数理解析研究所講究録
第 162 巻 1972 年 109-135

On the global existence of real analytic solutions of linear differential equations

Takahiro KAWAI

Research Institute for Mathematical Sciences
Kyoto University

§0. Introduction

Professor Sato initiated and developed the theory of sheaf $\mathcal{C}$ in 1969 (Sato [2] , [3]), and this theory has turned out to be a very powerful tool in analysis, especially in the study of linear (pseudo-)differential equations. (Cf. Kashiwara and Kawai [1] , [2] , Kawai [1] ~ [5] , Sato [2] ~ [6] . See also Hörmander [2] , [3]). The present speaker gave a survey lecture on these subjects at the symposium last March on the theory of hyperfunctions and differential equations (Kawai [3]), and listed there four problems to be solved. They were:

(i) the treatment of the case $k=\infty$, where k is the number appearing in Egorov [1] and Nirenberg and Treves [1] concerning the local solvability of linear (pseudo-)differential equations,

(ii) to extend our theory to the case where the assumption of simple characteristics is omitted,

(iii) to extend our theory to overdetermined systems,

and

(iv) to give global existence theorems.

Especially he placed emphasis on problems (iii) and (iv) at that occasion.

A complete result is given by Sato [6], concerning problem (iii) and a result is given by the present speaker concerning problem (iv) (Kawai [4], [5]).

Now in this lecture we will explain how problem (iv) is deduced from the local theory of linear differential equations.

More complete arguements should be given in our forthcoming papers (Kawai [6]) and this lecture should be regarded as a survey one.

§1. Global existence of real analytic solutions of single linear differential equation with constant coefficients.

As is well known the topological structure of the space of real analytic functions on an open set is rather complicated, hence even Professor Ehrenpreis, who initiated and completed the general theory of linear differential equations <u>with constant coefficients</u> in the framework of distributions with Professors Malgrange, Hörmander and Palamodov, seems at present to have abandoned to attack the problem of global existence of real analytic solutions. (Cf. Ehrenpreis [2], [3]). But we can treat this problem without much difficulty by the aid of the theory of hyperfunctions and that of sheaf $\mathcal{C}$, at least when we restrict ourselves to the consideration of the operators satisfying suitable regularity conditions which allow us to consider the problems geometrically. In a sense our method can be regarded as "method of algebraic analysis" contrary to "method of functional analysis", which is developed, for example, in Hörmander [1] , Palamodov [1] , Ehrenpreis [3] , etc. (The word "algebraic analysis" seems to go back to Euler but it has recently been endowed with positive meanings by Professor Sato, who aims at the Renaissance of classical analysis).

We first examine in the special case whether the theory of hyperfunctions is useful to investigate the problem of global existence of real analytic solutions. In fact we easily understand that it is very powerful in the following special case, i.e., the case when the operator P(D) is elliptic.

Of course in this case there is a decisive result due to Malgrange [1] , i.e.,

<u>Theorem (Malgrange [1]).</u> For any open set Ω in $\mathbb{R}^n$, $P(D)u=f$ has a solution $u(x)$ in $\mathcal{A}(\Omega)$ for any datum $f(x)$ in $\mathcal{A}(\Omega)$. Here $\mathcal{A}(\Omega)$ denotes the space of real analytic functions defined on Ω .

Now we show how we can prove this deep theorem with ease if we assume that Ω is relatively compact. The essence of the proof is, as described below, the flabbiness of sheaf of hyperfunctions, which we denote by $\mathcal{B}$ in the sequel.

Our proof is divided into two parts. First we remember the following lemma due to John [1] .

<u>Lemma 1.</u> If the linear differential operator $P(D)$ is elliptic, then we can find a hyperfunction $E(x)$ defined on $\mathbb{R}^n$ satisfying

(i) $P(D)E(x) = \delta(x)$

and

(ii) $E(x)$ is real analytic outside the origin.

This lemma can be proved by many methods: for example, one can use the fact that the non-characteristic Cauchy problem in the complex domain has the entire solution as far as all the data given are entire functions, the linear differential operator under consideration is of constant coefficients and the initial hypersurface is a hyperplane. (Cf. Leray [2] Lemma 9.1). Then one can use the celebrated reasonings of John [1] Chapter 3 to construct $E(x)$. (Cf. John [1] pp.66 —— 72). Another proof is given by the following way: First construct the elementary solution $E_o(x)$ of $P_m(D)$, the principal part of $P(D)$, in the form

$$\frac{1}{(-2\pi i)^n} \int_{|\xi|=1} \frac{1}{(P_m(\xi)+i0)} \Phi_{n-m}(\langle x, \xi\rangle + i0)\, \omega(\xi),$$

