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Relative Hamiltonian for States of von Neumann Algebras

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§1. Introduction

The present work is concerned with some general analysis of type III von Neumann algebras. It is known that type III von Neumann algebras appear in connection with equilibrium states of statistical mechanics as well as algebras of local observables. Type III von Neumann algebras are distinguished from other types of von Neumann algebras by the property that it does not have a trace and have been considered pathological by mathematicians at the beginning.

Works in the past year or so indicates that one finds something which can replace the role of trace even in type III von Neumann algebras. A very beautiful structure theorem for type III algebra was reported on by Takesaki [10], [11]. The theorem says that any type III algebra can be built on a semi-finite algebra which has a trace.

In the present work, you will again see some structure for a general von Neumann algebra which replaces the role of trace and as an example of its consequence we shall write down a generalization of some inequalities in statistical mechanics, which contain trace in its ususl form.

§2. Tomita-Takesaki Theory

For the sake of non-specialist, we begin with an introductory account of famous Tomita-Takesaki theory [9]. It deals with a von Neumann algebra M on a Hilbert space $\mathcal{H}$ with a cyclic and separating vector Ψ. (Ψ is cyclic if $M\Psi$ is dense in $\mathcal{H}$ and Ψ is separating if non-zero x in M never annihilates Ψ.) Since any type III von Neumann algebra on a separable Hilbert space has (many) cyclic and separating vectors in that Hilbert space, this deals with quite a general situation.

Notation: $\psi(x) = (\Psi, x\Psi)$ for $x \in M$.

The basic operators in this theory are S_ψ, Δ_ψ and J_ψ defined in the following manner:

$$(2.1) \qquad S_\psi(x\Psi) = x^*\Psi , \qquad x \in M.$$

This equation defines an antilinear operator S_ψ, which can be shown to be closable. The polar decomposition of its closure $\overline{S_\psi}$ given by

$$(2.2) \qquad \overline{S_\psi} = J_\psi \Delta_\psi^{1/2}$$

defines Δ_ψ and J_ψ. More explicitly

$$(2.3) \qquad \Delta_\psi = S_\psi^* \overline{S_\psi} ,$$

$$(2.4) \qquad J_\psi = (\overline{S_\psi}\Delta_\psi^{-1/2})^-$$

where the adjoint S_ψ^* of an antilinear operator S_ψ is defined in a similar manner as the case of linear operators: $(f, S_\psi^* g) = (g, S_\psi f)$.

The positive selfadjoint operator Δ_ψ is called the modular operator and has the following properties.

(1) $\Delta_\psi^* = \Delta_\psi$, $\Delta_\psi \geq 0$.

(2) $\Delta_\psi \Psi = \Psi$.

(3) $\Delta_\psi^{it} M \Delta_\psi^{-it} = M$.

(4) KMS condition.

Conversely, an operator satisfying these conditions is unique and is a modular operator. (The uniqueness of modular automorphism is in [9], which determines Δ_ψ^{it} up to operators in M'. (2) then uniquely determines Δ_ψ^{it} .)

Among these 4 properties, the property (3) is most difficult to prove. It allows one to define a one-parameter group of $*$-automorphisms of M called modular automorphisms:

(2.5) $$\sigma_t^\psi(x) \equiv \Delta_\psi^{it} x \Delta_\psi^{-it} .$$

It depends only on the expectation functional ψ and not on how ψ is represented by a vector Ψ.

The fourth condition, bearing the name of three physicists Kubo, Martin and Schwinger who found these properties in connection with equilibrium states of statistical mechanics, was first recognized its importance in mathematical study of statistical mechanics by Haag, Hugenholtz and Winink [7],[8] just at the time when Tomita has completed his theory.

It states that two functions

$$F_1(t) \equiv \psi(x\sigma_t^\psi(y)),$$
$$F_2(t) \equiv \psi(\sigma_t^\psi(y)x)$$

are connected by an analytic continuation:

(2.6) $$F_2(t+i) = F_1(t).$$

This property of modular operators brings statistical mechanics of equilibrium states and theory of type III von Neumann algebras close together.

We call the operator J_ψ as modular conjugation operator. It has the following 5 properties, which in turn characterize J_ψ as in the case of 4 properties for Δ_ψ [2].

