数理解析研究所講究録
第 170 巻 1973 年 104-112

ISOLATED ENDS OF OPEN LEAVES OF CODIMENSION-ONE FOLIATIONS

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§0. Introduction

The purpose of this paper is to investigate the behaviour of open leaves of codimension-one foliations. We define the limit sets of leaves and of ends of leaves on the analogy of topological dynamics. The main theorem describes how an end of an open leaf approaches to a closed leaf in the simplest case and shows the periodicity of the end in that case.

We work in the differentiable category throughout this paper.

§1. Ends of open manifolds

We recall the definition of ends in the case we concern. Those who are interested in ends can find the general theory in Siebenmann [3].

Definition 1.1 (Intrinsic definition) Let F be an open manifold without boundary. A family $\mathcal{E}$ of non-empty connected open subsets of F is called an end of F if $\mathcal{E}$ satisfies the following:

(1) $\overline{U} - U$ is compact for all $U \in \mathcal{E}$.

(2) If $U, U' \in \mathcal{E}$, there is $U'' \in \mathcal{E}$ with $U'' \subset U \cap U'$.

(3) $\cap \{ \overline{U} \mid U \in \mathcal{E} \} = \emptyset$.

(4) $\mathcal{E}$ is a maximal family with respect to (1), (2) and (3).

To clarify the concept "ends", we give an intuitive definition. At first we can find a covering $\{ K_i \}_{i=1}^{\infty}$ of F such that

(1) K_i is a compact submanifold of F with boundary.

(2) $K_i \subset \text{Int } K_{i+1}$ for all i.

(3) F - Int K_i does not contain compact connected components.

Then an <u>end</u> of F is a sequence $\{ V_i \}_{i=1}^{\infty}$ such that

(1) V_i is a connected component of F - K_i for all i.

(2) $V_i \supset V_{i+1}$ for all i.

If such a sequence $\{ V_i \}_{i=1}^{\infty}$ is given, $\{ V_i \}_{i=1}^{\infty}$ satisfies (1), (2) and (3) of Definition 1.1 and there is an end $\mathcal{E}$ of the intrinsic definition which contains $\{ V_i \}_{i=1}^{\infty}$. We can identify these definitions by this correspondence.

<u>Definition 1.2</u> An end $\mathcal{E}$ is <u>isolated</u> if $\mathcal{E}$ has a member U which does not belong to the other ends.

Now we give two simple examples.

<u>Example 1.3</u> Let F be the real line $\mathbb{R}$. There are just two ends $\omega = \{ (x, \infty) \mid x \in \mathbb{R} \}$ and $\alpha = \{ (-\infty, x) \mid x \in \mathbb{R} \}$.

<u>Example 1.4</u> Let F $= \{ (x,y) \in \mathbb{R}^2 \mid (x - 1)^2 + y^2 < 1,$ $(x - 1/n)^2 + y^2 > 1/9n^2(n + 1)^2$ for all $n = 1, 2, 3, \cdots \}$.

F has countable isolated ends which correspond to the circles

$\{ (x,y) \in \mathbb{R}^2 \mid (x - 1/n)^2 + y^2 = 1/9n^2(n + 1)^2, n = 1, 2, 3, \cdots \}$ and just one non-isolated end which corresponds to the circle $\{ (x,y) \in \mathbb{R}^2 \mid (x - 1)^2 + y^2 = 1 \}$.

Definition 1.5 An end $\mathcal{E}$ is periodic if there are a member $U \in \mathcal{E}$, a compact connected manifold P whose boundary consists of two connected components B_1 and B_2, and a diffeomorphism $f: B_1 \longrightarrow B_2$ such that $\overline{U}$ is diffeomorphic to $P \underset{f}{\smile} P \underset{f}{\smile} P \underset{f}{\smile} \cdots$ (countable union). We call P a period.

We give an easy proposition and omit the proof.

Proposition 1.6 (1) Every open manifold has at least one end. (2) Every periodic end is isolated.

§2. Limit sets of open leaves

Let M^n be a connected orientable closed manifold of dimension n, $\mathcal{F}$ a transversely orientable foliation of codimension one on M^n, and F^{n-1} an open leaf of $\mathcal{F}$. We fix them from now to the end of §4.

Definition 2.1 Let $L(F) = \bigcap_{i=1}^{\infty} (F - K_i)^a$ where $\{ K_i \}_{i=1}^{\infty}$ is a covering of F such that K_i is compact and $K_i \subset K_{i+1}$ for all i and $(\ )^a$ means the closure in M^n. We can easily show that L(F) is well-defined and omit the proof. We call L(F) the limit set of F.

Definition 2.2 Let ε be an end of F. Let $L_\varepsilon(F) = \bigcap \{U^a \mid U \in \varepsilon\}$ where $(\ )^a$ means again the closure in M^n. We call $L_\varepsilon(F)$ ε-limit set of F.

Now we write down the fundamental properties of the limit sets. The proof is left to the reader.