where

$$\Phi_j(\tau) = \begin{cases} (-1)^{j-1}(j-1)!\, \tau^{j} & (j \quad 0) \\ \dfrac{\tau^j}{j!} \log \tau - \dfrac{1}{j!}\left(1 + \text{---} + \dfrac{1}{j}\right) \tau^j & (j \leq 0) \end{cases}$$

and $\omega(\xi)$ denotes the volume element of the unit sphere, i.e., $\omega(\xi)=\sum_{j=1}^{n}(-1)^{j-1}\xi_j d\xi_1\wedge\cdots\wedge d\xi_{j-1}\wedge d\xi_{j+1}\wedge\cdots\wedge d\xi_n$. Next construct the required $E(x)$ by the successive approximation starting from $E_0(x)$, or more precisely from

$$\frac{1}{(-2\pi i)^n P_m(\zeta)}\,\Phi_{n-m}(\langle z,\zeta\rangle)\quad,$$

where z and ζ denote the complexifications of x and ξ respectively. Note that $P_m(\zeta)$ never vanishes as far as ζ is sufficiently near to the real unit ball $\{\xi\in\mathbb{R}^n \mid |\xi|=1\}$ by the assumption of ellipticity. The convergence of the successive approximation is easy to check, and it is also easy to verify that $E(x)$ has all the required properties.

Secondly we use the flabbiness of sheaf $\mathcal{B}$ to obtain a hyperfunction $\tilde{f}(x)$, which is defined on $\mathbb{R}^n$ and satisfies the following conditions:

(i) Its support is contained in $\overline{\Omega}$, the closure of Ω.

(ii) It coincides with $f(x)$ in Ω.

Then using $\tilde{f}(x)$ we define $u(x)$ by the integration $\int E(x-y)\tilde{f}(y)dy$. This integration is well defined as an integration along fiber (Sato [1]), since the support of $\tilde{f}(y)$ is compact by the definition. On the other hand by property (i) of $E(x)$ we have $P(D)u(x)=\tilde{f}(x)$ and by property (ii) of $E(x)$ and the property of $\tilde{f}(x)$ we see that $u(x)$ is real analytic in Ω. Thus if we consider the restriction of $u(x)$ to Ω , which we denote by $u(x)$ again, $u(x)$ is a real analytic solution of the equation $P(D)u=f$.

This proof of the existence theorem in the elliptic case teaches us the following facts:

(i) Flabbiness of sheaf $\mathcal{B}$ allows us to pass the technical difficulties by, especially it reduces all the problems to the boundary.

and

(ii) The informations which the "good" elementary solutions have (property (ii) in the above case) are used in the course of integrations and give us a good solution of $P(D)u=f$.

These observations oblige us to want to consider more general differential operators, not necessarily elliptic: in fact we have "good" elementary solutions for the differential operator $P(x,D_x)$ satisfying the following conditions (1) and (2), which exist globally if the operator $P(x,D_x)$ is of constant coefficients. (Kawai [1]). We also remark that we can treat more general class of operators first considered in Andersson [1] (see also Kawai [3] , [5]), since in this section we restrict ourselves to the case where the differential operators are with constant coefficients, which is a easy case from the view-point of construction of elementary solutions.

(1) The principal symbol $P_m(x,\xi)$ of $P(x,D_x)$ is real.

(2) $P_m(x,\xi)$ is of simple characteristics, i.e.,

$\text{grad}_\xi P_m(x,\xi)$ does not vanish whenever $P_m(x,\xi)=0$ for any point (x,ξ) in the real cotangential sphere bundle.

Now, what is the good property presented by the elementary solutions constructed in Kawai [1]? It is described in the following lemma.

Lemma 2. Let $P(D)$ be a linear differential operator with constant coefficients satisfying conditions (1) and (2). Then there exist two hyperfunctions $E_+(x)$ and $E_-(x)$ such that

(i) $P(D)E_\pm(x)=\delta(x)$ holds

and

(ii) $S.S.E_\pm(x)$ is contained in $\{(x,\xi)\in S^*\mathbb{R}^n \mid x=0$ or $x=\pm t\,\text{grad}_\xi P_m(\xi)$ with $t\geqq 0$ and $P_m(\xi)=0\}$ respectively, where $S^*\mathbb{R}^n$ denotes the cotangential sphere bundle of $\mathbb{R}^n$ and $S.S.E_\pm(x)$ denotes the support of $E_\pm(x)$ regarded as sections of sheaf $\mathcal{C}$. (Cf. Sato [4]).

