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ON A PROBLEM IN ADDITIVE NUMBER THEORY

by

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Let $g > 1$ be a fixed integer. Any positive integer n can be uniquely written in the from

$$n = \varepsilon_1 g^{k-1} + \varepsilon_2 g^{k-2} + \dots\ .\ . + \varepsilon_{k-1} g + \varepsilon_k$$

where $\varepsilon_i = \varepsilon_i(n)$, $1 \leqq i \leqq k$, are integers such that $0 \leqq \varepsilon_i \leqq g-1$. We put

$$\alpha(n) = \sum_{i=1}^{k} \varepsilon_i .$$

R.Bellman and H.Shapiro [1] proved the relation

$$\alpha(n) = \frac{x \log x}{2 \log 2} + O(x \log \log x)$$

in the case of $g = 2$. L.Mirsky [2] and S.C.Tang [4] extended independently this result to the general case of $g \geqq 2$, by establishing

$$\alpha(n) = \frac{(g-1)\ x \log x}{2 \log g} + O(x) .$$

In this paper we shall make a refinement on this result for the particular case of $g = 2$. Indeed, we prove the following

Theorem. We have

(A) $$\liminf_{x\to\infty} \frac{1}{x}\left(\frac{x \log x}{2 \log 2} - \sum_{n\le x} \alpha(n)\right) = 0 ,$$

and

(B) $$\limsup_{x\to\infty} \frac{1}{x}\left(\frac{x \log x}{2 \log 2} - \sum_{n\le x} \alpha(n)\right) = 1 - \frac{\log 3}{2 \log 2} .$$

1. Preliminaries

For any positive integer x we define

$$k = k(x) = \left[\frac{\log x}{\log 2}\right] + 1 ,$$

where $[z]$ is the integral part of z. Then it is easy to see that

$$k = \frac{\log x}{\log 2} + \frac{1}{\log 2} \log \frac{2^k}{x} .$$

Put

$$\sigma(x) = \sum_{i=1}^{k} (1 - 2\varepsilon_i(x)) ,$$

then

$$\sum_{n\le x} \sigma(n) = \sum_{n\le x} k(n) - 2 \sum_{n\le x} \alpha(n) .$$

But clearly

$$\sum_{n\le x} k(n) = xk - 2^k + k + 1$$

$$= \frac{x \log x}{\log 2} + \frac{x}{\log 2} \log \frac{2^k}{x} - 2^k + k + 1$$

where $k = k(x)$. Hence we have

$$\frac{x \log x}{\log 2} - 2 \sum_{n \leq x} \alpha(n)$$

$$= 2^k - \frac{x}{\log 2} \log \frac{2^k}{x} + \sum_{n \leq x} \sigma(n) - k - 1$$

$$= x\, h(x) + S(x) - k - 1 ,$$

where

$$h(x) = \frac{2^k}{x} - \frac{1}{\log 2} \log \frac{2^k}{x} ,$$

and

$$S(x) = \sum_{n \leq x} \sigma(n) .$$

It is readily seen that $h(x)$ decreases for $2^{k-1} \leqq x < 2^k \log 2$ and increases for $2^k \log 2 < x < 2^k$, and that

$$\min_{\substack{2^{k-1} \leqq x < 2^k \\ x\,:\,\text{integer}}} h(x) > h(2^k \log 2) = \frac{1 + \log \log 2}{\log 2} \tag{1}$$

and

$$\max_{2^{k-1} < x < 2^k} h(x) < h(2^{k-1}) = 1 . \tag{2}$$

In order to estimate $S(x)$ we need some notations and lemmas. Let $k \geq 1$. Denote by $\{c_{k,j} ; 1 \leq j \leq 2^k\}$ the set consisting of the 2^k possible arrengements $c_{k,j}$ of k digits formed with 0's and 1's and ranked in ascending order of magnitude ; e.g. $c_{2,1} = 00$, $c_{2,1} = 01$, $c_{2,3} = 10$ and $c_{2,4} = 11$. For any 0-1 sequence $\Delta = \delta_1 \delta_2 \ldots \delta_\ell$ of length ℓ we write

