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SOME ASPECTS OF BOUNDARY-LAYER TRANSITION

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Introduction

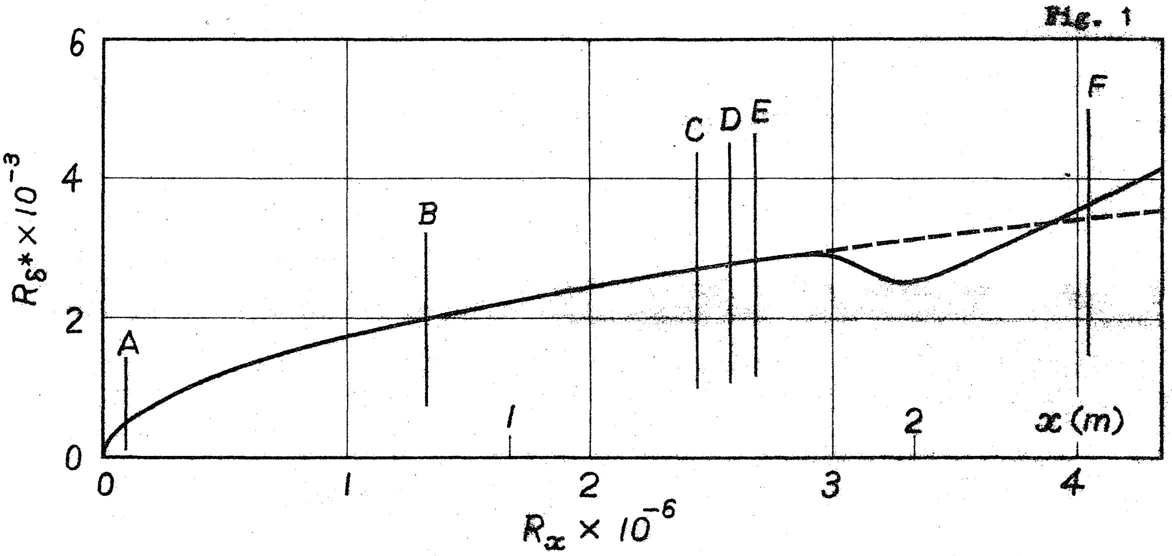
The problem of transition from laminar to turbulent flow in the boundary layer has absorbed the interest of investigators for so many years. Through considerable efforts devoted to it the physical processes involved in transition are now reasonably well understood. There remain, however, many details which need clarification, and which are called in question particularly when it is required to predict the onset of transition. Most of the difficulty in prediction lies in the large number of factors which affect transition, but which are not independent of each other.

The present paper reviews the current state of knowledge of boundary-layer transition, with emphasis on the possibility of its prediction or at least elucidation in incompressible flows with the aid of stability theories. Effects of important factors affecting transition, such as pressure gradient, free-stream turbulence and surface roughness, are discussed.

Processes leading to transition on a flat plate

Before entering into detailed discussion of the main subjects, it seems adequate to illustrate the sequence of processes leading to transition in the boundary layer on a flat plate in zero pressure gradient. As is well known, the transition is preceded by the appearance of weak oscillations in the form of a two-dimensional traveling wave (Tollmien-Schlichting wave) as predicted by the linear stability theory, provided all sources of disturbance such as free-stream turbulence and surface roughness are sufficiently small (Schubauer and Skramstad, 1948). For the boundary layer on a flat plate in zero pressure gradient the free-stream velocity U_1 is constant and the two-dimensional wave is to appear when the Reynolds number based on the displacement thickness δ^* of the boundary layer, $R_{\delta^*} = U_1\delta^*/\nu$, exceeds a value of about 520 (Jordinson, 1970). Referring to Fig. 1, which is based on the experimental results of Schubauer and Klebanoff (1955) at a free-stream velocity U_1 of 24 m/s and a free-stream turbulence level of 0.03 per cent, supplemented by similar experimental data due to Bennett (1953)

Fig. 1



and Klebanoff and collaborators (1959, 1962), and in which R_{δ^*} is plotted as function of the distance x from the leading edge of the plate, or the Reynolds number $R_x = U_1x/\nu$, the critical value $R_{\delta^*} = 520$ would be reached at x = 5.4 cm (station A), but the wave first appears with an amplitude discernible on the oscillograph only at the station B. Meanwhile, the wave develops in the manner as described by the linear stability theory, until a non-linear effect manifests itself at the station C, where the wave becomes three-dimensional, namely exhibits nearly periodic variation in amplitude in the spanwise direction. This makes it possible for vorticity in the spanwise direction to be locally intensified to such an extent that an unstable velocity profile persists for a considerable fraction of time, resulting in the generation of disturbances of a frequency of one order of magnitude higher than the original oscillation (station D). As these high-frequency disturbances travel downstream, they break down into disturbances of still higher frequency and of smaller scale. Through this cascade process of breakdown, however, the originally periodic structure of oscillations is obliterated, until eventually random oscillations characteristic of turbulent flow burst forth in a small localized region which is called a turbulent spot (station E). The turbulent spots grow as they travel downstream, until they merge into a fully developed turbulent flow (station F).

The transition region commonly referred to is the region beginning with the first appearance of turbulent spot and the first discernible deviation of mean velocity distribution from Blasius profile, and ending up in a fully developed turbulent flow where the spots have eventually merged together. In this sense the station E is called as the beginning of transition, or simply transition point, although considerable growth of disturbances has already been made in the region upstream of this point.

The development of disturbances leading to turbulent flow is thus very much complicated, but a clear impression carried away from the presentation of a sequence of events in this form is that the region that can be described by the linear stability theory occupies almost 90 per cent of the distance up to the beginning of transition. Therefore, it is quite natural to think of the possibility of predicting transition by simply applying the linear theory, without taking account of the non-linear and other complicated effects.

Application of linear stability theory

As a matter of fact, the linear stability theory has been applied by Smith and collaborators (1956, 1970) in order to calculate the amplification of unstable disturbances on a passage from the theoretical neutral point up to the experimentally observed transition point. They found that for a number of experimental results for relatively low turbulence levels the maximum amplifications obtained were close to each other, the average value being e^9 or 8×10^3. Similar conclusion was reached independently by Ingen (1956).

