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Gentzen-Style Formulation of Systems of Set-Calculus

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We shall give Gentzen-style formulation for several systems of infinitary set-calculus. Most simple system will be given in §1. This system contains only the operators for constructing regular sets, that is, '·' (concatenation), '+' (sum) and '$\bigcup_{n=0}^{\infty}$' (infinite sum). However, basic sets for the present system are arbitrary. It is easily proved that this system include the system given by A. Saloma [3]. And in this system we can define the fixed point operator and prove the computational induction for suitable predicate without dificulty. In § 2 we shall prove plausibility, completeness and elimination of redundance. Elimination of redundance corresponds to Gentzen's cut-elimination [1]. The method given by K. Schutte [4] will be applied. In § 3, the system is somewhat extended so that the interpolation-theorem holds. In § 4 we shall extend the system by combining formula. The extended system will be able to contribute to axiomatic basis for programming. In § 5 we shall some comments to the system given by C. A. R. Hoare [2].

§1 Gentzen-style formulation I

Basic symbols are following: (1) Constant symbols λ, ϕ, a, b, c,... (2) variables x, y, z,... (3) operation symbols +, $\cdot$, $\bigcup$, and (4) the symbol $\subset$, Regular <u>expressions</u> and their <u>degrees</u> which are ordinal numbers $< \omega^{\omega}$, are defined recursively as follows, where the degree of the expression η is denoted by d(η). (1) A constant symbol or a variable is a regular expression with the degree 0. A constant or a variable is called a literal.
(2) If $\alpha_1, \ldots, \alpha_n, \ldots,$ α and β are regular expressions, so are $\alpha \cdot \beta$, $\alpha + \beta$, and $\bigcup_{n=0}^{\infty} \alpha^n$, where d($\alpha \cdot \beta$) = d(α) + d(β), d($\alpha + \beta$) = d(α) + d(β) +1 and d($\bigcup_{n=0}^{\infty} \alpha^n$) = (d(α) + 1)$\cdot\omega$. (2) (extreme clause) Regular expressions are obtained only by (1) and (2). We apply the following abbreviations: α^0 for the constant λ , α^n for $\underbrace{\alpha \cdot \alpha \cdots \alpha}_{n}$ and $\alpha\beta$ for $\alpha \cdot \beta$. We denote an expression $\alpha_1 \alpha_2 \cdots \alpha_n$ by term, where α_j's are a literal. We say that an expression α has the empty word property (ewp) when α is of the form λ , $\bigcup_{n=0}^{\infty} \eta^n$, or α is of the form $\eta + \xi$ or $\eta\xi$ and η and ξ have ewp. Ingerently, ' a regular expression α has ewp ' means that the set represented by α contains the element λ .

When $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n$ are regular expressions then the figure of the following form is called a <u>sequent</u>:

$$\alpha_1, \ldots, \alpha_m \subset \beta_1, \ldots, \beta_n$$

Inherently, it means that $\alpha_1 \cdot \cdots \cdot \alpha_m \subset \beta_1 + \ldots + \beta_n$, where ' $\subset$ ' is the set-inclusion and ' + ' or ' $\cdot$ ' is the set sum or product.

Now we give axioms and rules of inference of our system. In what follows, by Greek capital letters such as Γ, Δ, Π etc. we denote finite set of expressions.

Axioms are sequents of the following form:

$$\alpha_1, \ldots, \alpha_m \subset \Delta_1, \alpha, \Delta_2 \quad \text{or} \quad \phi \subset \Delta$$

where α is an arbitrary regular expression and $\alpha_1 \cdot \cdots \cdot \alpha_m$ is α and ϕ is a particular constant (inherently, denote the empty set).

Rules of inference are of the following form

$$\frac{S_1,\ S_2, \ldots,\ S_n, \ldots}{S}$$

where S_1, $S_2, \ldots$, $S_n, \ldots$ and S are sequents. S_1, $S_2, \ldots$ are called the upper sequents and S the lower sequent of this rule. They are the followings:

I. (1) We can replace an arbitrary expression α in a sequent by $\lambda\alpha$ or $\alpha\lambda$ and conversely, where λ is the particular constant (inherently, denoting the empty word). We can also replace $\phi\alpha$ or $\alpha\phi$ by ϕ and conversely, where ϕ is a particular constant (inherently, denoting the empty set). If the left hand-side contains ϕ, we can replace the left hand-side by ϕ.