(1) $(J_\psi f\ ,\ J_\psi g) = (g,f)$.

(2) $J_\psi^2 = 1$

(3) $J_\psi \Psi = \Psi$.

(4) $J_\psi M J_\psi = M'$.

(5) $(\Psi\ ,\ xj(x)\Psi) \geqq 0\ ,\quad x \in M,\quad j(x) \equiv J_\psi x J_\psi$.

The property (4) is most difficult to prove. The property (5) is an immediate consequence of the property (1) for Δ_ψ. Nevertheless, it is crucial in exploring structures which replace the role of trace.

One more important property of Δ_ψ and J_ψ are their relation given by

$$J_\psi \Delta_\psi J_\psi = \Delta_\psi^{-1}\ .$$

§3 Simple exapmle —— Finite matrix algebra.

Let M be a matrix algebra acting on a finite dimensional space $\mathcal{H}$ with a cyclic and separating normalized vector Ω. Then $\mathcal{H} = \{x\Omega;\ x \in M\}$ and

$$(3.1) \qquad (x\Omega, y\Omega) = tr(x^*y)$$

(We use physicist's convention for inner product.)

Any state ψ of M can be written as

$$(3.2) \qquad \psi(x) = tr(\rho_\psi x)$$

in terms of the density matrix $\rho_\psi \in M$, $\rho_\psi \geq 0$. ψ is faithful if and only if $\rho_\psi > 0$ (strictly positive). We can then write $\rho_\psi = e^{-H_\psi}$ where $H_\psi = -\log \rho_\psi$ is the Hamiltonian and ψ is the Gibbs state for this Hamiltonian (with the inverse temperature $\beta = 1$). If we set

$$(3.3) \qquad \Psi = \rho_\psi^{1/2}\Omega \quad ,$$

then we have $\psi(x) = (\Psi, x\Psi)$.

The modular conjugation operator J_Ψ in this example is common for all Ψ of the form (3.3) and is given by

$$(3.4) \qquad J_\Psi x\Omega = x^*\Omega \quad .$$

The modular operator is given by

(3.5) $\Delta_\psi = \exp\{-H_\psi + j(H_\psi)\} \quad (= \rho_\psi j(\rho_\psi^{-1}))$.

where $j(x) = J_\psi x J_\psi$. The modular automorphism is given by

(3.6) $\sigma_t^\psi(x) = e^{-itH_\psi} x e^{itH_\psi}$

σ_t^ψ is called the time translation automorphism in physics except for an unfortunate discrepancy in the sign (i.e. $-t$ is the time).

In this example, it is natural to define the relative Hamiltonian $h(\varphi/\psi)$ of φ and ψ as the difference of the Hamiltonian:

(3.7) $h(\varphi/\psi) \equiv H_\phi - H_\psi$.

Using the following definition of the right expansional (time-antiordered product)

(3.8) $$\mathrm{Exp}_r\left(\int_0^t ; h(s)ds\right) \quad \left(\equiv \overline{T} \exp\int_0^t h(s)ds\right)$$
$$\equiv \sum_{n=0}^\infty \int_0^t ds_1 \cdots \int_0^{s_{n-1}} ds_n h(s_n) \cdots h(s_1) ,$$

and the following convenient formula (for example, see [3])

(3.9) $$e^{t(A+B)}e^{-Bt} = \mathrm{Exp}_r\left(\int_0^t ; e^{As}Be^{-As}ds\right) ,$$

we can find the following properties of $h(\varphi/\psi)$ in this example:

(1) The Radon-Nikodym derivative satisfying chain rule [2]:

$$(3.10) \qquad \phi = (e^{-H_\phi/2} e^{H_\psi/2})\Psi = \mathrm{Exp}_r(\int_0^t ; \sigma^\psi_{-is}(-h(\varphi/\psi)))\Psi \quad .$$

(2) The intertwining operator for modular automorphisms [5]:

$$(3.11) \qquad \sigma^\varphi_t(x) = u^{\varphi\psi}_t \sigma^\psi_t(x)(u^{\varphi\psi}_t)^* \quad ,$$

$$(3.12) \qquad u^{\varphi\psi}_t = e^{-itH_\psi} e^{itH_\psi} = \mathrm{Exp}_r(\int_0^t ; \sigma^\psi_s(-h(\varphi/\psi))) \quad .$$

(3) Chain rule:

$$(3.13) \qquad h(\varphi_1/\varphi_2)+h(\varphi_2/\varphi_3) = h(\varphi_1/\varphi_3) \quad .$$

For a general von Neumann algebra M , Hamiltonians H_ψ can not necessarily be found and σ^φ_t (at least for some t) is an outer automorphism for type III M . However our result shows that we can introduce the notion of the relative Hamiltonian for a general M .

