfrak{M}$, on which there is defined a probability measure $P(d\omega)$. In this section we shall restrict ourselves to complex random variables with mean 0 and finite variance. The set of all such random variables constitutes of a complex Hilbelt space $\mathcal{H}$ with the following inner product:

(3.1) $\quad (X , Y) = E(X , \overline{Y})$; E: expectation.

Definition 3.1. A random variable $X_z(\omega)$ is said to be a stationary random hyperfunction, if it satisfies the conditions:

(3.2) $\quad X_z(\omega) \in \mathcal{O} (C^1 - R^1 , \mathcal{H})$,

(3.3) $\quad X_z(\omega)$ is translation invariant in the following sense, $E(X_{z_1+h} , \overline{X_{z_2+h}}) = E(X_{z_1} , \overline{X_{z_2}})$ for any $h \in R^1$.

From Definition 3.1, we may say that $X_z(\omega)$ defines hyperfunctions for almost all $\omega \in \Omega$, by means of the result due to Arnold [1] or Belyaev [2]. Namely we have

Proposition 3.1. For a random variable $X_z(\omega)$ which satisfies (3.2) , there exists a random variable $\hat{X}_z(\omega)$ defined on $z \in (C^1-R^1)$ with the properties:

(3.3) $\quad \hat{X}_z(\omega)$ is holomorphic for almost all $\omega \in \Omega$, that is, $\hat{X}_z(\omega) \in \mathcal{O} (C^1 - R^1)$ for almost all $\omega \in \Omega$.

(3.3) $\quad P(X_z(\omega) = \hat{X}_z(\omega)) = 1$ for every $z \in (C^1 - R^1)$.

Now let $X_z(\omega)$ be a stationary random hyperfunction.

Set $\psi(z) = E(X_{\frac{z}{2}}, \overline{X_{-\frac{z}{2}}})$, which is an element of $\mathcal{O}(C^1-R^1)$.

Proposition 3.2. The holomorphic function $\psi(z)$ is represented in the following form:

$$\psi(z) = \int_{-\infty}^{\infty} e^{i\lambda z}\, d\mu_1(\lambda) \quad (\operatorname{Im} z = y > 0)$$

$$\psi(z) = \int_{-\infty}^{\infty} e^{i\lambda z}\, d\mu_2(\lambda) \quad (\operatorname{Im} z = y < 0)),$$

where the measures satisfy

$$\int_{-\infty}^{\infty} e^{-\varepsilon|\lambda|}\, d\mu_i(\lambda) < \infty \quad (i = 1,2) \text{ for any } \varepsilon > 0.$$

Define a function $\varphi(z) \in \mathcal{O}(C^1 - R^1)$ as follows:

$$\varphi(z) = \int_{0}^{\infty} e^{i\lambda z}\, d\mu(\lambda) \quad (y > 0)$$

$$\varphi(z) = -\int_{-\infty}^{0} e^{i\lambda z}\, d\mu(\lambda) \quad (y < 0),$$

where $\mu = \mu_1 + \mu_2$.

The hyperfunction Φ defined by φ is a Fourier hyperfunction and the measure is unique as a Fourier hyperfunction. We call the measure μ the spectral measure of $X_z(\omega)$.

Put $f(z_1,z_2) = -E(X_{z_1}, \overline{X_{\bar{z}_2}})$, which is in $\mathcal{O}((C^1-R^1)\times(C^1-R^1))$.

Definition 3.2. The hyperfunction F on R^2 defined by $f(z_1,z_2)$ is said to be a covariance hyperfunction of $X_z(\omega)$.

Theorem 3.3. The covariance hyperfunction F of $X_z(\omega)$ is coincident with the hyperfunction G defined by $g(z_1,z_2) \in \mathcal{O}((C^1-R^1)\times(C^1-R^1))$ which is :

$$\begin{aligned}
g &= 0 \quad (y_1>0,\ y_2>0)\\
&= -\int_0^{\infty} e^{i(z_1-z_2)}\, d\mu(\lambda) \quad (y_1>0,\ y_2<0)\\
&= 0 \quad (y_1<0,\ y_2<0)\\
&= -\int_{-\infty}^{0} e^{i(z_1-z_2)}\, d\mu(\lambda) \quad (y_1<0,\ y_2>0).
\end{aligned}$$

Thorem 3.4. For a stationary random hyperfunctio $X_z(\omega)$ with the spectral measure μ, there is a stationary hyperfunction $\hat{X}_z(\omega)$ which is expressible in the following form:

$$\begin{aligned}
\hat{X}_z(\omega) &= \int_0^{\infty} e^{i\lambda z}\, dM(\lambda) \quad (y>0)\\
&= -\int_{-\infty}^{0} e^{i\lambda z}\, dM(\lambda) \quad (y<0),
\end{aligned}$$

where $M(d\lambda)$ is a random measure with respect to the measure μ, such that $X_z(\omega)$ and $\hat{X}_z(\omega)$ define a same $\mathcal{H}$-valued hyperfunction.

Remark 3.1. Theorem 3.3 corresponds to Khinchin's theorem and Theorem 3.4 to Kolmogoroff's representation formula in the theory of classical stationary processes. See H. Cramér and M.R. Leadbetter [4] for the classical case.

Appendix

Let F be a hyperfunction on R^n. Then there is a function $f(z) \in \mathcal{O}(\underbrace{\prod_{1}^{n}(C - R)}_{(z \in C^n)})$ such that F is represented by the boundary value of $f(z)$. So we can write

(4.1) $$F = \sum_{\varepsilon} \|\varepsilon\| f(x + i\varepsilon y),$$

where $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$, $\varepsilon_i = 1$ or -1 and $\|\varepsilon\| = \varepsilon_1 \times \varepsilon_2 \times \cdots \times \varepsilon_n$.

For such representations of hyperfunctions we refer to Schapira[13].

Definition. A hyperfunction F is said to be positive definite if there exists $f(z) \in \mathcal{O}(\prod_{1}^{n}(C-R))$ such that

(4.2) $$\sum_{i,j=1}^{k} \alpha_i \bar{\alpha}_j \|\varepsilon\| f(x_i - x_j + \sqrt{-1}\, y) \geq 0 \text{ holds}$$

for any $\alpha_i \in C$, in the set $\Sigma_\varepsilon = \{ z ; \varepsilon_i y_i > 0, i = 1,2 \dots n \}$ and any $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots \varepsilon_n)$

Theorem. Every positive definite hyperfunction F is represented by the Fourier transform of a measure, that is,

(4.3) $$F = \int_{R^n} e^{i\langle \lambda \cdot x \rangle} d\mu(\lambda),$$

where the measure μ is unique as a Fourier hyperfunction and it is tempered in the sense of hyperfunction, i.e.

(4.4) $$\int_{R^n} e^{-\varepsilon|\lambda|} d\mu(\lambda) < \infty \quad \text{for any } \varepsilon > 0.$$

Remark. For a positive definite distribution, the measure correspond to it is tempered in the sense of distribution, i.e.

(4.5) $$\int_{R^n} \frac{1}{(1+|\lambda|)^m} d\mu(\lambda) < \infty \quad \text{for some } m.$$ (See I.M. Gelfand and N.Y. Vilenkin [7])

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