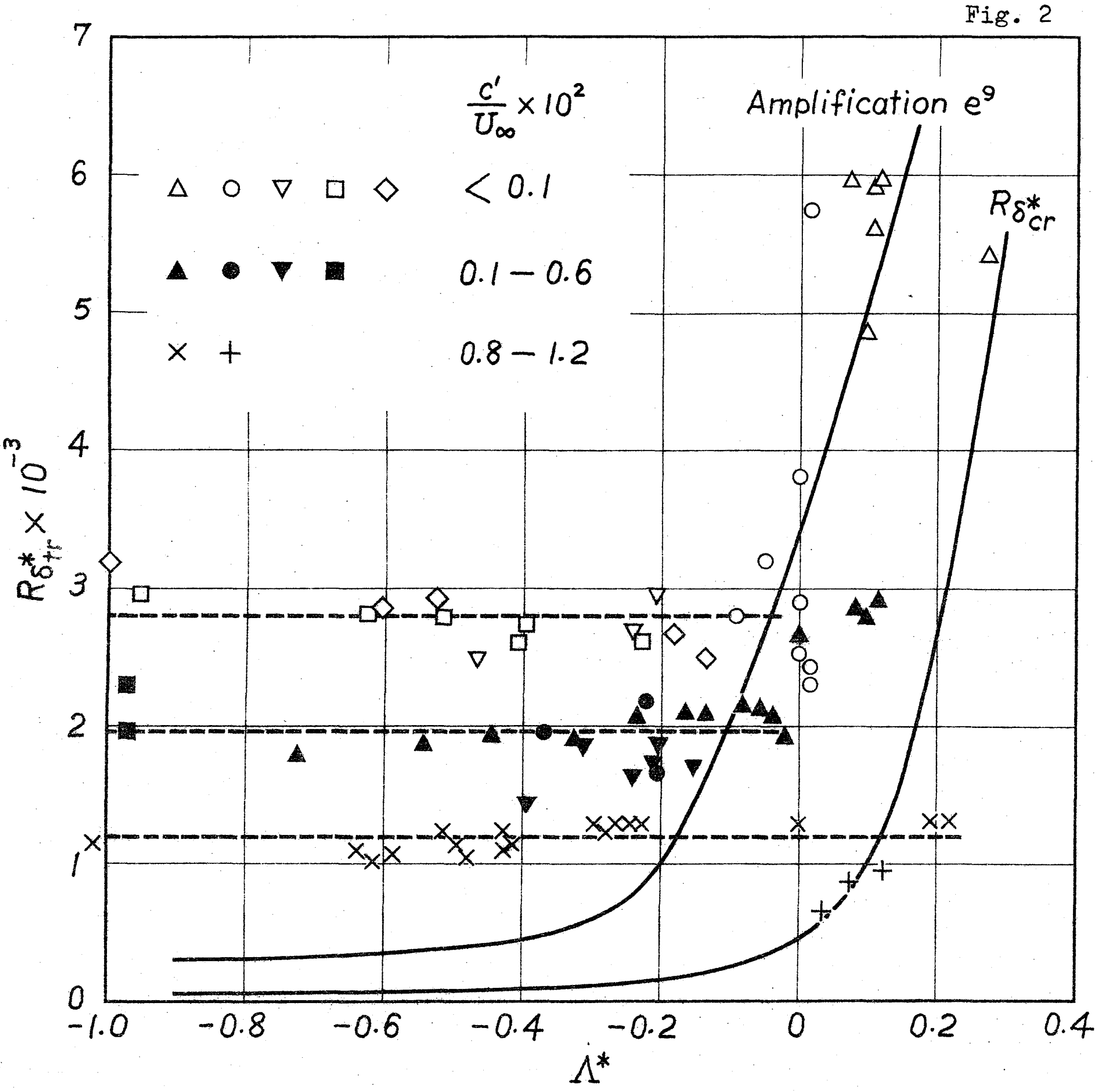
Based on this finding Smith and collaborators suggest that transition should occur when the maximum amplification of e^9 is reached. This method of prediction seems to have worked fairly well, particularly because it indicated good agreement with Michel's purely empirical correlation (1951) between the Reynolds number based on the momentum thickness and the Reynolds number based on the distance along the surface, both evaluated at transition point.

It is true that the experimental results on which Michel's correlation and Smith's prediction are based are for relatively low free-stream turbulence levels. Strictly speaking, however, the turbulence level of individual experiments is not exactly the same. In view of the strong effect of turbulence level on transition, particularly at low turbulence levels, a doubt is raised as to whether Michel's correlation expresses nothing but a self-evident fact that the value of the momentum-thickness Reynolds number at transition increases with increase in the surface-distance Reynolds number. In other words, Michel's correlation might represent a combined effect on transition, in which the effect of free-stream turbulence is unseparated from that of pressure gradient.

That the doubt proves true to some extent may be seen from the correlation of experimental data, which was recently made by Hall and Gibbings (1970) and is reproduced in Fig. 2. Here the displacement-thickness Reynolds number at transition $R_{\delta^*_{tr}}$ is plotted as function of Pohlhausen pressure-gradient parameter $\Lambda^* = (\delta^{*2}/\nu)(dU_1/dx)$ and free-stream turbulence level c'/U_∞, where U_∞ is the undisturbed velocity and c'^2 is the arithmetic mean of the mean-square values of the three components of fluctuating velocity. Curves representing maximum amplification of e^9 and also critical Reynolds number $R_{\delta^*_{cr}}$ are shown in Fig. 2. The curve of maximum amplification of e^9 agrees with the experimental results in the region of falling pressure gradient ($\Lambda^* > 0$, accelerated flow), but departs radically from the experimental data in the region of rising pressure gradient ($\Lambda^* < 0$, decelerated flow), in which $R_{\delta^*_{tr}}$ = constant for a given turbulence level appears to be a reasonable representation of the data. The comparison clearly suggests the need of taking account of the non-linear effects for predicting transition.

