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多孔質物体を過ぎる遅い流れ
―特に境界条件について―

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§ 1. Introduction

There are many interesting flows of viscous fluid past a porous and permeable body or wall, e.g. filtration, drift of down, suction or injection of fluid through a porous wall, etc. . In the investigation of such a kind of flow, the Navier-Stokes equation may be the fundamental equation in the pure fluid region, while Darcy's law[1,2] which expresses that the velocity is proportional to the pressure gradient is usually assumed to hold in a porous medium. As the boundary condition at the surface of the porous medium for these two equations, the following conditions have been used so far[3,4]: (i) the pressure is continuous, (ii) the normal mass flux is continuous, and (iii) the tangential velocity just outside the porous medium is zero.

When the porous body has large porosity, i.e. the fraction of void to total volume of the body is close to unity such as fiber glass, the effect of viscous stress at the surface is easy to penetrate into the body through the pores and seems to produce a flow near the surface in the body even if there is no pressure gradient. Darcy's law is then improper to describe the flow near the surface at least. We must use such kind of equation that takes into account the effect of the viscous stress. Further, the condition (iii) may not be imposed because the local velocity at the pores must be different from zero.

Beavers and Joseph[5] have proposed an empirical slip boundary condition as an improvement on the condition (iii). Saffman[6] derived a general govering equation for the flow in a porous medium by use of a statistical method. The resulting equation is complicated and has a term of viscous stress just as the so-called generalized Darcy's law. Using the equation, he gave a similar boundary condition to that of Beavers an Joseph. Taylor[7] and Richardson[8] invented a special model of porous material for which an exact analysis can be carried out, and examined the boundary condition suggested by Beavers and Joseph. These studies[5~8] treat the case of only one dimensional flow .

In a previous paper,[9] the present author investigated a slow flow of viscous fluid past a porous sphere and discussed the boundary condition for the Stokes and Darcy equations. The present study is an extension and generalization of that work. That is, we consider the flow of viscous fluid at small Reynolds numbers past a porous and permeable body of arbitrary but smooth shape. The flow in the pure fluid region is assumed to governed by the Stokes equation, besides, it is assumed that the generalized Darcy's law, which contains a term of viscous stress (cf.(2.5)), holds in the whole region of porous medium including neighbourhood of the boundary. We investigate the asymptotic behavior of the flow for small permeability of the porous medium.

§ 2. Fundamental Equations

We consider the steady flow of viscous fluid at small Reynolds numbers past a porous body. Let the representative speed and length of the flow be Q_0 and L , respectively. We take the rectangular coordinates $X=(X,Y,Z)$. It is assumed that the flow in the pure fluid region is governed by the Stokes equations:

$$\operatorname{div} q = 0, \tag{2.1}$$

$$\operatorname{grad} p = \Delta q, \tag{2.2}$$

$$\hat{p} = \frac{\mu Q_0}{L} p, \quad \hat{q} = Q_0 q = Q_0(q_x, q_y, q_z), \quad X = LX\ , \tag{2.3}$$

where $\hat{p}$ is the pressure, $\hat{q}$ the velocity and μ the viscosity.

We further assume that the generalized Darcy's law, which was first suggested by Brinkman,[2] holds in a whole region of the porous medium. Then, the flow equations in the porous region are*,

$$\operatorname{div} Q = 0, \tag{2.4}$$

$$\operatorname{grad} P = \Delta Q - S^2 Q\ , \tag{2.5}$$

$$S^2 = L^2 / k\ , \tag{2.6}$$

where k is the permeability of the porous medium. We take k to be constant for simplicity. The generalized Darcy equation is derived analytically by Tam[10] as an equation which describes the flow of viscous fluid at low Reynolds numbers past a swarm of small spherical particles** held stationally in space. It is to be expected that the generalized Darcy's law will give good results in the case of highly porous media such as fibers.

We now consider appropriate boundary conditions to be applied at the surface of the porous medium to the Stokes and generalized Darcy equations. It may be required that the tangential velocity at the surface should be continuous because of the viscosity. The normal velocity should also be continuous, for the mass flux across the surface is conserved. Taking into account that the force applied to the fluid by the porous medium is a body force, we may take that the shear and normal stresses are also continuous.

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* Upper-case letters refer to the quantities evaluated in the porous medium.