The proof of this lemma was rather implicit in Kawai [1], especially concerning the global existence of $E_\pm(x)$, however it is easy to prove this lemma using the successive approximation method as is sketched in the proof of Lemma 1, since the operator $P(D)$ has constant coefficients.

We believe that such an elementary solution as is given by Lemma 2 is very good and that all the informations about the operator P(D) should be deduced from it, and the belief in the good elementary solution has its reward as is described in this report.

We first consider the solvability in $\mathcal{A}(K)$ for compact set K in $\mathbb{R}^n$. Here $\mathcal{A}(K)$ denotes the space of real analytic functions on K, i.e., $\varinjlim_{V \supset K} \mathcal{O}(V)$, where V denotes a complex neighbourhood of K and $\mathcal{O}(V)$ denotes the space of holomorphic functions on V. This problem has its own interests as well as it plays a role as a lemma to our final object of solving the equation P(D)u=f in $\mathcal{A}(\Omega)$ for an open set Ω.

__Theorem 3.__ Assume that K is the closure of relatively compact open set $\Omega = \{x \mid \varphi(x) < 0\}$, where $\varphi(x)$ is a real valued real analytic function defined near K satisfying $\mathrm{grad}_x \varphi \neq 0$ on $\partial\Omega$, the boundary of Ω. Suppose that the compact set K satisfies the following geometrical condition (3) and that the differential operator P(D) satisfies conditions (1) and (2). Then for any f(x) in $\mathcal{A}(K)$ we can find u(x) in $\mathcal{A}(\Omega)$ such that P(D)u=f holds in Ω.

(3) For any x_0 in $\partial\Omega$ the bicharacteristic curve of P(D) $b(x_0, \mathrm{grad}_x \varphi|_{x=x_0})$ issuing from $(x_0, \mathrm{grad}_x \varphi|_{x=x_0})$ never intersects Ω.

The proof of this theorem is given just in the same way as in the second part of our proof of the existence theorem in elliptic case by the use of either one of the good elementary solutions given in Lemma 2. In fact the smoothness of the boundary and the regularity of $f(x)$ permit us to extend $f(x)$ to $\mathbb{R}^n$ by $f(x)\theta(-\varphi(x))$, where θ denotes the 1-dimensional Heaviside function. Note that $S.S.(f(x)\theta(-\varphi(x)))$ is contained in $\{(x,\xi)\in S^*\mathbb{R}^n \mid x\in\partial\Omega, \xi=\pm\,\mathrm{grad}_x\varphi(x)\}$. Then we can apply Sato's lemma on the regularity of the integration along fiber (Sato [4] Corollary 6.5.3) to the integration

$\int E(x-y)f(y)\theta(-\varphi(y))dy$ and obtain the required result.

This proof of Theorem 3 needs only one of good elementary solutions given in Lemma 2, but this contradicts our sense of symmetry: We must use both good elementary solutions, because neither one is better than the other. This belief in both good elementary solutions is rewarded again, i.e., we can improve Theorem 3 as follows.

<u>Theorem 4.</u> In Theorem 3 the condition (3) on Ω can be weakened to the following.

(4) For any x_0 in $\partial\Omega$ the bicharacterestic curve of $P(D)$ $b(x_0, \mathrm{grad}_x\varphi|_{x=x_0})$ issuing from $(x_0, \mathrm{grad}_x\varphi|_{x=x_0})$ intersects Ω in an open interval.

<u>Proof of Theorem 4.</u> We denote $f(x)\theta(-\varphi(x))$ by $\tilde{f}(x)$. By condition (4) we have the decomposition of $N=\{(x,\xi)\in S^*\mathbb{R}^n \mid x\in\partial\Omega,\ P_m(\xi)=0,\ \xi\neq\pm\operatorname{grad}\varphi(x)\}$ into the form $N_+\cup N_-$, where $N_+=\{(x,\xi)\in S^*\mathbb{R}^n \mid x\in\partial\Omega,\ P_m(\xi)=0,\ \xi\neq\pm\operatorname{grad}\varphi(x)$ and half of the bicharacteristic curve $t\operatorname{grad}_\xi P_m(\xi)(t\geq 0)$ does not intersect $\Omega\}$ and $N_-=\{(x,\xi)\in S^*\mathbb{R}^n \mid x\in\partial\Omega,\ P_m(\xi)=0,\ \xi\neq\pm\operatorname{grad}\varphi(x)$ and half of the bicharacteristic curve $t\operatorname{grad}_\xi P_m(\xi)(t\leq 0)$ does not intersect $\Omega\}$. Since sheaf $\mathcal{C}$ is flabby (Kashiwara [1]), we can find hyperfunctions $\tilde{f}_+(x)$ and $\tilde{f}_-(x)$ such that $S.S.(\tilde{f}(x)-\tilde{f}_+(x)-\tilde{f}_-(x))=\emptyset$, $S.S.\tilde{f}_+(x)\cap N\subset N_+$ and $S.S.\tilde{f}_-(x)\cap N\subset N_-$. Then applying Sato's lemma on the regularity of the integral along fiber to