Obviously, the shortcoming of the use of linear theory lies in the fact

Fig. 2



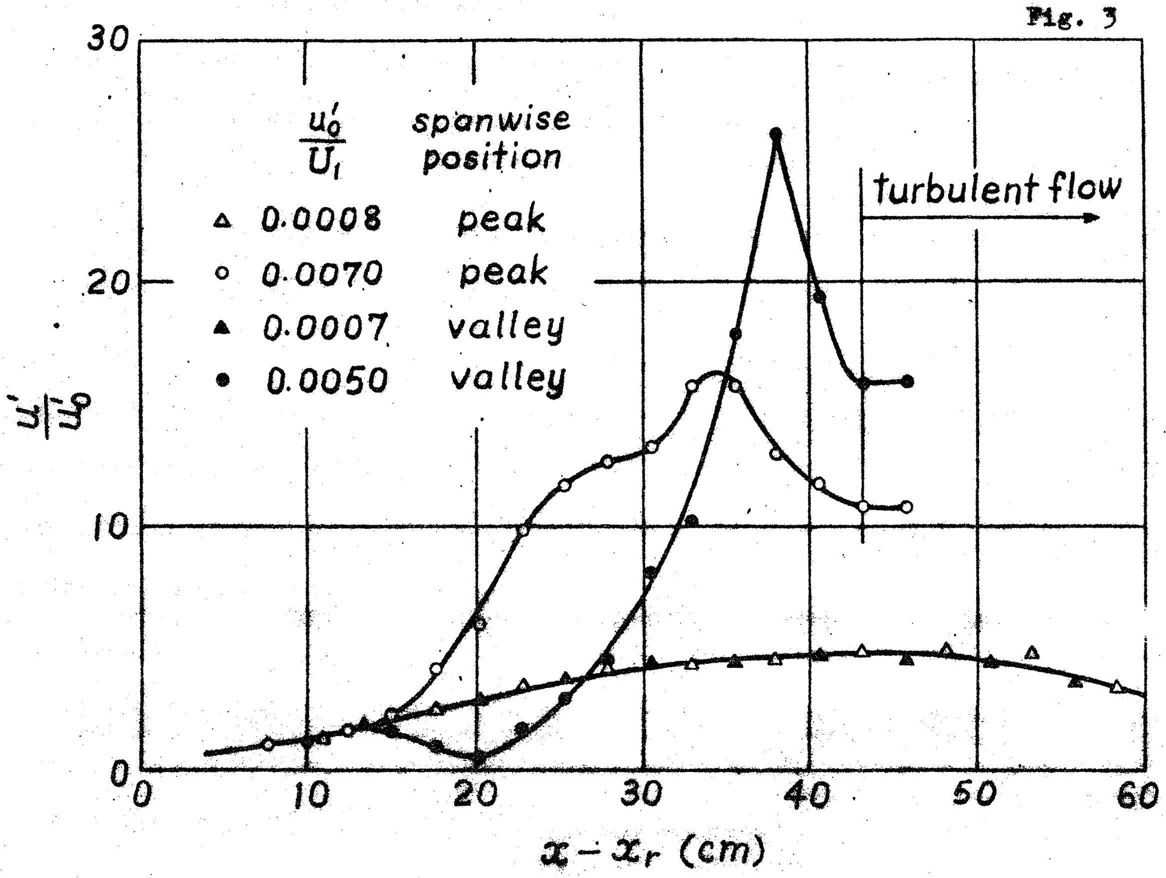
that no information is provided for the amplitude of the disturbance. It is certainly difficult to understand how the factor of amplification is relevant rather than the amplitude of the disturbance to the onset of transition. A step in the right direction should be taken on the basis of the non-linear theory.

In the region of falling pressure gradient the scarcity of data available makes any definite conclusions impossible. In falling pressure gradient corresponding to the stagnation point ($\Lambda^* = 0.42$), the critical Reynolds number R_{δ^*cr} attains a very high value of about 1.3×10^4, but no experimental data on transition are available. In this connection only a bare mention is made of a preliminary experimental investigation of Snedeker, Donaldson and Yates (1970), in which the flow in falling pressure gradient exhibits an instability against certain finite or three-dimensional disturbances at a Reynolds number an order of magnitude below the critical value given by the linear theory.

Consideration from non-linear stability theory

Experimental investigations were made by Klebanoff and collaborators (1962), with emphasis on the non-linear growth of the wave and the ensuing breakdown to turbulent flow. In view of their preceding observation (1959) that the initially two-dimensional wave exhibited a nearly periodic variation in amplitude of about 2.5 cm wavelength in the spanwise direction, a controlled wave of 145 Hz frequency and of 2.5 cm wavelength was generated by a vibrating ribbon in combination with cellophane tapes, the ribbon having been placed at 0.23 mm from the surface and 89 cm from the leading edge of the flat plate. With free-stream velocity U_1 of 15.2 m/s, frequency of 145 Hz and location of ribbon at x_r = 89 cm, the generated wave is to amplify according to the linear theory. In Fig. 3 the wave amplitude u' (root-mean-square of streamwise fluctuating velocity) at a distance of 1.14 mm, relative to u'_0, which is the value of u' at $x - x_r$ = 7.6 cm, is plotted against the distance downstream from the vibrating ribbon, $x - x_r$. When the initial wave amplitude is low (u'_0/U_1 = 0.0008 and 0.0007), the wave amplifies and damps in accordance with the prediction of the linear theory. It shows the same behavior at different spanwise positions. No transition to turbulence ensues. When the initial wave amplitude is high (u'_0/U_1 = 0.0070 and 0.0050), the wave development first follows the curve for low wave amplitude, and then exhibits characteristically different behavior, according as the spanwise position is at the peak (where the wave amplitude is maximum) or at the valley (where the wave amplitude is minimum). At the peak the wave amplitude increases rapidly once deviated from the curve for low amplitude, while at the valley it first decreases and then increases rapidly. In both cases transition to turbulent flow takes place after the ave amplitude attains a maximum.