** The porosity of this medium is unity.

Magnitude of permeability of ordinary porous media is very small and of the order of $10^{-5} \sim 10^{-8}$ cm^2.[2)] In this regard, we consider the flow past a porous medium with small permeability and investigate the asymptotic behavior of the flow for large S on the basis of the Stokes and generalized Darcy equations together with the boundary conditions mentioned above.

§ 3. Asymptotic Theory for Large S

3.1 Darcy's law

First, we discuss the flow in the main region* of the porous medium where the quantities of the flow field do not change abruptly. Considering that the parameter S is large, we may expand the velocity and the pressure in power series in S^{-1} as follows :

$$\left.\begin{aligned} P &= P_D = P_D^{(0)} + S^{-1}P_D^{(1)} + S^{-2}P_D^{(2)} + \dots, \\ Q &= Q_D = S^{-2}Q_D^{(2)} + S^{-3}\, Q_D^{(3)} + \dots . \end{aligned}\right\} \tag{3.1}$$

Inserting these expansions into eqs.(2.4) and (2.5) and equating the same order terms in S , we have the results :

$$\operatorname{div} Q_D^{(i)} = 0 \qquad (i=2,3,\dots), \tag{3.2}$$

$$\operatorname{grad} P_D^{(j)} = -\, Q_D^{(j+2)} \qquad (j=0,1)\ , \tag{3.3}$$

$$\operatorname{grad} P_D^{(i)} = \Delta Q_D^{(i)} - Q_D^{(i+2)} \qquad (i=2,3,\dots)\ . \tag{3.4}$$

From (2.4) and (2.5), the pressure in the porous medium is found to satisfy the Laplace equation :

$$\Delta P = 0\ . \tag{3.5}$$

Hence, we get

$$\Delta P_D^{(j)} = 0 \quad (j=0,1,2,\dots)\ . \tag{3.6}$$

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* We denote the quantities in this region by subscript D .

Applying the Laplacian operator to eq.(3.3) and using eq.(3.6), we have

$$\Delta Q_D^{(j)} = 0 \quad (j=2,3) \ . \tag{3.7}$$

Putting this into eq.(3.4), we have

$$\mathrm{grad}\ P_D^{(j)} = -Q_D^{(j+2)} \quad (j=2,3) \ .$$

Repeating the same procedure successively, we can easily show that

$$\mathrm{grad}\ P_D^{(i)} = -\ Q_D^{(i+2)} \quad (i=0,1,2,\ldots) \ . \tag{3.8}$$

This is nothing but Darcy's law. Thus, it turns out that Darcy's law governs the main or asymptotic field in the porous medium to any order of approximation in S. The Darcy equations (3.2) and (3.8) are rewritten by (3.1) as follows :

$$\mathrm{div}\ Q_D = 0 \quad , \quad \mathrm{grad}\ P_D = -S^2 Q_D \ . \tag{3.9}$$

In the asymptotic field, the fluid is pushed on through the porous medium by the action of the pressure against the body force. However, as is seen in the case of flow past a porous sphere,[9] there appears a boundary layer near the surface of the porous medium, where the viscosity affects the flow directly. We must appeal to the generalized Darcy's law in order to analyse this layer.

3.2 Fundamental equations in orthogonal curvilinear coordinates

We consider a porous wall of arbitrary but smooth shape. It is convenient for the analysis of the boundary layer adjacent to the surface to introduce orthogonal curvilinear coordinates. We take x_3 as a coordinate along the uint normal $\mathbf{n}$ to the boundary (pointed into the porous medium) and x_1, x_2 coordinates within a parallel surface x_3=const. , then

$$\mathbf{x} = x_3\,\mathbf{n}(x_1,x_2) + \mathbf{x}_W(x_1,x_2) \ , \tag{3.10}$$

where $\mathbf{x}_W = (x_W, y_W, z_W)$ is the position vector on the wall. For simplicity, we shall take the coordinates x_1 and x_2 in such a way that the coordinate lines coincide with the lines of principal curvature of the boundary. Then, the system of coordinates is triply orthogonal everywhere. Let R_1 and R_2 be the principal radii of curvature of the boundary*. We take $R_i > 0$ when the normal points towards the center of principal curvature. From eq.(3.10) and Rodrigues' formula,[11] the metrical coefficients[12] are given by

$$H_1^{-1} = (1 - \frac{x_3}{R_1})A_1^{-1},\ H_2^{-1} = (1 - \frac{x_3}{R_2})A_2^{-1},\ H_3^{-1} = 1\ , \tag{3.11}$$

where

$$\frac{1}{A_i^2} = \left(\frac{\partial x_W}{\partial x_i}\right)^2 + \left(\frac{\partial y_W}{\partial x_i}\right)^2 + \left(\frac{\partial z_W}{\partial x_i}\right)^2 . \tag{3.12}$$