(2) $\dfrac{\Gamma \subset \Delta}{\Gamma \subset \Sigma}$ for Σ containing every expression in Δ.

II. Rules with respect to connectives

(1) $$\frac{\Gamma_1, \alpha\eta\beta, \Gamma_2 \subset \Delta \qquad \Gamma_1, \alpha\xi\beta, \Delta_2 \subset \Delta}{\Gamma_1, \alpha(\eta + \xi)\beta, \Gamma_2 \subset \Delta}$$

$$\frac{\Gamma \subset \Delta_1, \alpha\eta\beta, \alpha\xi\beta, \Delta_2}{\Gamma \subset \Delta_1, \alpha(\eta + \xi)\beta, \Delta_2}$$

(2) $$\frac{\Gamma_1, \alpha_1\alpha_2\beta^n\gamma_1\gamma_2, \Gamma_2 \subset \Delta \qquad n=0,1,2,\ldots}{\Gamma_1, \alpha_1(\bigcup_{n=0}^{\infty}\alpha_2\beta^n\gamma_1)\gamma_2, \Gamma_2 \subset \Delta}$$

$$\frac{\Gamma \subset \Delta_1, \alpha_1\alpha_2\beta^n\gamma_1\gamma_2, \Delta_2}{\Gamma \subset \Delta_1, \alpha_1(\bigcup_{n=0}^{\infty}\alpha_2\beta^n\gamma_1)\gamma_2, \Delta_2}$$

III. (1) Cut $$\frac{\Gamma \subset \Delta_1, \gamma, \Delta_2 \qquad \gamma \subset \Lambda}{\Gamma \subset \Delta_1, \Delta_2, \Lambda}$$

is called the cut-expression.

(2) Elimination

$$\frac{\Gamma \subset \Delta_1, \gamma\beta^n\xi, \Delta_2 \qquad n=N,N+1,\ldots}{\Gamma \subset \Delta_1, \Delta_2}$$

where N is an arbitrary positive integer and β has not ewp.

(3) $$\frac{\Gamma \subset \Delta \qquad \Pi \subset \Sigma}{\Gamma, \Pi \subset \Delta, \Sigma} \qquad \frac{\Gamma_1, \alpha, \beta, \Gamma_2 \subset \Delta}{\Gamma_1, \alpha\beta, \Gamma_2 \subset \Delta}$$

where $\Gamma\Pi$ or $\Delta\Sigma$ is $\alpha_1 \cdots \alpha_h\, \beta_1 \cdots \beta_k$ or $\gamma_1\xi_1, \ldots, \gamma_1\xi_n, \gamma_2\xi_1, \ldots, \gamma_m\xi_n$ if Γ, Π, Δ or Σ is $\alpha_1, \ldots, \alpha_h$; $\beta_1, \ldots, \beta_k$; $\gamma_1, \ldots, \gamma_m$ or $\xi_1, \ldots, \xi_n$ respectively.

The proofs of our system are defined recursively as follows: (1) An axiom is a proof. (2) If $\mathfrak{P}_1, \mathfrak{P}_2, \ldots$ are proofs with the lowermost sequents $S_1, S_2, \ldots$ respectively, and if

$$\frac{S_1 \qquad S_2 \qquad \cdots\cdots}{S}$$

is a rule of inference then

$$\frac{\mathfrak{P}_1 \qquad \mathfrak{P}_2 \qquad \cdots\cdots}{S}$$

is a proof. (3) Proof is obtained only by applying the above (1) and (2). The proof to S is that which has S as the lowermost sequent. When we have a proof to S, we say that S is provable. In particular, we say that S is strictly provable when S is provable by applying only I and II.

§ 2. Plausibility, Completeness and Elimination of Redundance

First we give an interpretation.

Difinition 1. We make every expression α correspond to an mathematical entity $|\alpha|$. $|\alpha|$ is defined as follows:

(1) $|\phi|$ = the empty set. $|\lambda|$ = the set consisting only of the empty word.

(2) If a is a constant or a variable, then $|a| = \{a\}$.

(3) $|\alpha+\beta| = |\alpha| \cup |\beta|$, $|\alpha\cdot\beta| = |\alpha|\cdot|\beta|$ and $\bigcup_{n=0}^{\infty} \alpha_n = \bigcup_{n=0}^{\infty} |\alpha_n|$.