$$v(x)=\int E_+(x-y)\tilde{f}_+(y)dy+\int E_-(x-y)\tilde{f}_-(y)dy,$$

we find $S.S.v(x)\cap S^*\Omega=\emptyset$. Note that the above integration is well defined as that of the section of sheaf $\mathcal{C}$. Therefore we have $P(D)v(x)=\tilde{f}(x)+g(x)$, where $g(x)$ is real analytic in $\mathbb{R}^n$. Here we have used the fact that $H^1(\mathbb{R}^n,\mathcal{a})$ vanishes. Then restricting $g(x)$ to a closed ball B containing K in its interior, we can apply Theorem 3 to find $w(x)$ which is real analytic in the interior of B and satisfies $P(D)w(x)=g(x)$ there. Thus subtracting $w(x)$ from $v(x)$, we find the required $u(x)$, which is real analytic in Ω and satisfies $P(D)u(x)=f(x)$ there. This completes the proof of Theorem 4.

In an obvious way we can modify the form of Theorem 4 to obtain the results which assures the existence of the solution u(x) in $\mathcal{A}(K)$. We refer the reader to Kawai [4] Theorem 1' about the modifications.

Remark. Since the space $\mathcal{A}(K)$ has a natural structre as a topological vector space, i.e., $\mathcal{A}(K)$ is a DFS-space, Serre's duality theorem holds for the pair ($\mathcal{A}(K)$, $\mathcal{B}_K$), where $\mathcal{B}_K$ denotes the space of hyperfunctions with support in K. Then Serre's duality theorem shows that the existence of solutions in $\mathcal{A}(K)$ can be deduced by the unique continuation theorem concerning hyperfunction solutions. On the other hand the unique continuation theorem follows easily from Theorem 3.3' in Kawai [1] in a precise form using the notion of bicharacteristics. Thus we have the following theorem.

Theorem 5. Let K be a compact set in $\mathbb{R}^n$ and the operator P(D) satisfy conditions (1) and (2). Suppose that condition (5) below holds. Then $P(D)\mathcal{A}(K)=\mathcal{A}(K)$ holds.

(5) For any (x,ξ) in $S^*\mathbb{R}^n$ such that x belongs to ChK, the convex hull of K, but not to K, and such that ξ satisfies $P_m(\xi)=0$, there is a point y outside ChK for which the segment $\overline{xy}$ does not intersect K and is contained in the bicharacteristic curve of P(D) issuing from (x,ξ).

We omit the proof of this theorem in this lecture since it essentially uses "functional analysis". We only remark the following two facts which are related to Theorem 5.

(A) Analogue of Theorem 4 can be proved even if K is a closure of an open set Ω, whose regularity at the boundary is not necessarily assumed. In fact it is sufficient in this case to assume the following condition (6) instead of condition (4):

(6) Any bivharacteristic curve of P(D) intersects Ω in an open interval.

The validity of this statement is obvious from the method of the proof of Theorem 4, if we remark the fact that sheaf $\mathcal{B}$ is flabby. In this case, however, we need not assume f(x) belongs to $\mathcal{A}(K)$, since we extend f(x) to $\mathbb{R}^n$ using the flabbiness of sheaf $\mathcal{B}$. Hence this analogue of Theorem 4 should be regarded as an existence theorem for $\mathcal{A}(\Omega)$ rather than $\mathcal{A}(K)$. (Cf. Theorem 9 in the below).

(B) If we allow the principal symbol of P(D) to be complex valued, then we have the following Theorem 6. Before stating Theorem 6 we prepare a notion regarding bicharacteristics of P(D). In order to define the notion we assume in the sequel that the principal symbol $P_m(\xi)$ has the form $A_m(\xi)+iB_m(\xi)$, where A_m and B_m are real valued, and that

(7) $\mathrm{grad}_\xi A_m$ and $\mathrm{grad}_\xi B_m$ are linearly independent whenever $P_m(\xi)=0$, $\xi \neq 0$.

Using these assumptions on $P_m(\xi)$ we can define the bicharacteristic plane $\Lambda_{(x_0,\xi^0)}$ of P(D) though (x_0, ξ^0) by the 2-dimensional linear variety passing through x_0 which is spanned by $\mathrm{grad}_\xi A_m|_{\xi=\xi^0}$ and $\mathrm{grad}_\xi B_m|_{\xi=\xi^0}$, where $P_m(\xi^0)=0$ holds.