The fundamental equations (2.4) and (2.5) are now rewritten as follows** :

$$H_1 H_2 \left\{\frac{\partial}{\partial x_1}\left(\frac{Q_1}{H_2}\right) + \frac{\partial}{\partial x_2}\left(\frac{Q_2}{H_1}\right) + \frac{\partial}{\partial x_3}\left(\frac{Q_3}{H_1 H_2}\right)\right\} = 0\ , \tag{3.13}$$

$$H_1 \frac{\partial P}{\partial x_1} = H_2 \frac{\partial}{\partial x_3}\left[\frac{H_1}{H_2}\left\{\frac{\partial}{\partial x_3}\left(\frac{Q_1}{H_1}\right) - \frac{\partial Q_3}{\partial x_1}\right\}\right] - H_2 \frac{\partial}{\partial x_2}\left[H_1 H_2 \left\{\frac{\partial}{\partial x_1}\left(\frac{Q_2}{H_2}\right) - \frac{\partial}{\partial x_2}\left(\frac{Q_1}{H_1}\right)\right\}\right] - S^2 Q_1\ , \tag{3.14}$$

$$H_2 \frac{\partial P}{\partial x_2} = H_1 \frac{\partial}{\partial x_1}\left[H_1 H_2 \left\{\frac{\partial}{\partial x_1}\left(\frac{Q_2}{H_2}\right) - \frac{\partial}{\partial x_2}\left(\frac{Q_1}{H_1}\right)\right\}\right] - H_1 \frac{\partial}{\partial x_3}\left[\frac{H_2}{H_1}\left\{\frac{\partial Q_3}{\partial x_2} - \frac{\partial}{\partial x_3}\left(\frac{Q_2}{H_2}\right)\right\}\right] - S^2 Q_2\ , \tag{3.15}$$

$$\frac{\partial P}{\partial x_3} = H_1 H_2 \frac{\partial}{\partial x_2}\left[\frac{H_2}{H_1}\left\{\frac{\partial Q_3}{\partial x_2} - \frac{\partial}{\partial x_3}\left(\frac{Q_2}{H_2}\right)\right\}\right] - H_1 H_2 \frac{\partial}{\partial x_1}\left[\frac{H_1}{H_2}\left\{\frac{\partial}{\partial x_3}\left(\frac{Q_1}{H_1}\right) - \frac{\partial Q_3}{\partial x_1}\right\}\right] - S^2 Q_3\ , \tag{3.16}$$

where Q_i is the x_i-component of the velocity.

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* The principal radius of curvature R_i is also normalized by L.

** Details of the transformation from the Cartesian to the orthogonal curvilinear coordinates may be found in ref.12.

As previously stated, it may be required that the velocity and the shear and normal stresses are continuous across the surface. These boundary conditions are written by use of the continuity equations as follows :

$$q_i = Q_i \quad (i=1,2,3) \ , \tag{3.17}$$

$$p = P \ , \tag{3.18}$$

$$\frac{\partial q_1}{\partial x_3} = \frac{\partial Q_1}{\partial x_3} \ , \tag{3.19}$$

$$\frac{\partial q_2}{\partial x_3} = \frac{\partial Q_2}{\partial x_3} \ . \tag{3.20}$$

3.3 Analysis of the boundary layer and boundary conditions for the Stokes and Darcy equations

We now proceed to the analysis of the boundary layer. Considering that the thickness of this layer is of $O(S^{-1})$,[9] we introduce a new stretched coordinate η related to x_3 by

$$\eta = Sx_3 \ . \tag{3.21}$$

Taking into account that any solution of the generalized Darcy's law should tend to that of Darcy's law as $\eta \to \infty$, we assume the solution for the boundary layer in the following forms :

$$Q_i = Q_{i,B} + Q_{i,D} = S^{-1}Q^{(1)}_{i,B}(x_1,x_2,\eta) + S^{-2}\{Q^{(2)}_{i,D}(x_1,x_2,x_3) + Q^{(2)}_{i,B}(x_1,x_2,\eta)\} +.. \tag{3.22}$$

