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Recent advances of analytic method in the theory of numbers

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In this lecture, I shall state two analytic methods induced by Gallagher [2] and Fogels [1] and extend these so as to be able to apply in case of algebraic number fields. By virtue of these tools we shall have a prospect of natural extension of density theorem concerning the number $N(\sigma, T)$ of zeros $\rho = \beta + i\gamma$ in the rectangle

$$\sigma \leqq \beta \leqq 1, \quad -T \leqq \gamma \leqq T$$

of all Hecke's zeta-functions $\zeta(s, \chi)$ with same moduli $\tilde{\mathfrak{m}}$. Hirano is now working out this problem applying the extended Ingham's method [5, Chapt. 9].

Let k be an algebraic number field of degree n over the rational number field. Let k have r_1 real conjugates $k^{(l)}$ $(1 \leqq l \leqq r_1)$ and r_2 pairs of complex conjugates $k^{(m)}$, $k^{(m+r_2)}$ $(r_1+1 \leqq m \leqq r_1+r_2)$, namely $n = r_1 + 2r_2$. A formal product

$$\tilde{\mathfrak{m}} = \mathfrak{m}\, \mathfrak{p}_\infty^{(1)} \cdots \mathfrak{p}_\infty^{(q)} \qquad (0 \leqq q \leqq r_1)$$

is called a divisor of k, where $\mathfrak{m}$ is an integral ideal and $\mathfrak{p}_\infty^{(i)}$ is an infinite prime spot. Let $A(\tilde{\mathfrak{m}})$ be the group of all ideals prime to $\mathfrak{m}$ in k and $S(\tilde{\mathfrak{m}})$ be the group of all principal ideals (ν) generated by ν satisfying the multiplicative congruence

$$\nu \equiv 1 \mod \tilde{\mathfrak{m}},$$

namely

$$\nu \in k, \quad \nu \equiv 1 \mod \mathfrak{m} \quad \text{and} \quad \nu^{(1)}, \cdots, \nu^{(q)} > 0.$$

A class of the factor group $A(\tilde{\mathfrak{m}})/S(\tilde{\mathfrak{m}})$ is called a ray class mod $\tilde{\mathfrak{m}}$. We denote by $h(\tilde{\mathfrak{m}})$ the number of classes of this factor group. Let $\mathfrak{d}$ be the different and d be the discriminant of k, so that $N\mathfrak{d} = |d|$. Let R be the regulator of k, h be the number of absolute ideal classes of k and w be the number of roots of unity in k [4].

Throughout the paper, c or c with suffix will be used to denote a positive constant depending only on k, and c(*) will be used when c depends further on several parameters *. $A \ll B$

or $A = O(B)$, where B is positive, means that there exists c satisfying

$$|A| \leqq cB$$

in the region under consideration.

In the first place, we shall estimate the number $T(t)$ of integral ideals in a ray class C mod $\tilde{\mathfrak{m}}$ whose norms don't exceed t. The details of this subject will be stated another paper. We shall therefore only point out the main results.

Theorem 1. Let C be a ray class mod $\tilde{\mathfrak{m}}$. Take c sufficiently large, and suppose that

$$t \geqq cN\mathfrak{m}.$$

Then

$$T(t) = \sum_{N\mathfrak{a} \leqq t,\ \mathfrak{a} \in C} 1 = \frac{1}{h(\tilde{\mathfrak{m}})} \prod_{\mathfrak{p} | \mathfrak{m}} \left(1 - \frac{1}{N\mathfrak{p}}\right) \frac{2^{r_1}(2\pi)^{r_2}|R|h}{\sqrt{|d|}\ w} t$$
$$+ O\left\{ \frac{1}{h(\tilde{\mathfrak{m}})} \prod_{\mathfrak{p} | \mathfrak{m}} \left(1 - \frac{1}{N\mathfrak{p}}\right) N\mathfrak{m}^{\frac{1}{n}} t^{1-\frac{1}{n}} \right\}.$$

Theorem 2. If $0 \leqq y \ll x$, then

$$\sum_{x \leqq N\mathfrak{a} \leqq x+y,\ \mathfrak{a} \in C} 1 \ll \frac{1}{h(\tilde{\mathfrak{m}})} \left(y + N\mathfrak{m}^{\frac{1}{n}} x^{1-\frac{1}{n}} + N\mathfrak{m} \right).$$

Secondly, we shall introduce a new elegant real analytic method due to Gallagher.

Theorem 3. If $\sum_{n=1}^{\infty} |a_n| < \infty$, then

$$\int_{-T}^{T} \Big| \sum_{n=1}^{\infty} a_n n^{it} \Big|^2 dt \leqq \frac{\pi}{2} T^2 \int_{\tau^{-1}}^{\infty} \Big| \sum_{z \leqq n \leqq z\tau} a_n \Big|^2 \frac{dz}{z}$$

where $\tau = e^{\frac{\pi}{T}}$.