Preparing this notion, we have the following theorem.

Theorem 6. Let the operator P(D) satisfy condition (7) and let the compact set K in $\mathbb{R}^n$ satisfy the following condition (8). Then $P(D)\mathcal{A}(K)=\mathcal{A}(K)$ holds.

(8) For any bicharacteristic plane Λ of P(D), $\Lambda \cap (\mathrm{Ch}K-K)$ has no relatitively compact component.

We have not yet proved this theorem without using the duality theorem. A little weaker theorem is obtained by a direct method similar to the proof of Theorem 3 using the elementary solution in Kawai [4] Theorem 2.

Now we go on to the problem of global existence of solutions in $\mathcal{A}(\Omega)$ for an open set Ω. A complete result is obtained if Ω is in $\mathbb{R}^2$, hence we first state the theorem.

Theorem 7. For any linear differential operator with constant coefficients $P(D)$ we have $P(D)\mathcal{A}(\Omega)=\mathcal{A}(\Omega)$ if a relatively compact open set Ω in $\mathbb{R}^2$ satisfies the following condition:

(9) Any characteristic line of $P(D)$ intersects Ω in an open interval.

The proof of this theorem relies on the fact that explicit construction of elementary solutions of $P(D)$ is possible for any $P(D)$ in the 2-dimensional case.

We can also prove that the converse of the theorem is true at least if $P(D)$ is homogeneous. In fact we have the following theorem.

Theorem 8. Let $P(D)$ be a homogeneous linear differential operator with constant coefficients defined on $\mathbb{R}^n$. Assume that $P(D)\mathcal{A}(\Omega)=\mathcal{A}(\Omega)$ holds for a domain $\Omega=\{x \mid \varphi(x)<0\}$, where $\varphi(x)$ is a real valued real analytic function defined near $\overline{\Omega}$ satisfying $\mathrm{grad}_x\varphi(x)\neq 0$ on $\partial\Omega$. Then for any characteristic boundary point x_0, i.e., the boundary point where $P_m(\mathrm{grad}_x\varphi(x)|_{x=x_0})=0$ holds, the characteristic hyperplane through x_0, i.e.,

$\{x \mid \langle x-x_0, \operatorname{grad}_x \varphi(x) \mid_{x=x_0} \rangle = 0\}$, does not interset $(R^n - \Omega) \cap N$ in a compact set for any compact neighbourhood N of x_0.

The existence of a special null-solution of P(D) proves this theorem and we omit the details. We hope that the assumption on homogeneity of P(D) will be redundant and that the characteristics should be replaced by the bicharacterics, though we have not yet proved them because of some technical difficulties.

On the contrary, we have the following Theorem 9 as an affirmative answer to the global existence of real analytic solutions.

Theorem 9. Let the operator P(D) satisfy condition (1) and (2) and let a relatively compact open set with smooth boundary satisfy the following condition (10). Then $P(D)\mathcal{A}(\Omega) = \mathcal{A}(\Omega)$ holds.

(10) Any bicharacteristic curve of P(D) intersects Ω in an open interval.

The proof of this theorem is just the same as that of Theorem 4. (Cf. Remark (A) after Theorem 5).

Since Theorem 9 seems to require too much information concerning the global shape of Ω, we modify Theorem 9 as follows.

<u>Theorem 10.</u> Assume the same conditions on $P(D)$ as in Theorem 9. Let a relatively compact open set Ω have the form $\{x \mid \varphi(x) < 0\}$ for a real valued real analytic function $\varphi(x)$ defined near $\overline{\Omega}$ satisfying $\text{grad}_x \varphi(x) \neq 0$ on $\partial\Omega$. If the open set Ω satisfies both condition (4) in Theorem 4 and condition (11) below, then $P(D)\mathcal{A}(\Omega) = \mathcal{A}(\Omega)$ holds.

(11) There exists a family of open sets $\{N_j\}_{j=1}^{p}$ which satisfy the following: For any point x in $\partial\Omega$ we can find some j such that for any bicharacteristic curve $b_{(x,\xi)}$ of P(D) through (x, ξ) $b_{(x,\xi)} \cap (\overline{\Omega} - \{x\}) \cap N_j$ is connected, where N_j is a neighbourhood of x.

The proof of this theorem is similar to that of Theorem 4, so we omit the details.