Fig. 3



One of the important points of the experimental results is the development of pronounced three-dimensionality from an initially weak variation in wave amplitude in the spanwise direction. For example, there appears a defect in local mean velocity at the peak and an excess at the valley, indicating the existence of a system of alternately rotating streamwise vortices. This is accounted for by the non-linear theory of Benney and Lin (1960, 1961, 1964), in which the behavior of a primary oscillation consisting of a two-dimensional Tollmien-Schlichting wave and a superposed three-dimensional wave with spanwise periodicity is considered. The non-linear interaction of these two components when the two-dimensional component predominates gives rise to a system of slowly growing secondary vorticity in the streamwise direction with the same spanwise periodicity as the primary oscillation. The secondary vorticity redistributes momentum in such a direction as to produce a spanwise variation in mean velocity which is in good agreement with experimental results. In order to simplify the analysis, however, it is assumed that the velocity profile is piecewise linear in- and outside the boundary layer, and that the two-dimensional and three-dimensional components have the same phase velocity, although strictly speaking the phase velocity of the three-dimensional wave should be greater by 10 to 15 per cent. These restrictions are removed in the subsequent analysis by Menzel (1969), in which the solution is determined by numerical evaluation.

As pointed out by Menzel, there is a possibility for a sufficiently unstable two-dimensional Tollmien-Schlichting wave to produce amplifying streamwise vortices in collaboration with the three-dimensional component, which itself is stable. This appears to yield a possible explanation for the observed premature development of three-dimensional, spanwise dependent disturbances. In this connection mention is made of the resonance theory, which was originally put forward by Raetz (1959) and recently elaborated by Craik (1971). The analysis concentrates on the triad of unstable Tollmien-Schlichting waves, which consists of a two-dimensional wave and a pair of oblique waves traveling at equal and opposite angles to the streamwise direction, all three waves having the same phase velocity in the streamwise direction. It is found that for such a triad resonant interactions take place, resulting in rapid growth of the pair of oblique waves. This is interpreted as responsible for imparting characteristic spanwise periodicity to the disturbance. The theoretically estimated periodicity agrees very well with Klebanoff's experiments in one case, but not satisfactorily in another case, thus leaving the comparison inconclusive.

Another important point of Klebanoff's results is that transition ensues when the wave amplitude exceeds a certain threshold value. In this example the threshold value of u'_0/U_1 is estimated at about 0.0026. This corresponds to the threshold value $u'/U_1 = 0.013$ at the station where the wave development begins to deviate from the curve for low amplitude. Since the distance of 1.14 mm from the surface amounts to about one fifth of the boundary-layer thickness, the distribution of u'/U_1 nearly attains a maximum in this neighborhood, and the threshold value of 0.013 may be taken as representing the maximum value of the wave amplitude distribution. Seeing that the wave must develop into a three-dimensional structure in order to lead to transition, it is considered as a requisite to transition that the threshold amplitude be exceeded.

It seems to me that the existence of threshold amplitude may be explained by the weakly non-linear stability theory, which was first stated by Landau (1944) and subsequently developed by Meksyn (1951), Stuart (1951, 1958, 1960) and Watson (1960, 1962). The analysis centers about the equation

$$dA^2/dX = A^2(\alpha_0 + \alpha_2 A^2 + \cdot \cdot \cdot),$$

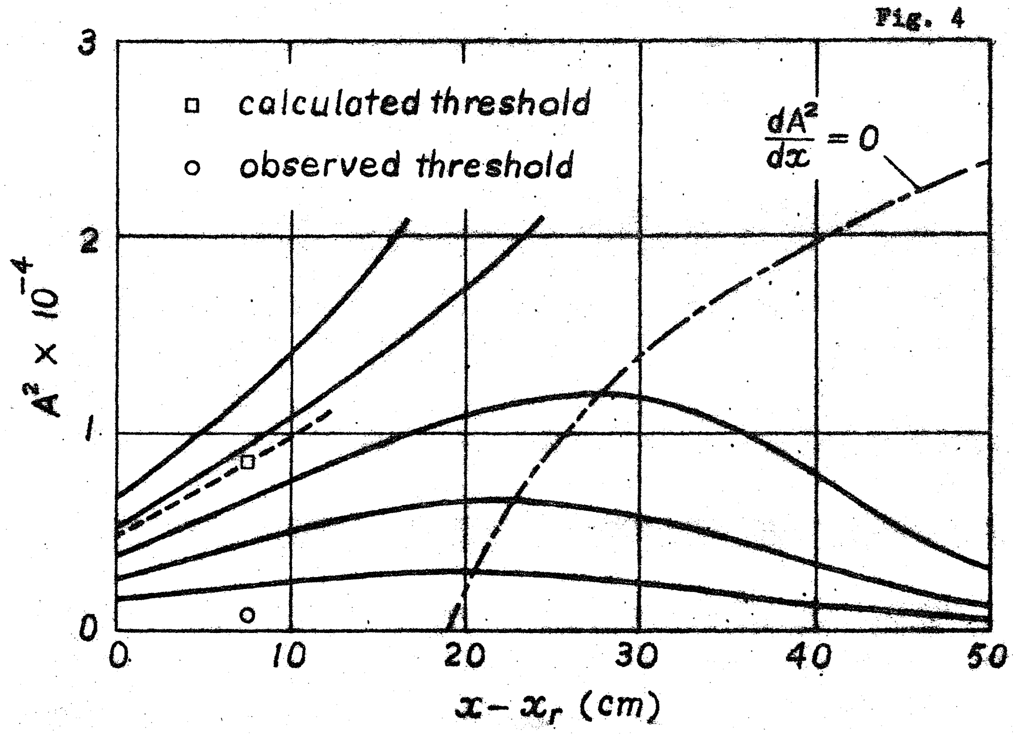
in which A is the non-dimensional amplitude of spatially growing primary wave, X is the non-dimensional streamwise distance, and α_0, α_2, . . . are numerical constants. The linear stability theory yields the constant α_0 as an eigenvalue of the Orr-Sommerfeld equation, while the non-linear stability theory provides the remaining constants, and particularly α_2. If α_0 is negative the flow is stable to infinitesimal disturbances, but the question remains as to whether disturbances of finite amplitude produce instability. If α_0 is positive the flow is unstable to infinitesimal disturbances, but there is a possibility that disturbances of finite amplitude lead to equilibrium state. The sign of α_2 is of importance when only the first two terms $\alpha_0 A^2$ and $\alpha_2 A^4$ are considered. For $\alpha_0 < 0$, $\alpha_2 > 0$ a subcritical threshold instability is produced, while for $\alpha_0 > 0$, $\alpha_2 < 0$ a supercritical equilibrium is attained.