$|\alpha_1,\ldots,\alpha_m \subset \beta_1,\ldots,\beta_n|$ is $|\alpha_1|\ldots|\alpha_m| \subset |\beta_1| \cup \cdots \cup |\beta_n|$

where '$\subset$' is the set inclusion.

Theorem 1. (Plausibility) Every provable sequent is true under the above interpretation.

Proof. Axioms are trivially true. It is clear that the rules of inference I, II and III (1) (3), transforms the true sequents to a true sequent. When β has not ewp, $\bigcap_{n=N}^{\infty} \gamma\beta^n\xi = \phi$. Hence we can see that the III (2) transforms a true sequent to a true one,

q.e.d.

Theorem 2. (Completeness and Elimination of redundance)

Let $\alpha_1,\ldots,\alpha_m$ and $\beta_1,\ldots,\beta_n$ are arbitrary regular expressions. If $|\alpha_1,\ldots,\alpha_m \subset \beta_1,\ldots,\beta_n|$ holds for an interprelation $|\ \ |$, then the sequent $\alpha_1,\ldots,\alpha_m \subset \beta_1,\ldots,\beta_n$ is strictly provable.

We shall give some preliminaries. First we can easily see the following :

$d(\alpha) = 0$ iff α is a term,

$d(\alpha\eta\beta),\ d(\alpha\xi\beta) < d(\alpha(\eta+\xi)\beta)$,

$d(F(\alpha^n)) < d(\bigcup_{n=0}^{\infty} F(\alpha^n))$

$d(F(\alpha^n)) < d\ (F(\bigcup_{n=0}^{\infty} \alpha^n))$

We define the degree of a sequent as the sum of the degrees of the expressions in the left hand-side.

Next we shall give the left decomposition of a sequent by the transfinite induction on the degree κ of the sequent.

(1) In the case where $\kappa = 0$, the expression in the left hand-side of the sequent is a sequence of terms. Then, if it contains ϕ, we replace the left hand-side ϕ. Then the decomposition terminates. In the case where $\kappa > 0$, we carry out as follows.

(2) If the sequent is of the form

$\Gamma_1,\ \alpha(\eta+\xi)\beta,\ \Gamma_2 \subset \Delta$, then we decompose it to the two sequents $\Gamma_1,\ \alpha\eta\beta,\ \Gamma_2 \subset \Delta$ and

$\Gamma_1,\ \alpha\xi\beta,\ \Gamma_2 \subset \Delta$ which degrees are less than that of the original.

(3) If the sequent is of the form

$\Gamma_1, \alpha_1(\bigcup_{n=0}^{\infty} \alpha_2 \beta^n \gamma_1)\gamma_2, \Gamma_2 \subset \Delta$, then we decompose it to infinitely many sequents $\Gamma_1, \alpha_1\alpha_2\beta^n\gamma_1\gamma_2, \Gamma_2 \subset \Delta$ $n=0,1,2,\ldots$ which degrees are less than that of the original.

We can easily see that, if every decomposed sequent is strictly provable, then the original is strictly provable, And we can see that the sequent with the degree 0 is of the form $\alpha_1, \alpha_2, \ldots, \alpha_n \subset \Delta$,

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are terms. And we can easily see that, if the original sequent is of the form $\Gamma \subset \Delta$, 11

then $|\alpha_1, \ldots, \alpha_n \subset \Gamma|$. When we continue (1) and (2), every sequent is decomposed to (infinitely or finitely) many sequents which degrees are 0.

Next we define the right decomposition of a sequent which degree is zero. From left to right in the right hand-side, we search an expression which contains + or $\overset{\infty}{\underset{n=0}{\cup}}$. If we have no such expression or the sequent is an axiom, then the decomposition terminates.

(1) Let such the first expression be $\alpha(\eta + \xi)\beta$ and the sequent be of the form $\Gamma \subset \Delta_1, \alpha(\eta + \xi)\beta, \Delta_2$. Then it is decomposed to $\Gamma \subset \Delta_1, \alpha\eta\beta, \alpha\xi\beta, \Delta_2$.