§4 The canonical cone.

We denote the weak closure of

$$\{\Delta_\Psi^{1/4} x\Psi ;\ x \in M ,\ x \geqq 0\}$$

by V_Ψ [2]. ($\mathcal{P}^\natural$ in the notation of A. Connes [6].) In the example of §3, $V_\Psi = \{x\Omega;\ x \in M,\ x \geqq 0\}$, as is easily verified. It plays the rôle of trace for type III M. The following is a partial list of important properties of V_Ψ :

(1) V_Ψ is independent of Ψ in the sense that for all cyclic and separating $\Phi \in V_\Psi$, we have $V_\Phi = V_\Psi$.

(2) All cyclic and separating $\Phi \in V_\Psi$ has the common modular conjugation operator $J_\Phi = J_\Psi$.

(3) All normal states has a unique representative in V_Ψ , i.e. for each $\varphi \in M_*^+$ there exists a unique $\xi_\varphi \in V_\Psi$ such that $(\xi_\varphi, x\xi_\varphi) = \varphi(x)$ for $x \in M$. The mapping $\varphi \to \xi_\varphi$ is bicontinuous.

(4) V_Ψ is selfdual. Namely $f, g \in V_\Psi$ implies $(f,g) \geqq 0$ and $(f,g) \geqq 0$ for all $g \in V_\Psi$ implies $f \in V_\Psi$.

§5 Multiple KMS property and basic estimate.

From the KMS condition (2.6), one can derive the following multiple KMS properties [1], [4]:

For any $x_1, \cdots, x_n \in M$, there exists a function $F(z_1 \cdots z_n)$ which is analytic in the tube domain

$$(5.1) \qquad T_n \equiv \{z;\ \mathrm{Im}\ z_1 \geqq \mathrm{Im}\ z_2 \geqq \cdots \geqq \mathrm{Im}\ z_n,\quad \mathrm{Im}(z_1 - z_n) \leqq 1\},$$

continuous and bounded in the closure of T_n and its value on Silov boundaries of T_n are given by

$$(5.2) \qquad F((t_1+i), \cdots, (t_j+i), t_{j+1}, \cdots, t_n)$$
$$= \psi(\sigma^{\psi}_{t_{j+1}}(x_{j+1}) \cdots \sigma^{\psi}_{t_n}(x_n)\, \sigma^{\psi}_{t_1}(x_1) \cdots \sigma^{\psi}_{t_j}(x_j))$$

where $t_1, \cdots, t_n$ are real and $j=0,1,\cdots,n-1$. (This property reduces to the KMS condition when $n=2$.)

Using this multiple KMS property and a multi-variable version of the three line theorem, we obtain the following estimates:

If $h^* = h \in M$, $t_1 \geqq 0, \cdots, t_n \geqq 0$ and $1/2 \geqq \sum_{j=1}^{n} t_j$, then $\Delta_\Psi^{t_n} h \Delta_\Psi^{t_{n-1}} \cdots \Delta_\Psi^{t_1} h\Psi$ is meaningful and

$$(5.3) \qquad \| \Delta_\Psi^{t_n} h \cdots \Delta_\Psi^{t_1} h\Psi \| \leqq \|h\|^n \|\Psi\|.$$

§6 Relative Hamiltonian

Due to the estimate (5.3), the following expression is absolutely and uniformly (for h in a bounded set) convergent in norm:

$$(6.1)\qquad \Psi(h) \equiv \sum_{n=0}^{\infty} \int_0^{1/2} ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{n-1}} ds_n \Delta_\Psi^{s_n} h \Delta_\Psi^{s_{n-1}-s_n} h \cdots \Delta_\Psi^{s_1-s_2} h \Psi .$$

In the example of §3, (6.1) reduces to (3.10), where $h=-h(\varphi/\psi)$. Hence we call a selfadjoint element h of M as a relative hamiltonian of φ and ψ if $\Phi = \xi_\varphi$ and $\Psi = \xi_\psi$ are related by $\Phi = \Psi(-h)$ and denote it by $h = h(\varphi/\psi)$. We also define (in view of (3.11) and (3.12))

$$(6.2)\qquad u_t^{\varphi\psi} \equiv \mathrm{Exp}_r\left(\int_0^t ; \sigma_s^\psi(h)ds\right) \quad (\in M)$$

if $\Phi = \Psi(h)$.