The calculation of α_2 was carried out by Reynolds and Potter (1967) and Pekeris and Shkoller (1967, 1969) for two-dimensional Poiseuille flow, namely parabolic distribution of velocity between parallel walls. Results of calculation are consistent to show that α_2 is positive in the neighborhood of the upper branch of the neutral curve. This suggests that the flow exhibits subcritical instability in the region slightly above the upper branch of the neutral curve.

There have been no published results on α_2 for boundary-layer flows. Seeing that the velocity profile of Blasius form for the boundary layer on a flat plate is not radically different from the parabolic profile for Poiseuille flow, there is a fair possibility that a subcritical instability exists also in this type of flow. On the other hand, however, due caution must be exercised against the effect of increasing thickness of the boundary layer. So far as infinitesimal disturbances are concerned, the effect can safely be assumed small, but there is no justification for validity of that assumption for finite disturbances.

In response to the author's suggestion, a non-linear analysis was recently made by N. Itoh (1972) of Japan National Aerospace Laboratory for the boundary

Fig. 4



layer in the region near the upper branch of the neutral curve and corresponding to the experiments of Klebanoff and collaborators. The method of analysis is essentially a modification of Watson's approach (1962), extended to include the effect of increasing thickness approximately. The results are shown in Fig. 4, in which the square of the non-dimensional amplitude A^2 is plotted against the distance from the vibrating ribbon $x - x_r$, A representing u'/U_1, where u' is the maximum root-mean-square of the streamwise fluctuating velocity. It is seen that a subcritical instability is produced when A^2 at $x - x_r = 7.6$ cm exceeds a value of 0.85×10^{-4}, which is to be compared to the observed threshold value of $0.0026^2 = 0.07 \times 10^{-4}$. Noting that the analysis is based on two-dimensional disturbances while the observed instability is due to three-dimensional disturbances, the result of analysis appears in favor of interpreting Klebanoff's experiments as indicative of subcritical threshold instability. There is some reason to believe that the superposition of a three-dimensional wave with spanwise periodicity on the two-dimensional wave increases the positive value of α_2, and therefore decreases the threshold amplitude. If this is the case, the three-dimensional disturbance of finite amplitude would be more unstable than the two-dimensional one, just opposite to the case of infinitesimal disturbance.

Seeing that the waves develop from small amplitudes and grow larger as they proceed downstream, it is highly desirable to be able to trace the spatial and temporal development of disturbances. Recently the weakly non-linear stability theory has been extended in this direction by Stewartson and Stuart (1971, 1972). The analysis has so far been carried out for two-dimensional Poiseuille flow.

Effect of free-stream turbulence

Since the Tollmien-Schlichting waves were first observed by Schubauer and Skramstad (1948) in the boundary layer at a very low free-stream turbulence level, it had been believed that for higher turbulence levels transition took place without the precedence of the Tollmien-Schlichting waves. Taylor's postulate (1936) for transition to result from momentary separation of the boundary layer due to the fluctuating pressure gradient of the free-stream turbulence had been considered most nearly correct. However, the experiments of Bennett (1953) indicated that even for a relatively high free-stream turbulence level of 0.42 per cent there was considerable amplification of that component of the frequency predicted by linear theory before transition developed. Moreover, there has been no experimental evidence that momentary separation occurred at or prior to transition, which has raised some doubt as to the adequacy of Taylor's postulate in describing the phenomena involved.

Very little is known about the mechanism of transition due to free-stream turbulence. For example, let us take up the phenomenon first noted by Dryden

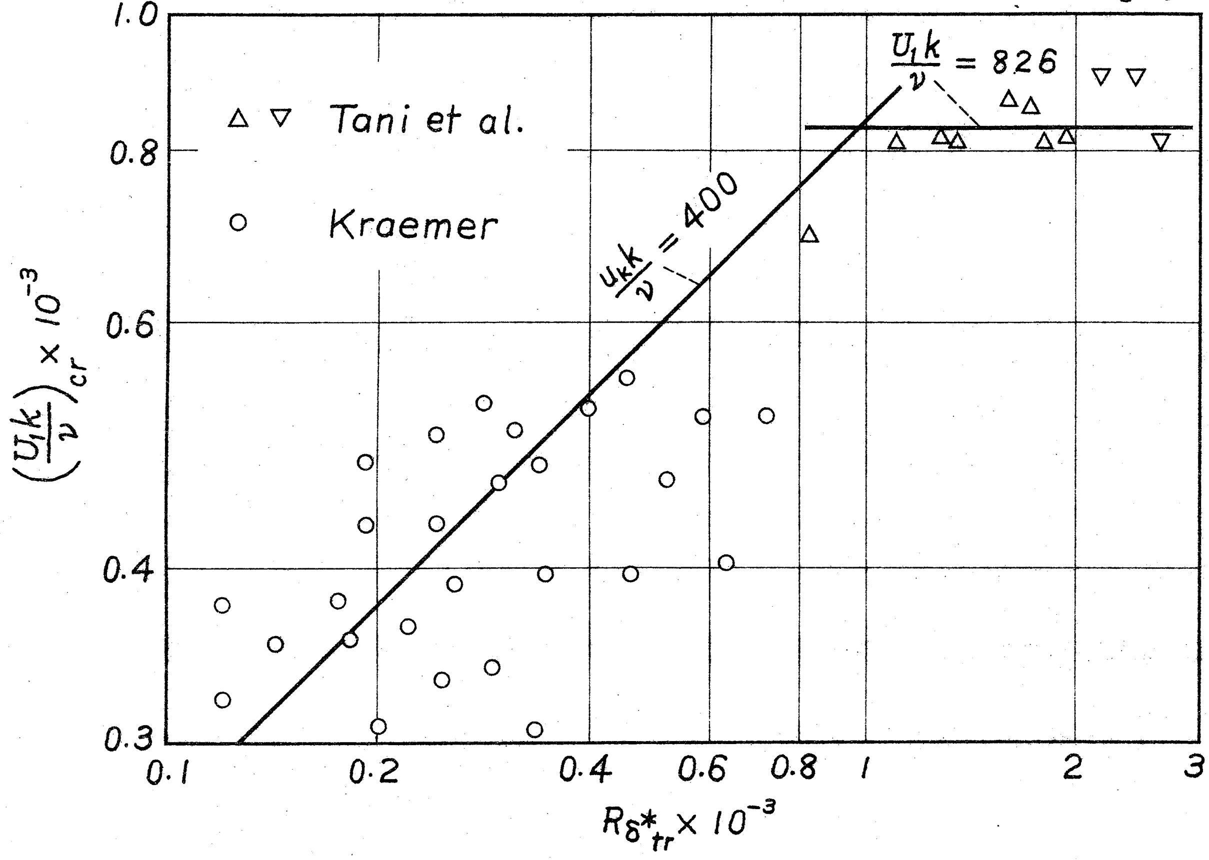
(1936) that at higher free-stream turbulence levels abnormally large fluctuations are observed in the boundary layer, affecting transition only to a limited extent. A plausible explanation for this phenomenon is that the free-stream turbulence induces a fluctuation in boundary-layer thickness with the frequency lying below those of Tollmien-Schlichting waves. However, there have been no detailed measurements except a preliminary investigation of Klebanoff (1964).