(2) Let such the first expression be $\alpha_1(\overset{\infty}{\underset{n=0}{\cup}} \alpha_2 \beta^n \gamma_1)\gamma_2$ and the sequent be of the form $\Gamma \subset \Delta_1, \alpha_1(\overset{\infty}{\underset{n=0}{\cup}} \alpha_2 \beta^n \gamma_1)\gamma_2, \Delta_2$. Then, if the decomposition is the n-th, it is decomposed to

$$\Gamma \subset \Delta_1, \alpha_1\alpha_2\beta^0\gamma_1\gamma_2, \alpha_1\alpha_2\beta^1\gamma_1\gamma_2, \ldots, \alpha_1\alpha_2\beta^n\gamma_1\gamma_2, \Delta_2$$

(3) We can easily see that, if the decomposed one is strictly provable, so is the original.

Proof of Theorem 2.

Now we shall give the proof of Theorem 2. In order to do so, it is sufficient to prove that, for every term t, if the sequent $t \subset \Delta$ is strictly not-provable, then $|t \subset \Delta|$ is not true.

If the sequent were strictly not provable, then we should have a branch which satisfies the following properties ;

(1) Let the left hand-sides of the sequents in this branch be $\Gamma_1, \ldots, \Gamma_m$. Then $\Gamma_1 \ldots \Gamma_m$ does not

occur in the right hand-sides of them.

(2) If an expression of the form $\beta(\eta + \xi)\gamma$ appears in the right hand-side of some sequent of this branch, then the expressions $\beta\eta\gamma$ and $\beta\xi\gamma$ appear in the right-hand-side of some sequent of this branch.

(3) If the right hand-side of the sequent in this branch contains an expression of the form

$$\alpha_1(\bigcup_{n=0}^{\infty} \alpha_2 \beta^n \gamma_1)\gamma_2 \quad , \quad \text{then} \quad \alpha_1\alpha_2\beta^n\gamma_1\gamma_2$$

for every $n=0,1,2,...$ is contained in the right hand-side of some sequent in this branch.

Let $\alpha_1,...,\alpha_m$ be the left hand-sides of this branch. It is sufficient to prove that $t \notin |\delta|$ for an arbitrary expression δ in the right hand-side of this branch, where t is $\alpha_1 \ldots \alpha_m$. We shall prove this by the transfinite induction on the degree $d(\delta)$ of δ. If $d(\delta) = 0$, then it is clear by the fact that δ is a term. In the case where δ is of the form $\beta(\eta + \xi)\gamma$, we have $t \notin |\beta\eta\gamma|$ and $t \notin |\beta\xi\gamma|$ by the induction-hypothesis, because

$d(\beta\eta\gamma), d(\beta\xi\gamma) < d(\beta(\eta + \xi)\gamma)$ and both $\beta\eta\gamma$ and $\beta\xi\gamma$ appear in the right by the property (2). Therefore $t \notin |\beta(\eta + \xi)\gamma|$. In the case where δ is of the form $\alpha_1(\bigcup_{n=0}^{\infty}\alpha_2\beta^n\gamma_1)\gamma_2$ we have $t \notin |\alpha_1\alpha_2\beta^n\gamma_1\gamma_2|$ $n=0,1,2,...$ by the induction-hypothesis because $d(\alpha_1\alpha_2\beta^n\gamma_1\gamma_2) < d(\alpha_1(\bigcup_{n=0}^{\infty}\alpha_2\beta^n\gamma_1)\gamma_2$ $n=0,1,2,...$ and every $\alpha_1\alpha_2\beta^n\gamma_1\gamma_2$ appears in the right by the property (3).

§ 3. Extended system and Interpolation Theorem.

We extend the definition of expressions and their degree as follows : we omit (3) and add the following (3)'.

(3)' If $\alpha_0, \alpha_1, \dots, \alpha_n, \dots,$ α and β are expressions, so are $\alpha \cdot \beta$, $\alpha + \beta$ and $\bigcup_{n=0}^{\infty} \alpha_0 \alpha_1 \cdots \alpha_n$ where $d(\alpha \cdot \beta) = d(\alpha) + d(\beta)$, $d(\alpha + \beta) = d(\alpha) + d(\beta) + 1$ and $d(\bigcup_{n=0}^{\infty}(\alpha_0 \alpha_1 \cdots \alpha_n)) = \sup_n (d(\alpha_0 \alpha_1 \cdots \alpha_n) + 1)$.

Moreover, we change $\alpha_1 \alpha_2 \beta^n \gamma_1 \gamma_2$ to $\alpha_1 \alpha_2 \beta_0 \beta_1 \cdots \beta_n \gamma_1 \gamma_2$ in the rules of inference II (2).