The correction terms of the boundary layer should vanish as $\eta \to \infty$, i.e.

$$Q^{(j)}_{i,B} \to 0 \qquad (\ \eta \to \infty \ , \ j=1,2,3,..). \tag{3.23}$$

The pressure is expected to have no abrupt change of the boundary layer type because it is a solution of the Laplace equation (3.5) :

$$P = P_D(x_1,x_2,x_3) = P^{(0)}_D + S^{-1}P^{(1)}_D +\dots \ . \tag{3.24}$$

The solution of the Stokes equation is also expanded in power series in S^{-1}:

$$q_i = q_i^{(0)} + S^{-1}q_i^{(1)} + S^{-2}q_i^{(2)} +.. \; (i=1,2,3) \;, \qquad (3.25)$$

$$p = p^{(0)} + S^{-1}p^{(1)} + \dots \;. \qquad (3.26)$$

Each order term of these expansions should satisfy the Stokes equations,i.e.

$$\operatorname{div} q^{(m)} = 0 \;\;, \;\; \operatorname{grad} p^{(m)} = \Delta q^{(m)} \;\; (m=0,1,\dots). \qquad (3.27)$$

Inserting eqs.(3.21),(3.22),(3.24)∿(3.26) in the boundary conditions (3.17)∿(3.20) and equating the same order terms in S on both sides, we have the next relations at $\eta = 0$ (or $x_3 = 0$) :

$$q_i^{(0)} = 0 \;, \;\; p^{(0)} = P_D^{(0)} \;, \qquad (3.28\text{ a,b})$$

$$q_i^{(1)} = Q_{i,B}^{(1)} \;, \quad \partial q_j^{(0)}/\partial x_3 = \partial Q_{j,B}^{(1)}/\partial\eta \;(j=1,2), \; p^{(1)}=P_D^{(1)} \;, \qquad (3.29\text{a,b,c})$$

$$q_i^{(2)}=Q_{i,D}^{(2)}+Q_{i,B}^{(2)} \;, \quad \partial q_j^{(1)}/\partial x_3 = \partial Q_{j,B}^{(2)}/\partial\eta \;\; (j=1,2), \;\; p^{(2)} = P_D^{(2)}. \qquad (3.30\text{a,b,c})$$

Equation (3.28a) is the boundary condition for the Stokes equations (3.27) of the zeroth order (m=0) and is seen to be the same condition as on a non-permeable wall. Once the solution of the Stokes equations subject to the boundary condition (3.28a)is known, the condition (3.28b)may be used to solve eq.(3.6) and to find the pressure in the porous medium. From eq.(3.8), we then obtain the velocity $Q_D^{(2)}$ in the asymptotic field.

We now substitute eqs.(3.21), (3.22) and (3.24) into eqs.(3.11), (3.13) ∿ (3.16). Expanding the results for large S and considering that Q_D and P_D satisfy the Darcy equations, we get the boundary layer equations for $Q_{i,B}^{(j)}$. The first order equations to be satisfied by $Q_{i,B}^{(1)}$ are given by

$$\frac{\partial^2 Q_{i,B}^{(1)}}{\partial\eta^2} - Q_{i,B}^{(1)} = 0 \;(i=1,2) \;, \qquad (3.31)$$

$$Q_{3,B}^{(1)} = 0 \;. \qquad (3.32)$$

The solution of eq.(3.31) subject to the conditions (3.29b)and (3.23) is

$$Q_{i,B}^{(1)} = -\left(\frac{\partial q_i^{(0)}}{\partial x_3}\right)_{x_3=0} \exp(-\eta) \quad (i=1,2). \qquad (3.33)$$

On substitution of this solution and (3.32) into the condition (3.29a) we have the following conditions at $x_3=0$ for the Stokes solution :

$$q_i^{(1)} = -\frac{\partial q_i^{(0)}}{\partial x_3} \quad (i=1,2) , \quad q_3^{(1)} = 0. \tag{3.34}$$

It is found that the normal velocity (q_3) at the surface is still zero up to this order of approximation. The boundary condition for the pressure $P_D^{(1)}$ is given by eq.(3.29c).