In this connection special mention is to be made of the new attempt of Benney and Bergeron (1969), in which the singularity of the non-viscous solution at the critical layer (where the flow velocity is equal to the phase velocity) is removed by taking account of the non-linear terms, instead of including the effect of viscosity as has been done for Tollmien-Schlichting waves. This gives rise to a class of wave motions, which differ from Tollmien-Schlichting waves particularly in the lack of a phase change across the critical layer. Since the theory is based on the assumption that the effect of non-linearity predominates over that of viscosity, it is suggested that the wave motion obtained from theory may have relevance to transition due to free-stream turbulence.

Effect of surface roughness

Surface roughness is known to affect transition because of the disturbances introduced by its presence. Geometrically simplest is a two-dimensional single roughness element, which consists of cylindrical wire placed on the surface normal to the streamwise direction. When a wire of diameter k (roughness height) is placed on a flat plate ahead of the transition point on a smooth plate, the transition point moves forward as the free-stream velocity U_1 is increased, beginning from the position on a smooth plate. Associated with this forward movement of transition, the displacement-thickness Reynolds number at transition $R_{\delta^*_{tr}}$ at first decreases, but then increases when transition occurs close enough to the roughness element (Tani, Hama and Mituisi, 1954). As has been observed by Tani and Sato (1956), the effect of roughness on transition is of different character according as $R_{\delta^*_{tr}}$ is decreasing or increasing in the course of the forward movement of transition. When $R_{\delta^*_{tr}}$ is decreasing the flow separated at the roughness element reattaches to the surface as a laminar boundary layer. Transition occurs in the reattached boundary layer, and the laminar oscillation observed is that characteristic of a boundary layer in zero pressure gradient.

Fig. 5



Thus roughness affects transition not only by increasing boundary layer thickness but also by supplying disturbances to the flow in the manner similar to the effect of free-stream turbulence. When $R_{\delta^*_{tr}}$ is increasing, on the other hand, transition occurs in the separated layer. Preceding to transition, a laminar oscillation is observed which is characteristic of the flow with an inflectional velocity profile. The effect of roughness for this condition is interpreted as modifying the existent boundary layer to such an extent that the instability of the modified flow is encountered.

These experimental evidences are sufficient to attach a significance to the critical condition, in which the Reynolds number at transition $R_{\delta^*_{tr}}$ attains its minimum such that transition occurs close enough to the roughness element. Tani and Sato tentatively gave a critical value of 840 characterizing this condition for the Reynolds number based on roughness height U_1k/ν, although the value was corrected to 826 later by Gibbings (1959). According to Kraemer (1961), however, the critical condition for lower values of $R_{\delta^*_{tr}}$ is better correlated by $u_kk/\nu = 400$, in which u_k is the velocity in the undisturbed boundary layer at the height of the roughness element. In Fig. 5 values of U_1k/ν for critical condition is plotted as function of $R_{\delta^*_{tr}}$ on the basis of the experimental results of Tani and collaborators (1954) as well as Kraemer.

The above mentioned effect of roughness is for a boundary layer in zero pressure gradient and at very low free-stream turbulence levels. Fig. 6 shows how the effect is modified by the falling pressure gradient and also by the increase in free-stream turbulence, based on the experimental results of Fage and Preston (1941), Tani, Iuchi and Yamamoto (1954) and Kraemer (1961). It is seen that the critical value of U_1k/ν is reduced in falling pressure gradient ($\Lambda^* > 0$, accelerated flow), both for low and high free-stream turbulence levels. Decrease in U_1k/ν with increase in turbulence level is easy to understand, since the effect of roughness is partly to supply additional disturbances to the flow. On the other hand, decrease in U_1k/ν in falling pressure gradient is difficult to interpret, bringing out a new question worthy of clarification.

For a three-dimensional single roughness element, such as a sphere of diameter k or an upright circular cylinder of height k, the experimental data appear to correlate well with a critical value of the Reynolds number u_kk/ν, in which u_k is again the velocity in the undisturbed boundary layer at the height of the roughness element. When the critical value is exceeded, a wedge-shaped region of turbulent flow originates at the roughness element and extends downstream. Fig. 7 shows the critical value of u_kk/ν plotted against the transition Reynolds number $R_{\delta^*_{tr}}$ for boundary layers in zero pressure gradient (Klebanoff, Schubauer and Tidstrom, 1955; Mochizuki, 1961; Tani, Komoda, Komatsu and Iuchi, 1962) and in falling pressure gradient (Peterson and Horton, 1959; Dobbinga, 1965). The