$$\sigma(\Delta) = \sum_{i=1}^{\ell} (1 - 2\delta_i) .$$

Now we put

$$T_k(j) = \sum_{i=1}^{j} \sigma(c_{k,j})$$

and define

$$T_k^* = \max_j T_k(j) .$$

Further we set

$$\ell_k = \min (j ; T_k^* = T_k(j))$$

and

$$L_k = \max (j ; T_k^* = T_k(j)) .$$

Lemma 1. For $k \geq 1$ and j, $1 \leq j \leq 2^k$, we have

$$T_k(j) = T_k(2^k - j) \geq 0 .$$

It will be convenient to understand that $T_k(0) = 0$ for any $k \geq 1$.

Lemma 2. For $k \geq 1$, we have

$$\ell_{k+1} = L_k = \frac{1}{3}(2^{k+1} - (-1)^{k+1}) .$$

Lemma 3. For $k \geq 1$, we have

$$T_k^* = \frac{1}{3}(2^{k+1} - \theta(k))$$

where

$$\theta(k) = \begin{cases} 1 & \text{if } k \text{ is odd,} \\ 2 & \text{if } k \text{ is even.} \end{cases}$$

Suppose $k \quad 5$. We set

$$j_k = \begin{cases} \frac{k-3}{2} & \text{for } k \text{ is odd,} \\ \frac{k-4}{2} & \text{for } k \text{ is even,} \end{cases}$$

and define for $0 \leqq j \leqq j_k - 1$

$$A_{k,j} = \frac{1}{3}(2^k - 2^{k-2j}) = \sum_{i=1}^{j} 2^{k-2i} ,$$

$$B_{k,j} = A_{k,j} + 2^{k-2j-3} = \frac{1}{3}(2^k - 5 \cdot 2^{k-2j-3}),$$

$$C_{k,j} = B_{k,j} + 2^{k-2j-4} = \frac{1}{3}(2^k - 7 \cdot 2^{k-2j-4})$$

and further

$$D_k = A_{k,j_k} = \begin{cases} \ell_k - 3 & \text{if } k \text{ is odd,} \\ \ell_k - 5 & \text{if } k \text{ is even.} \end{cases}$$

Lemma 4. If $A_{k,j} < \ell \leqq B_{k,j}$, then

$$T_k(\ell) \leqq \frac{1}{3}(2^{k+1} - 3 \cdot 2^{k-2j-2} - \theta(k)) ;$$

if $B_{k,j} < \ell \leqq C_{k,j}$, then

$$T_k(\ell) \leqq \frac{1}{3}(2^{k+1} - 7 \cdot 2^{k-2j-4} - \theta(k+1)) ;$$

and if $C_{k,j} < \ell < A_{k,j+1}$, then

$$T_k(\ell) \leqq \frac{1}{3}(2^{k+1} - 3 \cdot 2^{k-2j-3} - \theta(k)) .$$

For proofs of the above four lemmas see [3].

Let x be any integer such that $2^{k-1} \leqq x < 2^k$. Put $j = x - 2^{k-1} + 1$, $1 \leqq j \leqq 2^{k-1}$. Then it is easy to see that the dyadic expansion $\varepsilon_1\varepsilon_2 \ldots \varepsilon_k$ of x can be written as $1c_{k-1,j}$. Hence we have

$$\sigma(x) = -1 + \sigma(c_{k-1,j})$$

and so

$$S(x) = S(2^{k-1} - 1) - j + T_{r-1}(j) .$$

Especially we have from Lemma 1

$$S(2^k - 1) = -2^k + 1 \tag{3}$$

Hence we have

$$S(x) = -2^{k-1} + 1 - j + T_{k-1}(j) \tag{4}$$

Note that $T_{k-1}(j) = j + T_{k-2}(j)$, provided $1 \leqq j \leqq 2^{k-2}$. Thus we have also

$$S(x) = -2^{k-1} + 1 + T_{k-2}(j) \tag{5}$$

we notice that $-x < S(x) < 0$ for all $x \geqq 1$.