It is easy to proceed to second order approximation. The tangential velocity in the boundary layer is

$$Q_{i,B}^{(2)} = -\left\{\left(\frac{\partial q_i^{(1)}}{\partial x_3}\right)_{x_3=0} + \frac{1}{2}\{R_1^{-1} + R_2^{-1}\}\left(\frac{\partial q_i^{(0)}}{\partial x_3}\right)_{x_3=0} (1+\eta)\right\} e^{-\eta} \quad (i=1,2). \tag{3.35}$$

Inserting this solution in (3.30a), we have the following relation at $x_3=0$

$$q_i^{(2)} = Q_{i,D}^{(2)} - \frac{\partial q_i^{(1)}}{\partial x_3} - \frac{1}{2}(R_1^{-1} + R_2^{-1}) \frac{\partial q_i^{(0)}}{\partial x_3} \quad (i=1,2) . \tag{3.36}$$

The normal velocity is as follows :

$$Q_{3,D}^{(2)} = -\left(\frac{\partial q_3^{(1)}}{\partial x_3}\right)_{x_3=0} \exp(-\eta) . \tag{3.37}$$

Putting this expression into (3.30a), we obtain the following relation

$$q_3^{(2)} = Q_{3,D}^{(2)} - \frac{\partial q_3^{(1)}}{\partial x_3} \quad \text{at } x_3 = 0 . \tag{3.38}$$

Equations (3.36), (3.38) and (3.30c) are the relationships to be satisfied at the surface by the solutions of the Stokes and Darcy equations and thus constitute the second order boundary conditions for these equations.

Using eqs.(3.1), (3.25) and (3.26) and considering that the boundary conditions obtained so far are correct up to the order of S^{-2}, we get the refined forms of these conditions as follows :

$$p = P_D , \tag{3.39}$$

$$q_i = Q_{i,D} - S^{-1} \frac{\partial q_i}{\partial x_3} - \frac{1}{2} S^{-2} (R_1^{-1} + R_2^{-1})\frac{\partial q_i}{\partial x_3} \quad (i=1,2), \tag{3.40}$$

$$q_3 = Q_{3,D} - S^{-1} \partial q_3 / \partial x_3 . \tag{3.41}$$

It may be noted that a term representing the effect of wall curvature appears in the slip condition for the tangential velocity (3.40). When we take $R_1 = R_2 = \infty$ in (3.40), this equation reduces to the form which is suggested by Beavers and Joseph empirically. It may also be mentioned that the condition of continuity of the normal velocities q_3 and $Q_{3,D}$ does not hold, but there occurs a jump between them. The jump is due to the change of the tangential velocity* along the surface in the boundary layer.

The velocity in the boundary layer which is given by eqs.(3.33), (3.35) and (3.37) may also be refined in the forms :

$$Q_{i,B} = -S^{-1}\left(\frac{\partial q_i}{\partial x_3}\right)_{x_3=0}\left\{ 1 + \frac{1}{2}S^{-1}(R_1^{-1}+R_2^{-1})(1+\eta)\right\} \exp(-\eta) \ , \qquad (3.42)$$

$$Q_{3,B} = -\ S^{-1}\left(\frac{\partial q_3}{\partial x_3}\right)_{x_3=0} \exp(-\eta) \ . \qquad (3.43)$$

When a solution of the Stokes and Darcy equations (2.1), (2.2) and (3.9) subject to the boundary conditions (3.39) $\sim$(3.41) is obtained, the flow in the boundary layer is found from eqs.(3.42) and (3.43).

In this study, we have investigated the steady flow at low Reynolds numbers past a porous body and developed an asymptotic theory for large S on the basis of the linearized equations in which the convection terms are neglected. In view of practical purposes, it seems worth while to extend the analysis to the case of large Reynolds numbers. Then the full Navier-Stokes equation and a generalized Darcy's law which has a convection term must be used. The subject will however be left for the future work.

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* By use of the continuity equation, $\partial q_3 \ / \ \partial x_3$ may be related to the change of the tangential velocity.

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