<u>Remark.</u> As is remarked before Lemma 2, we can generalize Theorems 4, 9 and 10 for a wider class of linear differential operators with constant coefficients not necessarily satisfying conditions (1) and (2). We omit the details here and refer to Kawai [5] for it. We however emphasize the fact that one of the advantages of hyperfunction theory appears when one tries to state the theorems using conditions on the principal part of P(D) only. Thus in Kawai [5] no condition on lower order terms is needed. This fact is sometimes remarkably useful in treating

overdetermined systems with constant coefficients.

§2. Global existence of real analytic solutions of single linear differential equations with real analytic coefficients.

The reasonings of §1 depends on the global existence of good elementary solutions of the differential operator P(D). But if we want to treat the operators with variable coefficients, then there appears a difficulty: the arguements of Kawai [1] , [2] show only local existence of elementary solutions except some trivial cases, e.g. a linear differential operator with its principal part being of constant coefficients and the coefficients of lower order terms being entire functions. By this reason in the variable coefficient case we must content ourselves with the semi-global versions of Theorems 4,9 and 10 at present, i.e., we must consider all the problems in subsets of a fixed open set V in $\mathbb{R}^n$, not $\mathbb{R}^n$ itself, even if the coefficients of $P(x,D_x)$ are real analytic in a larger set than V. Of course the open set V depends on the operator under consideration. Such results are unsatisfactory, hence we will not discuss them any more here. However there is a case where the elementary solutions exist globally, hence all arguements in §1 succeed: globally hyperbolic operators in the sense of Leray [1]. (Cf. also Bruhat [1]). If we combine our construction of local elementary solutions and investigations of their properties developed in Kawai [1] with Leray's penetrating study of emissions, which are closely related to bicharacteristics, then we have the following Lemma 11. (Concerning the definition of global hyperbolicity and the related topics we refer to Leray [1] and Bruhat [1]. See also Kawai [5]).

<u>Lemma 11.</u> Assume that the linear differential operator $P(x,D_x)$ is globally hyperbolic on real analytic complete Riemanian manifold V. Then we have an elementary solution $E(x,y)$ for $(x,y) \in V \times V$ satisfying the following conditions:

(12) supp $E(x,y) \subset \mathcal{E}(y)$, where $\mathcal{E}(y)$ denotes the emission of y.

(13) $S.S.E(x,y) \subset \{(x,y;\xi,\eta) \in S^*(V \times V) \mid x=y,\ \xi=-\eta\} \cup \{(x,y;\xi,\eta) \in S^*(V \times V) \mid (x,\xi)$ and $(y,-\eta)$ are on the same bicharacteristic strip of $P(x,D_x)$ with $x \in \mathcal{E}(y)\}$.

Thus we have a global elementary solution in this case. Therefore we can prove analogues of Theorems 4,9 and 10. We omit the details and refer to Kawai [5]. Of course the assumption of hyperbolicity also allows us to treat the Cauchy problems for such operators both in the framework of real analytic functions and in that of hyperfunctions. A remarkable fact which appears in our treatment of Cauchy problems in the framework of hyperfunctions is firstly that bicharacteristics play no part when we decide the existence domain of solutions and secondly that they play their own essintial role only when we decide the domains where the uniqueness of solutions holds. About the details we also refer to Kawai [5].

§3. Global existence of real analytic solutions of systems of linear differential equations with constant coefficients.

The investigations of the problems stated in the title of this section are still progressing, hence we cannot give the final theorems but only sketch two methods which are expected to give the complete results and, in fact, have given results in some special cases. Since we want to explain the main ideas and do not try to give complete arguements in this section, we assume some additional conditions concerning the algebraic structure of the systems under consideration in order to avoid the technical difficulties. That is, in Theorem 12 we assume that the system of compatibility conditions has one generator and in Theorems 13 and 14 we assume that the system under consideration has only one unknown function. We remark that some trivial cases which can be treated by just the same method as developed in §1 may be omitted by these assumptions: the typical example is a system whose adjoint operator is an (over-)determined system of linear differential operators. But we hope the most typical features of the system of linear differential operators appear clearly even if we assume these conditions.

The first approach is the one concerning the existence of solutions in $\mathcal{A}(K)$ for compact set K in R^n. This method is essentially due to Ehrenpreis [1] , [3] and is a direct extension of the proof of Theorem 5. That is, it uses the pairing of $(\mathcal{A}(K), \mathcal{B}_K)$ and Serre's duality theorem. Then it is easy to reduce the existence theorem to the problem of support of solutions and we obtain the following:

Theorem 12. Denote by M_o the system of linear differential operators with constant coefficients and by M_1 the system which gives its compatibility conditions. Assume that ${}^tM_1$, the adjoint operator of M_1, has only one unknown function. Let K be a comact set in $\mathbb{R}^n$ satisfying the following conditions (14) and (15). Then $\mathrm{Ext}^1(M_o, \mathcal{A}(K))=0$ holds.