Proposition 2.3 (1) $L(F) \supset \bigcup_\varepsilon L_\varepsilon(F)$. If the number of ends of F is finite, $L(F) = \bigcup_\varepsilon L_\varepsilon(F)$.

(2) $L(F)$ and $L_\varepsilon(F)$ are non-empty compact invariant subsets of M^n where "invariant" means that to contain x implies to contain the leaf which contains x.

(3) $L_\varepsilon(F)$ is connected (not necessarily path-connected).

§3. Statement of the result

We are in the situation of the first paragraph of §2. The main theorem of this paper is the following

Theorem 3.1 Let ε be an isolated end of F. If $L_\varepsilon(F) \cap F = \emptyset$ and $L = L_\varepsilon(F)$ consists of just one leaf of $\mathcal{F}$, ε is a periodic end with a period P = L - (bicollar of N) for some connected submanifold N^{n-2} of L^{n-1}.

Definition 3.2 In the above case we will see, in the proof, that the behaviour of ε is very simple and we say that ε approaches tamely to L.

It seems to us that the condition of Theorem 3.1 is redundant.

<u>Conjecture 3.2</u> Let ε be an isolated end of F. If $L_{\varepsilon}(F) \cap F = \emptyset$, $L_{\varepsilon}(F)$ consists of just one leaf of $\mathcal{F}$.

If the conjecture is true, Theorem 3.1 gives a complete description of the behaviour of isolated ends of proper open leaves.

§4. Proof of Theorem 3.1

Let $x_0 \in L$. Since L is a compact leaf we can find a small segment s such that s is transverse to $\mathcal{F}$ and $s \cap L = \{x_0\}$. x_0 separates s into two parts s_+ and s_-. Since ε is isolated there is $U \in \varepsilon$ which does not belong to the other ends. Then $U \cap s_+$ or $U \cap s_-$ contains countable poits, say $A = U \cap s_+$ does so.

<u>Lemma 4.1</u> We can number the elements of A so that x_j is near to x_0 than x_i if $i < j$.

<u>Proof.</u> If it is impossible, we can show that $s \cap L - \{x_0\}$ is non-empty, which is a contradiction.

Let G be the group of the germs of diffeomorphisms $f: (U_f, x_0) \longrightarrow (V_f, x_0)$ at x_0 where U_f and V_f are connected open subsets of s which contain x_0. Let H be the group of the germs of $g = f \mid U_f \cap A : U_f \cap A \longrightarrow V_f \cap A$ at x_0 where f is as above and in addition $f(U_f \cap A) \subset V_f \cap A$. Let $\Phi : \pi_1(L,x_0) \longrightarrow G$ be the holonomy homomorphism of the leaf L. There is the natural homomorphism $\Psi : \operatorname{Im} \Phi \longrightarrow H$ which maps the germ of f to the germ of $f \mid U_f \cap A$.

Lemma 4.2 Im $\Psi\Phi$ is non-trivial.

Proof. If Im $\Psi\Phi$ is trivial, we can show that $L_\varepsilon(F) \cap L = \emptyset$, which is a contradiction.

Lemma 4.3 For all $a \in \pi_1(L,x_0)$ there are positive integers N_1, N_2 and an integer p such that $\Psi\Phi(a)$ is the germ of $g: \{x_i \mid i \geqq N_1\} \longrightarrow \{x_i \mid i \geqq N_2\}$ at x_0 where $g(x_i) = x_{i+p}$ for all i.

Proof. Let $\Psi\Phi(a)$ be the germ of $g: \{x_i \mid i \geqq N_1\} \longrightarrow \{x_i \mid i \geqq N_2\}$. Let $g(x_i) = x_{i+p}$ for some $i \geqq N_1$. Then $g(x_{i+1}) = x_j$ for some $j \geqq i + p + 1$. Suppose $j > i + p + 1$ and let $g^{-1}(x_{i+p+1}) = x_k$. Then $i < k < i + 1$, which is a contradiction.

Let H' be the group of the germs of $g: \{x_i \mid i \geqq N_1\} \longrightarrow \{x_i \mid i \geqq N_1 + p\}$ where $g(x_i) = x_{i+p}$ for some p and for all i. Then H' is an infinite cyclic group. Since Im $\Psi\Phi$ is a non-trivial subgroup of H', Im $\Psi\Phi$ is so. Since Im $\Psi\Phi$ is abelian, there is a homomorphism $u: H_1(L) \longrightarrow \operatorname{Im} \Psi\Phi$ such that $\Psi\Phi = uh$ where $h: \pi_1(L,x_0) \longrightarrow H_1(L)$ is the Hurewicz homomorphism.

$$\begin{array}{ccc} \pi_1(L,x_0) & \xrightarrow{\quad\Phi\quad} & \operatorname{Im}\Phi \subset G \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \Psi} \\ H_1(L) & \xrightarrow{\quad u\quad} & \operatorname{Im}\Psi\Phi \subset H' \subset H \end{array}$$