Our main results on relative hamiltonian are as follows [4]:

(1) For all $h^* = h \in M$, there exist φ and ψ in M_*^+ such that $h(\varphi/\psi) = h$. (We have seen this above as a consequence of (5.3).)

(2) For given φ and ψ, the relative Hamiltonian $h(\varphi/\psi)$ is unique if it exists.

(3) If $\Phi = \Psi(h)$, then $u_t^{\varphi\psi} \in M$ is an intertwining operator for modular automorphisms:

$$(6.3)\qquad \sigma_t^\varphi(x) = u_t^{\varphi\psi}\sigma_t^\psi(x)(u_t^{\varphi\psi})^*,$$

(6.4) $u_t^{\varphi\psi}(u_t^{\varphi\psi})^* = (u_t^{\varphi\psi})^* u_t^{\varphi\psi} = 1,$

(6.5) $(u_t^{\varphi\psi})^* = u_t^{\psi\varphi}.$

(4) If $\varphi = \psi(h)$, then

(6.6) $\log \Delta_\varphi = \log \Delta_\psi + h - j(h)$

where $j(h) = J_\psi h J_\psi$. (Cf. (3.5) and (3.7).)

(5) Chain rule holds:

(6.7) $h(\varphi_1/\varphi_2) + h(\varphi_2/\varphi_3) = h(\varphi_1/\varphi_3)$

where if two of $h(\varphi_i/\varphi_j)$ exist , then the third also exists and satisfies (6.7). Since $h(\varphi/\varphi) = 0$, a special case of (6.7) yields

(6.8) $h(\varphi/\psi) = -h(\psi/\varphi)$.

(6) If $\ell_1\psi(x) \geqq \varphi(x) \geqq \ell_2\psi(x)$ for all $x \in M$, $x \geqq 0$, then there exists $h(\varphi/\psi)$ and

(6.9) $\log \ell_1 \geqq -h(\varphi/\psi) \geqq \log \ell_2$.

(7) The relative modular operator [6] can be expressed as

$u_t^{\varphi\psi}\Delta_\psi^{it} = \exp(-it\{\log \Delta_\psi - h\}) = j(u_t^{\psi\varphi})\Delta_\phi^{it}$.

§7 Golden-Thompson and Peierls-Bogolubov Inequalities

Although there is no trace for type III M, we can still find inequalities which reduces to Golden-Thompson and Peierls-Bogolubov inequalities for finite matrix algebra M. Namely

$$(7.1) \qquad \psi(e^h) \geqq \|\Psi(h)\|^2 \geqq \exp \psi(h)$$

where $h^* = h \in M$, $\psi(x) = (\Psi, x\Psi)$ and the second inequality holds when $\|\Psi\| = 1$.

For finite matrix algebra M, we have

$$\psi(x) = tr(e^H x),$$

$$\psi(e^h) = tr(e^H e^h),$$

$$\|\Psi(h)\|^2 = tr(e^{H+h}), \; (\Psi(h) = e^{(H+h)/2}\Omega),$$

and hence the first inequality reduces to the Golden-Thompson inequality

$$tr(e^H e^h) \geqq tr(e^{H+h}).$$

Similarly, we have for $\Psi = e^{H/2}\Omega/(tre^H)^{1/2}$

$$\psi(x) = tr(e^H x)/tr(e^H),$$

$$\|\Psi(h)\|^2 = tr(e^{H+h})/tr(e^H),$$

$$\exp \psi(h) = \exp\{tr(e^H h)/tr(e^H)\},$$

and hence the second inequality reduces to the Peierls-Bogolubov inequality

$$tr(e^{H+h})/tr(e^H) \geqq \exp\{tr(e^H h)/tr(e^H)\}.$$

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