(14) There exists a real valued real analytic function $\varphi(x)$ which is defined in a neighbourhood of ChK, the convex hull of K, and satisfies

(a) $\{x \mid \varphi(x) \leqq 1\} = K$, $\{x \mid \varphi(x) \leqq 2\} = \mathrm{ChK}$,

and

(b) $\mathrm{grad}_x \varphi(x) \neq 0$ in ChK-K.

(15) The system ${}^tM_1$ is hyperbolic with respect to $\mathrm{grad}_x \varphi(x)\big|_{x=x_o}$ for any x_o satisfying $\varphi(x_o)=t$ with $1 < t \leqq 2$.

The proof of this theorem is obtained by the method of pie-nibbling due to Ehrenpreis [1] , [3] . By the way of the proof condition (b) can be weakened but we will not discuss it any more in this lecture.

The second approach is concerning the existence of real analytic solutions on an open set Ω and it can be summarized schematically as follows: if we can solve the system of linear differential equations in the space of hyperfunctions (or that of distributions or that of C^∞ functions etc.), then using the flabbiness of sheaf $\mathcal{B}$ and that of sheaf $\mathcal{C}$ we can solve the system in the space of real analytic functions assuming some additional "convexity" conditions on the boundary of Ω. We hope that the solvability in the space of hyperfunctions will be obtained under the least restrictive conditions on the "convexity" of Ω and that this method will give us the complete result, though we have not arrived there. Note that, for example, we need no "local convexity" conditions to solve the system of linear differential equations with constant coefficients if the space dimension n is equal to 2.

By this method we have the following theorems:

<u>Theorem 13.</u> Consider an overdetermined system M_0 with one unknown function. Let Ω be a relatively compact convex open set in $\mathbb{R}^n$. Then we have $\mathrm{Ext}^1(M_0, \mathcal{A}(\Omega))=0$, if we can find a polynomial P_0 whose homogeneous part satisfies conditions (1) and (2) in §1 in the generators of the ideal in the polynomial ring $A=C[\xi_1, ---, \xi_n]$ corresponding to the system M_0 under consideration, i.e., assume that, representing M_0 as $A/\mathcal{J}$, where $\mathcal{J}$ is an ideal in A, we can find polynomials $P_0, ---, P_k$ so that the ideal generated by them coincides with $\mathcal{J}$ and that P_0 satisfies conditions (1) and (2).

<u>Theorem 14.</u> For any overdetermined system M_0 of

linear diferential operators with constant coefficients and one unknown function, we can find a nowhere dense subset S of S^{n-1}, the (n-1)-dimensional co-sphere, such that the following holds: If a relatively compact open set Ω in $\mathbb{R}^n$ has the form $\bigcap_{j=1}^{N} \{x \mid \langle x, \xi^j \rangle < c_j,\ \xi^j \in S^{n-1}-S,\ c_j > 0\}$ for some positive integer N, then $\mathrm{Ext}^1(M_o, \mathcal{A}(\Omega))=0$ holds.

The proof of these theorems is given by the method analogous to that employed in the proof of Theorem 4, if we take into account of Komatsu's result that $\mathrm{Ext}^1(M_o, \mathcal{B}(\Omega))=0$ holds for any M_o and for any convex open set Ω in $\mathbb{R}^n$. (Cf. Komatsu [1] , [2]). Of course these forms of presentations of the theorems are very unsatisfactory from the aethetical viewpoint. In fact we have some recipes for generalizing these results using the notion of the bicharacteristics concerning the overdetermined systems, but we cannot make them applicable at present since we have almost no results concerning the global existence of hyperfunction solutions except for Komatsu's one or those which can be easily deduced from it only by the algebraic arguemens. Hence the present speaker wishes to return to these problems at the occasion of the next symposium, which will be held in next March. Please give him time enough until then.

References

Andersson, K.G., [1], Propagation of analyticity of solutions of partial differential equations with constant coefficients, Ark. Mat., 8, 277 —— 302, (1971)

Bruhat, Y., [1], Hyperbolic partial differential equations on a manifold, Proc. of Battelle Rencontres, Benjamin, 1968, pp.84 —— 106.

Ehrenpreis, L., [1], Some applications of the theory of distributions to several complex variables, Seminar on Analytic Functions, Princeton, 1957, pp.65 —— 79.

———, [2], Solutions of some problems of division IV, Amer. J. Math., 82, 522 —— 588, (1960).

———, [3], Fourier Analysis in Several Complex Variables, Wiley-Interscience, 1970.

Egorov, Yu.V., [1], Conditions for the solvability of pseudo-differential operators, Dokl, Akad. Nauk USSR, 187, 1232 —— 1234, (1969).