Fig. 6

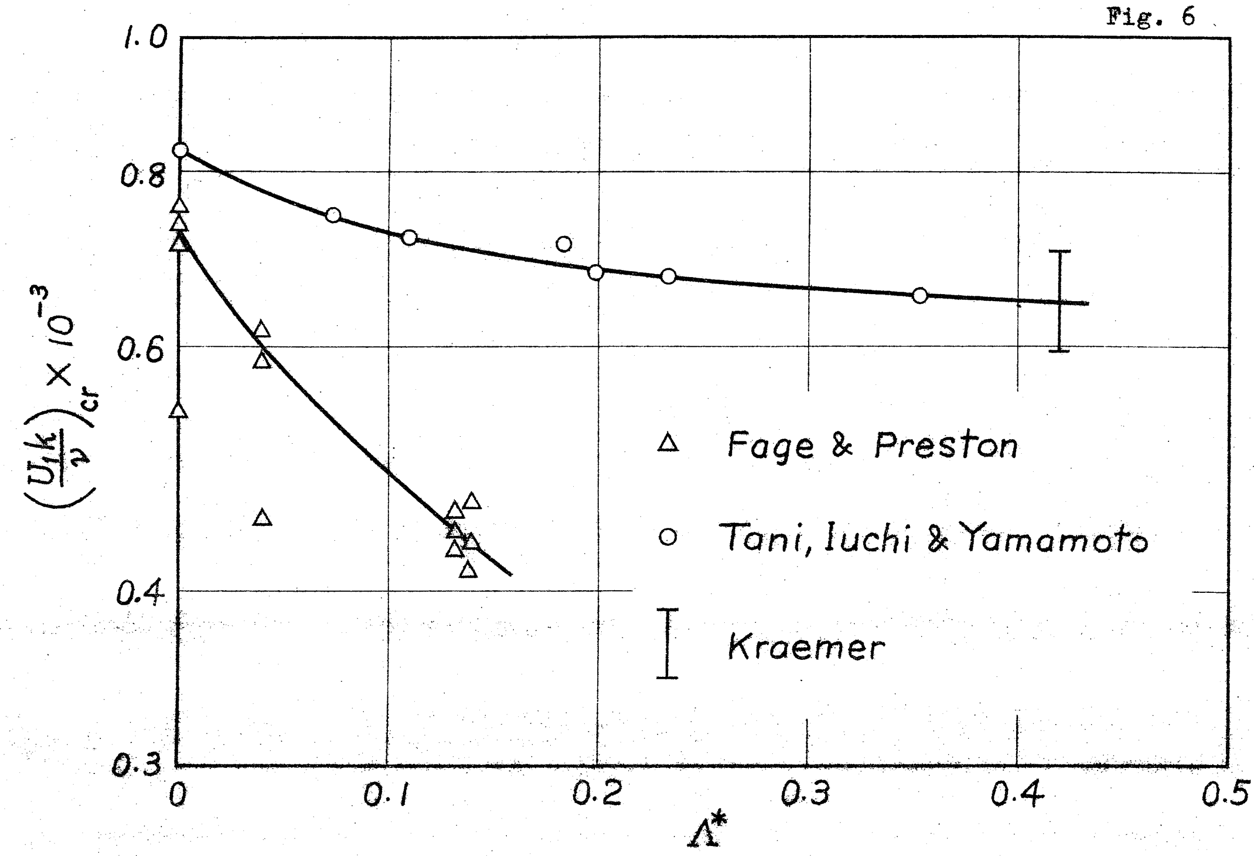
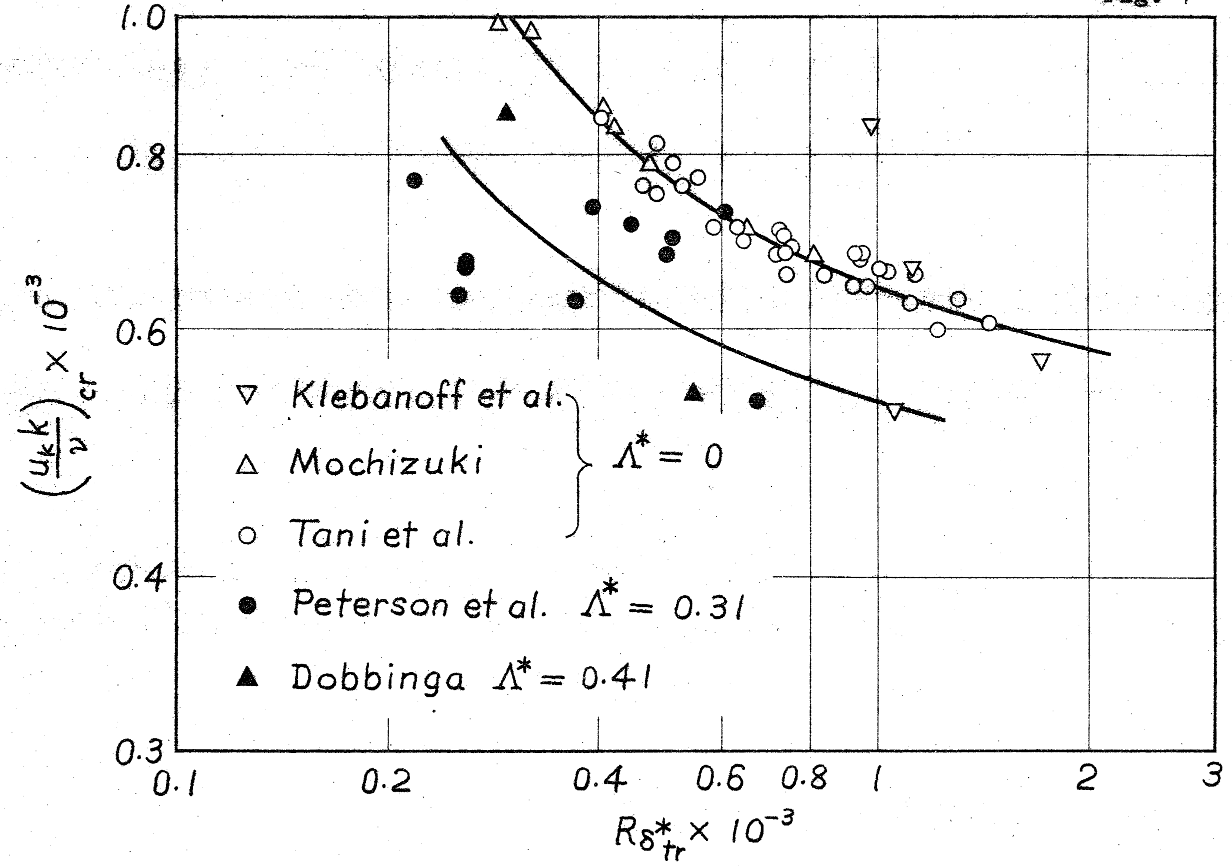