Throughout in the rest of this section we suppose that $k > 5$. Put

$$x_i = 2^{k-1} - 1 + i\, 2^{k-5} , \qquad 0 \leqq i \leqq 2^4$$

Let x be a positive integer such that $x_{i-1} < x \leqq x_i$, for some i, $1 \leqq i \leqq 2^4$, and write $j = x - x_{i-1} + L$, $1 \leqq j \leqq 2^{k-5}$. Then the dyadic expansion $\varepsilon_1 \varepsilon_2 \ldots \varepsilon_k$ of x is $1 c_{4,i} c_{k-5,j}$. Hence we have

$$S(x) = -x + T_4(i-1)\, 2^{k-5} + j \cdot \sigma(c_{4,i}) + T_{k-5,j} \tag{6}$$

The following table gives the values of the $\sigma(c_{4,i})$ and the $T_4(i)$ for $0 \leqq i \leqq 2^4$.

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\sigma(c_{4,1})$	0	4	2	2	0	2	0	0	-2	2	0	0	-2	0	-2	-2	-4
$T_4(i)$	0	4	6	8	8	10	10	10	8	10	10	10	8	8	6	4	0

2. Proof of (A)

By (3) we have

$$0 > h(2^k - 1) + \frac{S(2^k - 1)}{2^k - 1} > -\frac{1}{2^k} \tag{7}$$

and by (2) and (4)

$$h(2^k) + \frac{S(2^k)}{2^k} = \frac{k - 1}{2^{k-1}} \quad . \tag{8}$$

Now we wish to show that if $k > 5$ and $2^{k-1} + 1 \leqq x \leqq 2^k - 2$, then

$$h(x) + \frac{S(x)}{x} > 0 \quad . \tag{9}$$

Case 1. $2^{k-1} + 1 \leqq x \leqq x_1$. Let ℓ , $1 \leqq \ell \leqq k - 5$, be an integer such that $2^{k-1} - 1 + 2^{-1} < x \leqq 2^{k-1} - 1 + 2$. Put $j = x - 2^{k-1} + 1$ then $2^{\ell-1} < j \leqq 2^{\ell}$.

Then we have

$$T_{k-1}(j) = T_{k-1}(2^{\ell-1}) + (k-\ell-4)(j-2^{\ell-1}) + T_{\ell-1}(j-2^{\ell-1})$$

$$\geqq T_{k-1}(2^{\ell-1}) = (k-\ell-2)2^{\ell-1} \tag{10}$$

From this and (4) we get

$$\frac{S(x)}{x} \geqq \frac{-2^{k-1} + 2^{\ell-1}}{2^{k-1} + 2^{\ell-1}}$$

Since h(x) is decreasing we have

$$h(x) > h(2^{k-1} + 2\ ) > \frac{2^k}{2^{k-1} + 2} + \frac{2^\ell}{2^{k-1}}$$

Accordingly

$$h(x) + \frac{S(x)}{x} > \frac{2^\ell}{2^{k-1} + 2} + \frac{2^\ell}{2^{k-1}} - \frac{2^{\ell+1}}{2^{k-1} + 2} > 0\ .$$

Case 2. $x_1 \leqq x \leqq x_{15}$. Let $x_{i-1} \leqq x \leqq x_i$, $2 \leqq i \leqq 15$. Then by (6) and the table given in the preceding section we have

$$\frac{S(x)}{x} > -1 + \frac{2^{k-5}\min(T_4(i-1),\ T_4(i))}{2^{k-1} + i\ 2^{k-5}} \geqq -\frac{27}{31}$$