Hörmander, L., [1], Linear Partial Differential Operators, Springer, 1963.

———, [2], Uniqueness theorems and wave front sets for solutions of linear differential equations with analytic coefficients, to appear in Comm. Pure Appl. Math.

——, [3], On the existence and the regularity of solutions of linear pseudo-differential equations, Lectures at the Institute for Advanced Study, (1971).

John, F., [1], Plane Waves and Spherical Means Applied to Partial Differential Equations, Interscience, 1955.

Kashiwara, M., [1], On the flabbiness of sheaf $\mathcal{C}$, Reports of the symposium on the theory of hyperfunction, R.I.M.S. Kyoto Univ., 1970, pp.1 —— 4. (In Japanese).

Kashiwara, M., and T. Kawai, [1], Pseudo-differential operators in the theory of hyperfunctions, Proc. Japan Acad. 46, 1130 —— 1134, (1970).

——, [2], Ibid. (II), in prep.

Kawai, T., [1], Construction of local elementary solutions for linear partial differential operators with real analytic coefficients (I) —— The case with real principal symbols ——, to appear in Publ. R.I.M.S. Kyoto Univ. Its summary is given in Proc. Japan Acad., 46, 912 —— 916, (1970) and Proc. Japan Acad., 47, 19 —— 24, (1971).

——, [2], Construction of local elementary solutions for linear partial differential operators with real analytic coefficients II. —— The case with complex principal symbols ——. to appear in Publ. R.I.M.S. Kyoto Univ. Its summary is given in Proc. Japan Acad., 47, 147 —— 152, (1971).

———, [3], A survey of the theory of linear (pseudo-) differential equations from the view point of phase functions —— existence, regularity, effect of boundary conditions, transformations of operators, etc., Reports of the symposium on the theory of hyperfunctions and differential equations, R.I.M.S. Kyoto Univ., 1971, pp.84 —— 92. (In Japanese).

———, [4], On the global existence of real analytic solutions of linear differential equations (I), Proc. Japan Acad. 47, 537 —— 540, (1971).

———, [5], Ibid.(II), to appear in Proc. Japan Acad.

———, [6], On the global existence of real analytic solutions of linear differential equations (I), (II), ---- , in prep.

Komatsu, H., [1], Relative cohomology of sheaves of solutions of differential equations, Sem. Lions and Schwartz, 1966. Reprinted in Reports of Katata symposium on algebraic geometry and hyperfunctions, 1969, pp.1 —— 59.

———, [2], Resolutions by hyperfunctions of sheaves of solutions of differential equations with constant coefficients, Math. Ann., 176, 77 —— 86, (1968).

Leray, J., [1], Hyperbolic Partial Differential Equations, Mimeographed note, Princeton, 1952.

———, [2], Uniformalization de la solution du problème linéaire analytique de Cauchy près de la variété qui porte les données de Cauchy (Problème de Cauchy I), Bull. Soc. Math. Fr., 85, 389 —— 429, (1957).

Malgrange, B., [1], Existence et approximation des solutions des équations aux dérivées partielles et des équations de convolution, Ann. Inst. Fourier Grenoble, 6, 271 —— 355, (1955).

Nirenberg, L., and F. Treves, [1], On local solvability of linear partial differential equations. Part II: Sufficient conditions, Comm. Pure Appl. Math., 23, 459 —— 501, (1970).

Palamodov, V.P., [1], Linear Differential Operators with Constant Coefficients, Springer, 1970. Translation from the Russian original, which was pubbished in 1967 by Nauka.

Sato, M., [1], Theory of hyperfunctions II, J. Fac. Sci. Univ. Tokyo, 8, 387 —— 437, (1960).

————, [2], Hyperfunctions and partial differential equations, Proc. Int. Conf. on Functional Analysis and Related Topics, Univ. of Tokyo Press, 1969, pp.91 —— 94.

————, [3], Structure of hyperfunctions, Reports of the Katata symposium on algebraic geometry and hyperfunctions, 1969, pp.4-1 —— 4-30. (Notes by Kawai, in Japanese).

———, [4], Structure of hyperfunctions, Sûgaku no Ayumi, 15, 9 —— 72, (1970), (Notes by Kashiwara, In Japanese).

———, [5], Regularity of hyperfunction solutions of partial differential equations, to appear in Proceedings of Nice Congress (1970).

———, [6], Talks at this symposium. See also Sato, M, T. Kawai and M. Kashiwara: On pseudo-differential equations in hyperfunction theory, to appear in Proceedings of the symposium on partial differential equations held by A.M.S. at Berkeley (1971).