Since Im u = Im $\Psi\Phi$ is free abelian, the exact sequence

$$0 \longrightarrow \text{Ker } u \longrightarrow H_1(L) \xrightarrow{u} \text{Im } u \longrightarrow 0$$

splits and there is a homomorphism v: Im u $\longrightarrow H_1(L)$ with uv = 1. Then $H_1(L)$ = Ker u + Im v. Let a_0 be a generater of Im v. By Poincaré duality, there is a homology class $b \in H_{n-2}(L^{n-1})$ such that $a_0 \cdot b = 1$ and $a \cdot b = 0$ for all $a \in$ Ker u. By Nakatsuka's representation theorem [2], there is a connected oriented two-sided submanifold N^{n-2} of L^{n-1} such that $[N] = b$ and $x_0 \in N$.

Lemma 4.4 The images of $h \circ i_*$: $\pi_1(N,x_0) \longrightarrow \pi_1(L,x_0) \longrightarrow H_1(L)$ and $h \circ j_*$: $\pi_1(L-N,x_0) \longrightarrow \pi_1(L,x_0) \longrightarrow H_1(L)$ are contained in Ker u where i: $N \subset L$ and j: $L - N \subset L$.

Proof. Let $a_1 \in \text{Im } h \circ i_*$ and $a_2 \in \text{Im } h \circ j_*$. Consider their intersection numbers with b. We see that $a_1 \cdot b = a_2 \cdot b = 0$. Therefore a_1, $a_2 \in$ Ker u, which completes the proof.

By Lemma 4.4, there is an imbedding f: $N \times [0,1] \longrightarrow M$ which is transverse to $\mathcal{F}$ and satisfies the following conditions:

(1) $f(x,0) = x$ for all $x \in N$ and $f(x_0,1) = x_q$ for some q.

(2) For each $i \geqq q$, $f(x_0,t_i) = x_i$ for some $t_i \in (0,1]$.

(3) $f(N \times [0,1]) \cap U' = f(N \times \{t_i \mid i \geqq q\})$ where $U' \in \mathcal{E}$ such that $U' \subset U$ and $\overline{U'} - U' = f(N \times 1)$.

Let L^* be the compact connected manifold with boundary obtained from L-N by attaching two copies N_1, N_2 of N as boundary.

By Lemma 4.4, there is an immersion $g: L^* \times [0,1] \longrightarrow M$ such that

(1) $g \mid \mathrm{Int}\, L^* \times [0,1]$, $g \mid N_0 \times [0,1]$ and $g \mid N_1 \times [0,1]$ are imbeddings.

(2) $g \mid N_0 \times [0,1] = f$ where we identify N_0 and N.

(3) $g(x,t_i) = f(x,t_{i+k})$ for all $x \in N_1 = N$ and all $i \geqq q$ and for some positive integer k. $g(N_1 \times [0,1]) \subset f(N \times [0,1])$.

(4) $g(L^* \times t_i) \subset \overline{U'}$ for all $i \geqq q$.

Then we can identify $\overline{U'}$ and $L^* \cup L^* \cup L^* \cup \cdots$. Therefore $\mathcal{E}$ is a periodic end with a period L^*. This completes the proof of Theorem 3.1.

Remark 4.5 Consequently we see that $U \cap s_-$ is a finite set and $\mathrm{Im}\, \Psi\Phi = H'$ and $k = 1$.

§5. An example

We construct a foliation on $S^1 \times S^1 \times S^1$. Let D be a 2-disk in $S^1 \times S^1$ which does not intersect $S = S^1 \times x_0$. At first we consider a foliation on $(S^1 \times S^1 - \partial D) \times S^1$ whose leaves are $\overset{\circ}{D} \times x$ and $(S^1 \times S^1 - D) \times x$, $x \in S^1$. By making a whirlpool at $\partial D \times S^1$, we obtain a foliation on $S^1 \times S^1 \times S^1$ with a compact leaf $\partial D \times S^1$. We cut $S^1 \times S^1 \times S^1$ by $S \times S^1$ and we glue there by the diffeomorphism $f: S \times S^1 \longrightarrow S \times S^1$ such that $f(x,y) = (x, g(y))$ where

$g: S^1 \longrightarrow S^1$ is a diffeomorphism such that $g(e^{ti}) = e^{\frac{1}{2}ti}$ for all $t \in [0, \pi]$. Let $\mathfrak{F}$ be the obtained foliation. Let F be the compact leaf $\supset D \times S^1$, F_0 the leaf containing $S \times 1$, F_1 the leaf containing $S \times (-1)$. Then F_0 is diffeomorphic to $S^1 \times S^1 - D$ and F_0 has just one end ε_0. $L_{\varepsilon_0}(F_0) = L(F_0) = F$. F_1 has at least one non-isolated end ε_1 and countable isolated ends ε_j. $L_{\varepsilon_1}(F_1) = F \cup F_0$. $L_{\varepsilon_j}(F_1) = F$.

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