1.0
0.8
0.6
0.4
0.3
0
0.1
0.2
0.3
0.4
0.5
Λ^*
$\left(\frac{U_1 k}{\nu}\right)_{cr} \times 10^{-3}$
Fage & Preston
Tani, Iuchi & Yamamoto
Kraemer

Fig. 7


1.0
0.8
0.6
0.4
0.3
0.1
0.2
0.4
0.6
0.8
1
2
3
$R_{\delta^*_{tr}} \times 10^{-3}$
$\left(\frac{u_k k}{\nu}\right)_{cr} \times 10^{-3}$
Klebanoff et al.
Mochizuki
Tani et al.
$\Lambda^* = 0$
Peterson et al. $\Lambda^* = 0.31$
Dobbinga $\Lambda^* = 0.41$

experimental values are somewhat scattered, but it is evident that the critical Reynolds number is reduced in falling pressure gradients. The result thus agrees with that for two-dimensional roughness in suggesting that the transition caused by roughness is hastened by the effect of falling pressure gradient.

Disturbances in stagnation region

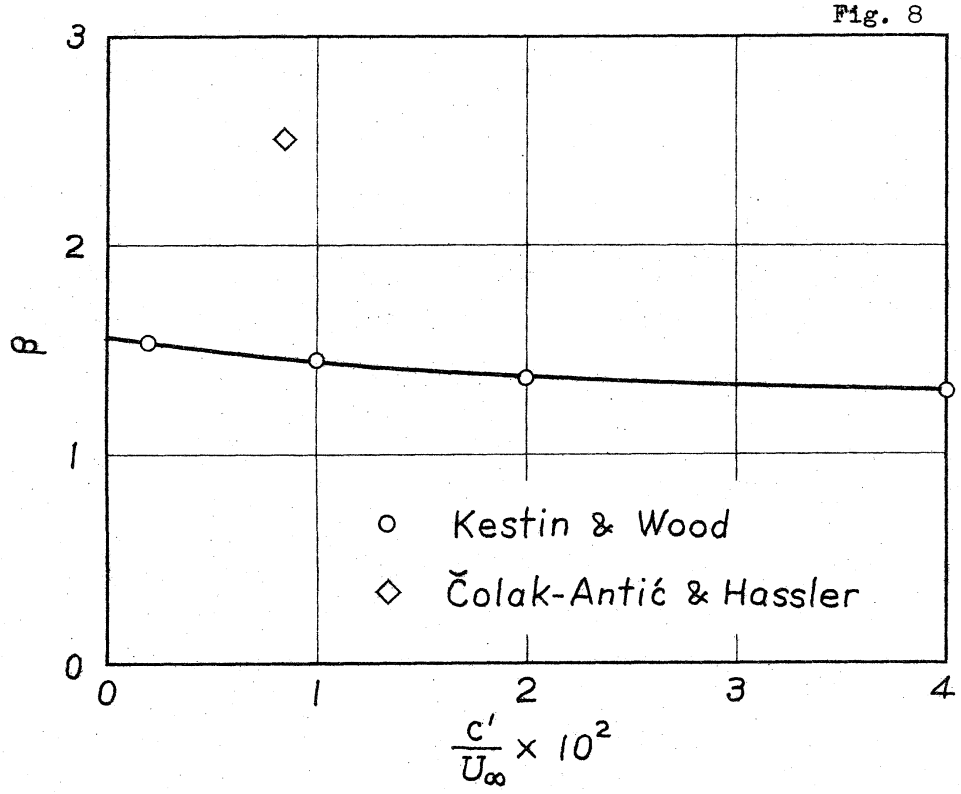
In connection with the premature transition due to roughness in falling pressure gradients, attention is directed to the earlier observation of velocity fluctuations near the stagnation point of a two-dimensional cylinder (Piercy and Richardson, 1928, 1930). Noting that the streamlines near the stagnation point are concave, Görtler (1955) conjectured that such flows might become unstable to disturbances in the form of a system of alternately rotating streamwise vortices similar to those predicted by him (1940) to appear on a concave wall. The conjecture was supported by the analysis of Hämmerlin (1955) on the stagnation-point boundary layer with free-stream velocity components $U_1 = cx$ and $V_1 = -cy$. Instability was indicated to occur for a certain range of the spanwise wavelength, namely the distance between two vortices rotating in the same direction. However, no critical wavelength was obtained, because the associated eigenvalue problem yielded a continuous spectrum of eigenvalues.

Recently Kestin and Wood (1970) reconsidered the problem, taking the view that the inconclusive nature of the analysis of Görtler and Hämmerlin originates in the assumption for the free-stream velocity field, which in fact represents the stagnation flow toward an infinite flat plate. When a uniform flow of velocity U_∞ approaches a two-dimensional cylinder of finite radius of curvature a, the free-stream velocity field is modified such that $-V_1$ for large values of y approaches U_∞ instead of increasing indefinitely. With this modification it became possible for Kestin and Wood to obtain a single discrete eigenvalue $\beta = (L/2\pi a)(2aU_\infty/\nu)^{\frac{1}{2}} = 1.72$, in which L is the spanwise wavelength of the streamwise vortices.

Experimental determination of the wavelength was made also by Kestin and

Wood (1969) and recently by Colak-Antić and Hassler (1971). The experimental results are shown in Fig. 8, in which the non-dimensional wavelength β is plotted as function of the free-stream turbulence level c'/U_∞. The results of Kestin and Wood extrapolate to $\beta = 1.56$ to the limit of zero turbulence level, yielding good agreement with the theoretical result. The value of β obtained by Colak-Antić and Hassler appears comparatively large.

Fig. 8



The decrease in wavelength (at constant Reynolds number $2aU_\infty/\nu$) with increase in free-stream turbulence level appears to be consistent with the experimentally observed enhancement of both skin friction and heat transfer in the stagnation region by free-stream turbulence, particularly for turbulence levels exceeding about 1 per cent (Smith and Kuethe, 1966; Kestin, 1966; Kayalar, 1969). It also suggests the possibility of subcritical threshold instability for this type of flow, because the wavelength would have increased with increase in turbulence level if there were supercritical equilibrium state. It appears worthwhile to examine the behavior of flow in the stagnation region on the basis of weakly non-linear stability theory.

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