From this and (1) we have

$$h(x) + \frac{S(x)}{x} > \frac{1 + \log\log 2}{\log 2} - \frac{27}{31} > 0$$

Case 3. $x_{15} < x \leqq 2^k - 2$. Let ℓ , $1 \leqq \ell \leqq k-5$, be an integer such that $2^k - 1 - 2^\ell < x < 2^k - 1 - 2^{\ell-1}$. Put $j = x - 2^{k-1} + 1$, then $2^{\ell-1} \leqq 2^{k-1} - j < 2^\ell$. From (4) and (10) we have

$$\frac{S(x)}{x} \geqq \frac{-2^k + (k- \quad -1)2^{\quad -1}}{2^k - 2}\ .$$

Since h(x) is increasing we have

$$h(x) \geq h(2^k - 2) > \frac{2^k - 2^{\ell+1}}{2^k - 2^\ell}$$

Hence

$$h(x) + \frac{S(x)}{x} > \frac{2^{\ell-1}}{2^k - 2}(k-5-\ell) \geq 0 \quad .$$

From (7), (8) and (9) we obtain

$$\liminf_{x \to \infty} \left(h(x) + \frac{S(x)}{x}\right) = 0 \; .$$

This proves (A).

3. **Proof of (B).**

We estimate the magnitude of the function $2^{r-1} \max_{x} 2^k \left(h(x) + \frac{S(x)}{x}\right)$. In what follows we suppose that $k > 7$.
If $x = x_{i-1} + j - 1$, $1 \leqq j \leqq 2^{k-5}$, then from (6) and Lemma 3 we have

$$\frac{S(x)}{x} \leqq -1 + \frac{3 \max(T_4(i-1), T_4(i)) + 2}{3(15 + i)} \tag{11}$$

Case 1. $x_{i-1} < x \leqq x_i$, $i \leqq 2$ or $i \geqq 11$. By (2), (11) and the table in 2 we readily find

i	=	1	2	11	12	13	14	15	16
$h(x) + \frac{S(x)}{x}$	<	$\frac{14}{48}$	$\frac{20}{51}$	$\frac{32}{78}$	$\frac{32}{81}$	$\frac{32}{84}$	$\frac{26}{87}$	$\frac{20}{90}$	$\frac{14}{93}$

Hence we have

$$h(x) + \frac{S(x)}{x} < \frac{32}{78} < 0.411 \quad .$$

Case 2. $x_{i-1} < x \leqq x_i$, $7 \leqq i \leqq 10$. Since $h(x)$ is increasing in the interval $x_7 < x \leqq x_{10}$ we have

$$h(x) < h(x_i + 1) \ , \quad x_{i-1} < x \leqq x_i \ , \quad 8 \leqq i \leqq 10 \qquad (12)$$

But if $x_6 < 2^k \log 2 < x_7$ then

$$h(x) < \max(h(x_6+1), h(x_7+1)) \ , \quad x_6 < x \leqq x_7 \qquad (13)$$

Here

$$h(x_i+1) = \frac{32}{16 + i} + \frac{\log(16+i)}{\log 2} - 5 \ , \quad 0 \leqq i \leqq 2^4 \qquad (14)$$

From (11), (12), (13) and (14) we get

$$h(x) + \frac{S(x)}{x} < \frac{32}{66} + \frac{32}{23} + \frac{\log 23}{\log 2} - 6 \qquad (x_6 < x \leqq x_7),$$

$$\frac{32}{69} + \frac{32}{24} + \frac{\log 24}{\log 2} - 6 \qquad (x_7 < x \leqq x_8),$$

$$\frac{32}{72} + \frac{32}{25} + \frac{\log 25}{\log 2} -66 \qquad (x_8 < x \leqq x_9),$$

$$\frac{32}{75} + \frac{32}{26} + \frac{\log 26}{\log 2} - 6 \qquad (x_9 < x \leqq x_{10}) \ .$$

Hence we have

$$h(x) + \frac{S(x)}{x} < \frac{32}{66} + \frac{32}{23} + \frac{\log 23}{\log 2} - 6 < 0.402 .$$

Case 3. $x_2 < x < 2^{k-1} + \ell_{k-2}$. From (5) and Lemma 3 we have (noticing that $k > 7$)

$$\frac{S(x)}{x} \leqq \frac{-2^{k-1} + 1 + T^*_{k-2}}{2^{k-1} + \ell_{k-2} - 1}$$

$$< -\frac{4}{7} + \frac{1}{3 \cdot 2^{k-2} + 2^{k-3}}$$

$$< -\frac{127}{224} .$$

Since $h(x)$ is decreasing in this interval we have (using (14))

$$h(x) + \frac{S(x)}{x} < \frac{7}{9} + \frac{2 \log 3}{\log 2} - 3 - \frac{127}{224} < 0.4$$

