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POLAR DECOMPOSITION FOR ISOMORPHISMS OF C*-ALGEBRAS

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We say that an automorphism ρ of a C*-algebra is positive if it is self-adjoint, that is, $\rho^* = \rho$ and its spectrum is contained in the positive half axis, where the adjoint isomorphism ρ^* of an isomorphism ρ of a C*-algebra A onto another B means an isomorphism of B onto A defined by the relation

$$\rho(y)^* = \rho^{-1}(y^*)$$

for $y \in B$. Several facts which assure the propriety of these terms we are met by are referred to [2].

It was proved in [2] (See also [1]) that if A and B are C*-algebras, the former has property (D) and if ρ is an isomorphism of A onto B, then there are a *-isomorphism π of A onto B and a positive automorphism φ of A, in the unique way, which satisfy the relation

$$\rho = \pi\varphi ,$$

and that the mapping $\rho \longrightarrow (\pi, \varphi)$ is bi-continuous under the norm topology, where that a C*-algebra A has property (D) means that any derivation of A is inner.

The main purpose of this note is to report that in the above statement a part of the assumption "A has property (D)" can be taken off, leaving the conclusion invariant:

Polar decomposition theorem for isomorphisms of C*-algebras. Let A and B be C*-algebras, ρ an isomorphism of A onto B. Then, there are a *-isomorphism π of A onto B and a positive automorphism φ of A, in the unique way, which satisfy the relation

$$\rho = \pi\varphi ,$$

and that the mapping $\rho \longrightarrow (\pi, \varphi)$ is bi-continuous under the norm topology.

We will only point out here a key to reduce this theorem to the preceding statement. It is so simple and fundamental.

Lemma. If the spectrum of a bounded linear operator σ on a Banach space X is simply connected and if a closed subspace Y of X is invariant under σ, then the spectrum of the restriction of σ on Y is contained in the spectrum of σ.

Thus, we know that if the restriction of a positive automorphism of a C*-algebra B on a sub-C*-algebra A of B becomes an automorphism of A, then it is also positive.

Positive automorphisms are so significant that we can prove the following theorems on them.

Theorem. Let A be a sub-C*-algebra which has property (D) of a C*-algebra B with identity. An automorphism ρ of A is positive if and only if there is a regular positive element h in B such that ρ is the restriction on A of the automorphism Adh of B defined by the formula

$$\mathrm{Adh}(x) = hxh^{-1}$$

for $x \in B$.

Theorem. Let A be a C*-algebra, Φ a faithful *-representation of A. An automorphism ρ of A is positive if and only if there is a positive automorphism $\tilde{\rho}$ of the von Neumann algebra $\widetilde{\Phi(A)}$ generated by $\Phi(A)$ which satisfies the relation

$$\tilde{\rho}\,\Phi = \Phi\,\rho\ .$$

Detailed arguments on this subject shall be published somewhere.

REFERENCES

[1]. T. Okayasu, A structure theorem of automorphisms of von Neumann algebras, Tôhoku Math. Journ. 20(1968), 199-206.

[2]. 岡安隆照, C^*-代数の同型写像について, 数研講究録166(1972), 8-17.