Thus we have an extended system. In this system we have the theorem 1 and 2 in § 2. Applying theorem 2 we have the interpolation theorem in the extended system.

§ 4. Extended system obtained by combining formulas

We extend the system given in § 1 by combining formulas. As basic symbols we add the following other than given in § 1; bound variables (constants and free variables are given in § 1) ; predicate symbols; logical connectives $\neg$ (negation), $\exists$ (exist), $\forall$ (for all); constant for formula $\top$. Besides the above, ϕ (the constant for regular expression) is also used as the constant for formula. Moreover, the connective '.' or '+' is also used as the logical connective and as the connective combining extended expressions.

Formulas are defined as usual, where '.' is used for 'and' and '+' for 'or'.

Then expressions are defined as follows : (1) A regular expression or a formula is an expression. (2) If A and B are expressions, so is $A \cdot B$, $A+B$ or $\bigcup_{n=0}^{\kappa} A^n$.

(3) Expression are obtained only by applying (1) and (2).

As the axioms, the sequents of the form

$$\Gamma \subset \Delta_1, T, \Delta_2$$

are added.

Rules of inference are added or modified as follows:

I (1) is extended as follows:

$$\frac{\Gamma \subset \Delta}{\Phi, \Gamma, \Psi \subset \Lambda}$$

where Φ and Ψ are consist only of formulas, and every expression in Δ is contained in Λ .

To the group I, we add the following I (4).

I (4) If α is an expression, then $T\alpha$ or αT can be replaced by α , and conversely. If P is a formula, then $(\neg P \cdot P$ or $P \cdot (\neg P)$ is replaced by ϕ .

To the group II, we add the following II (4).

II (4) A formula $\neg(P+Q)$, $\neg(PQ)$ $\neg\exists xF$ or $\neg\forall xF$ is replaced by $\neg P \cdot \neg Q$ $\neg P + \neg Q$ $\forall x \neg F$ or $\exists x \neg F$ respectively, and conversely.

II (5) $$\frac{\Gamma_1, \alpha \cdot F(a) \cdot \beta, \Gamma_2 \subset \Delta}{\Gamma_1, \alpha \cdot \exists xF(x) \cdot \beta, \Gamma_2 \subset \Delta} \qquad \frac{\Gamma \subset \Delta, \alpha \cdot F(b) \cdot \beta, \alpha \cdot \exists xF(b) \cdot \beta}{\Gamma \subset \Delta, \alpha \cdot \exists xF(b) \cdot \beta}$$

a is eigen-variable

b is an arbitrary constant or a free variable

II (6) $$\frac{\Gamma_1, \alpha \cdot F(b) \cdot \forall xF(x) \cdot \beta, \Gamma_2 \subset \Delta}{\Gamma_1, \alpha \cdot \forall xF(x) \cdot \beta, \Gamma_2 \subset \Delta} \qquad \frac{\Gamma \subset \Delta, \alpha \cdot F(a) \cdot \beta}{\Gamma \subset \Delta, \alpha \cdot \forall xF(x) \cdot \beta}$$

b is an arbitrary constant or a free variable

a is eigen-variable

III (1) is modified as follows:

$$\frac{\Gamma \subset \Delta_1, \gamma, \Delta_2 \qquad \Gamma_1, \gamma, \Gamma_2 \subset \Delta}{\Gamma, \Gamma_1, \Gamma_2 \subset \Delta_1, \Delta_2, \Delta}$$

where Γ_1 and Γ_2 are consist only of formulas.

To III(3), the following is added.

$$\frac{\Gamma \subset \Delta, \alpha P \beta \qquad \Gamma \subset \Delta, \alpha Q \beta}{\Gamma \subset \Delta, \alpha PQ \beta}$$

where P and Q are formulas.

We shall give the interpretation of an expression α . If α is a regular expression, then $||\alpha|| = |\alpha|$ given in § 2. If G is a formula, then $||G||$ is defined as follows. We denote the set of all constants and free variables by T^*.

$||T|| = T^*$, $||\phi|| = \phi$ (the empty set)

$||P(a_1,\dots,a_n)|| \in \{\phi, T^*\}$ for predicate symbol P and constants or free variables $a_1,\dots,a_n$.