Case 4. $2^{k-1} + \ell_{k-2} \leqq x < 2^{k-1} + L_{k-2}$. We have from (5)

$$h(x) + \frac{S(x)}{x} \leqq \frac{2^k}{x} - \frac{1}{\log 2} \log \frac{2^k}{x} + \frac{-2^{k-1} + 1 + T^*_{k-2}}{x}$$

$$= w(x) , \quad \text{say.}$$

Since $x \geq 2^{k-1} + \ell_{k-2}$ we have

$$w'(x) = \frac{1}{x}\left(\frac{1}{\log 2} - \frac{2^{k-1} + 1 + T^*_{k-2}}{x} \right) > 0$$

and so

$$h(x) + \frac{S(x)}{x} \leqq w(2^{k-1} - 1 + L_{k-2})$$

$$= h(2^{k-1} - 1 + L_{k-2}) + \frac{S(2^{k-1} - 1 + L_{k-2})}{2^{k-1} - 1 + L_{k-2}} \qquad (15)$$

$$= 2 - \frac{\log 3}{\log 2} + O(\frac{1}{2^k}) .$$

Case 5. $2^{k-1} + L_{k-2} < x \leqq x_6$. Put $\ell = 2^{k-1} - 1 + 2^{k-2} - x$, so that $2^{k-4} \leqq \ell < \ell_{k-2}$. By (5) and Lemma 1 we have

$$S(x) = -2^{k-1} + 1 + T_{k-2}(x - 2^{k-1} + 1)$$

$$= -2^{k-1} + 1 + T_{k-2}(\ell) .$$

Now we appeal to Lemma 4. For $A_{k-2,j} < \ell \leqq B_{k-2,j}$ we have

$$\frac{S(x)}{x} \leqq \frac{-2^{k-1} + 1 + T_{k-2}(\ell)}{2^{k-1} + 2^{k-2} - A_{k-2,j}}$$

$$\leqq \frac{-2^{k-1} + 1 + \frac{1}{3}(2^{k-1} - 3 \cdot 2^{k-2j-4})}{2^{k-1} + 2^{k-2} + \frac{1}{3}(2^{k-2} - 2^{k-2j-2})}$$

$$= -\frac{1}{2} + O(\frac{1}{2^k})$$

Similarlly we can easily verify that the inequality

$$\frac{S(x)}{x} \leqq -\frac{1}{2} + O(\frac{1}{2^k}) \qquad (16)$$

holds for $B_{k-2,j} < \ell \leqq c_{k-2,j}$ and for $c_{k-2,j} < \ell \leqq A_{k-2,j+1}$. It is also clear that (16) holds true for $D_{k-2} < \ell < \ell_{k-2}$, since $D_{k-2} = \ell_{k-2} + O(1)$. On the other hand $h(x)$ is decreasing in this interval, and so

$$h(x) < h(2^{k-1} - 1 + L_{k-2})$$

$$< \frac{5}{2} - \frac{\log 3}{\log 2} + O\left(\frac{1}{2^k}\right) .$$

From this and (16) we obtain

$$h(x) + \frac{S(x)}{x} \leqq 2 - \frac{\log 3}{\log 2} + O\left(\frac{1}{2^k}\right) .$$

As the result we have

$$h(x) + \frac{S(x)}{x} \leqq 2 - \frac{\log 3}{\log 2} + O\left(\frac{1}{2^k}\right)$$

for all x , $2^{k-1} < x < 2^k$, since $2 - \frac{\log 3}{\log 2} > 0.415$. But by (15) the equality holds here at least fo $x = 2^{k-1} - 1 + L_{k-2}$. Therefore

$$\limsup_{x \to \infty} \left(h(x) + \frac{S(x)}{x}\right) = 2 - \frac{\log 3}{\log 2} ,$$

which proves (B).

This completes the proof of our theorem.

References

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