$||A+B|| = ||A|| \cup ||B||$ for every expressions A and B

$$||A\cdot B|| = \begin{cases} ||A|| \cap ||B|| & \text{if } A \text{ or } B \text{ is a formulas} \\ ||A||\cdot||B|| & \text{otherwise} \end{cases}$$

$||\neg A|| = T^* - ||A||$

$||\exists x F(x)|| = \bigcup_{a \in T^*} ||F(a)||$, $||\forall x F(x)|| = \bigcap_{a \in T^*} ||F(a)||$

Then we have the following usual truth table.

$\|\|P\|\|$	$\|\|Q\|\|$	$\|\|P+Q\|\|$	$\|\|P\cdot Q\|\|$	$\|\|\neg P\|\|$
T^*	T^*	T^*	T^*	ϕ
T^*	ϕ	T^*	ϕ	ϕ
ϕ	T^*	T^*	ϕ	T^*
ϕ	ϕ	ϕ	ϕ	T^*

$||\exists x F(x)|| = T^*$ iff we have a such that $||F(a)|| = T^*$

$||\forall x F(x)|| = T^*$ iff $||F(a)|| = T^*$ for all a

For every expression α , β or γ , we have

$$||(\alpha+\beta)+\gamma|| = ||\alpha+(\beta+\gamma)||$$
$$||(\alpha\beta)\gamma|| = ||\alpha(\beta\gamma)||$$

We shall prove the second. If more than two of α , β and γ are formulas, then

$$||(\alpha\beta)\gamma|| = ||\alpha|| \cap ||\beta|| \cap ||\gamma|| = ||\alpha(\beta\gamma)||.$$

If all of α , β and γ are not formulas, then

$$||(\alpha\beta)\gamma|| = ||\alpha||\cdot||\beta||\cdot||\gamma|| = ||\alpha(\beta\gamma)||.$$

If only one of α , β and γ is a formula, e.g. β , then

$$||(\alpha\beta)\gamma|| = (||\alpha|| \cap ||\beta||)\cdot||\gamma||$$
$$||\alpha(\beta\gamma)|| = ||\alpha||\cdot(||\beta|| \cap ||\gamma||).$$

In the case that $||\beta|| = T^*$,

$$||(\alpha\beta)\gamma|| = ||\alpha||\cdot||\gamma|| = ||\alpha(\beta\gamma)||.$$

In the case that $||\beta|| = \phi$,

$$||(\alpha\beta)\gamma|| = \phi = ||\alpha(\beta\gamma)||$$

We have the next two theorems. However we omit the proof here.

Theorem 1 (Plausibility) If a sequent $\Gamma \subset \Delta$ is provable, then it is valid.

Theorem 2 (Completeness and elimination of redundance). If a sequent $\Gamma \subset \Delta$ is valid, then it is provable by applying rules of inference of I and II.

§5 Comment to Hoare's rules

Hoare gave an axiomatic basic for computer programming [2]. We shall give some comments to the system. Under a translation, his rules are included in the present system. However, we do not treat the assignment statement, Hoare's axiom DO, which will be discussed in the forthcoming paper.

We translate his formal statement into the present system as follows.

$P\{Q\}R$ is translated to $QP \subset R$

$P\{Q_1;Q_2\}R$ to $Q_2Q_1P \subset R$

'while B do S' to $(\neg B)\bigcup_{n=0}^{\infty}(SB)^n$

$R \supset S$ to $R \subset S$

$P \wedge S$ to PS

Then D1 is obtained by the following.

$$\frac{QP \subset R \quad R \subset S}{QP \subset S}\ \text{cut}$$

$$\frac{\dfrac{S \subset P \quad Q \subset Q}{QS \subset QP}\ \text{III} \qquad QP \subset R}{QS \subset R}\ \text{cut}$$

D2 is obtained by the follosing.

$$\frac{\dfrac{Q_1P \subset R_1 \quad Q_2 \subset Q_2}{Q_2Q_1P \subset Q_2R_1}\ \text{III} \qquad Q_2R_1 \subset R}{Q_2Q_1P \subset R}\ \text{cut}$$

D3 is obtained by the following.

$$\frac{\dfrac{P \subset P \quad SBP \subset P \quad \dfrac{SBP \subset P \quad SB \subset SB}{(SB)^2P \subset P} \quad \cdots\cdots}{\bigcup_{n=0}^{\infty}(SB)^nP \subset P} \qquad \neg B \subset \neg B}{\neg B\cdot\bigcup_{n=0}^{\infty}(SB)^nP \subset \neg B\cdot P}$$

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