Proof. Write

$$a_n = c\left(\frac{\log n}{2\pi}\right).$$

Therefore,

$$S(t) = \sum_{n=1}^{\infty} a_n n^{it} = \sum_{\nu} c(\nu) e^{2\pi i \nu t}$$

where ν runs over

$$0, \quad \frac{\log 2}{2\pi}, \quad \frac{\log 3}{2\pi}, \quad \cdots\cdots$$

Put

$$F_\delta(x) = \delta^{-1} \quad \text{or} \quad 0,$$

according as $|x| \leqq \frac{1}{2}\delta$ or not, and write

$$f(x) = \sum_{\nu} c(\nu) F_\delta(x - \nu) = \frac{1}{\delta} \sum_{|\nu - x| \leqq \frac{1}{2}\delta} c(\nu) \tag{1}$$

Accordingly,

$$\int_{-\infty}^{\infty} |f(t)|^p\,dt = \frac{1}{\delta^p}\int_{-\infty}^{\infty} \Big|\sum_{|\frac{\log n}{2\pi}-t|\le\frac{1}{2}\delta} a_n\Big|^p\,dt$$

$$= \frac{1}{2\pi\delta^p}\int_{\tau^{-1}}^{\infty} \Big|\sum_{z\le n\le z\tau} a_n\Big|^p \frac{dz}{z} \qquad (2)$$

for p=1 and 2, where

$$\delta = \frac{1}{2T} \quad \text{and} \quad \tau = e^{\frac{\pi}{T}}.$$

Assume that the right side of Theorem 3 is finite, since the result is trivially true otherwise. Hence, f(x) belongs to L^2 because of (2). We have also

$$\int_{-\infty}^{\infty} |f(t)|\,dt \le \frac{1}{2\pi\delta}\int_{\tau^{-1}}^{\infty} \sum_{z\le n\le z\tau} |a_n| \frac{dz}{z}$$

from (1), where the coefficient of $|a_n|$ of the left side turns out to be

$$\frac{1}{2\pi\delta}\int_{n/\tau}^{n} \frac{dz}{z} \ll \frac{1}{2\pi\delta}\log\tau = 1$$

It follows therefore from the assumption of the theorem that f(x) belongs to L^1.

By these properties,

$$\hat{f}(x) = \int_{-\infty}^{\infty} e^{2\pi i x t} f(t)\,dt = \sum_{\nu} c(\nu)\int_{-\infty}^{\infty} e^{2\pi i x t} F_\delta(t-\nu)\,dt$$

$$= \sum_{\nu} c(\nu)\, e^{2\pi i x\nu}\,\frac{\sin \pi\delta x}{\pi\delta x} \qquad (3)$$

belongs to L^2 [3] and

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} |\hat{f}(x)|^2\,dx.$$

This implies from (1) and (3) that

$$\frac{1}{\delta^2}\int_{-\infty}^{\infty} \Big|\sum_{|\nu-x|\le\frac{\delta}{2}} c(\nu)\Big|^2 dx = \int_{-\infty}^{\infty} \Big|\sum_{\nu} c(\nu) e^{2\pi i x\nu}\frac{\sin\pi\delta x}{\pi\delta x}\Big|^2 dx$$

$$\ge \left(\frac{2}{\pi}\right)^2 \int_{-T}^{T} \Big|\sum_{\nu} c(\nu) e^{2\pi i x\nu}\Big|^2 dx.$$

In view of (1) and (2), it follows therefore

$$\int_{-T}^{T} \Big|\sum_{n=1}^{\infty} a_n n^{it}\Big|^2 dt \le \frac{\pi^2}{4}\int_{-\infty}^{\infty} \Big|\frac{1}{\delta}\sum_{|\nu-x|\le\frac{\delta}{2}} c(\nu)\Big|^2 dx = \frac{\pi}{8\delta^2}\int_{\tau^{-1}}^{\infty} \Big|\sum_{z\le n\le z\tau} a_n\Big|^2 \frac{dz}{z}.$$

Now we shall apply Theorem 3 to the sum of the form

$$S(\chi, t) = \sum A(\mathfrak{a})\chi(\mathfrak{a}) N(\mathfrak{a})^{it},$$

where $\mathfrak{a}$ runs over all integral ideals of k, χ is a character mod $\tilde{\mathfrak{m}}$ and A($\mathfrak{a}$) is a function defined over all integral ideals in k. Put

$$a_n = \sum_{N\mathfrak{a}=n} A(\mathfrak{a})\chi(\mathfrak{a}),$$

and make use of Theorem 1, we have

$$\sum_{\chi \bmod \tilde{\mathfrak{m}}} \int_{-T}^{T} |S(\chi, t)|^2 dt = \sum_{\chi \bmod \tilde{\mathfrak{m}}} \int_{-T}^{T} \Big|\sum_{n=1}^{\infty} a_n n^{it}\Big|^2 dt$$

$$\leq \frac{\pi}{2} T^2 \int_{\tau^{-1}}^{\infty} \sum_{\chi \bmod \tilde{\mathfrak{m}}} \Big|\sum_{z \leq N\mathfrak{a} \leq z\tau} A(\mathfrak{a})\chi(\mathfrak{a})\Big|^2 \frac{dz}{z}. \qquad (4)$$

Divide integral ideals $\mathfrak{a}$ satisfying $z \leq N\mathfrak{a} \leq z\tau$ into several classes in such a way that each pair of integral ideals in a class $\mathfrak{K}$ is never congruent mod $\tilde{\mathfrak{m}}$ with each other and that the number M of classes is less than

$$O\Big\{\frac{1}{h(\tilde{\mathfrak{m}})}(z\tau - z + z^{1-\frac{1}{n}} N\mathfrak{m}^{\frac{1}{n}} + N\mathfrak{m})\Big\},$$

in view of Theorem 2. Accordingly,

$$\sum_{\chi \bmod \tilde{\mathfrak{m}}} \Big|\sum_{z \leq N\mathfrak{a} \leq z\tau} A(\mathfrak{a})\chi(\mathfrak{a})\Big|^2$$

$$\leq \sum_{\chi \bmod \tilde{\mathfrak{m}}} M \sum_{\mathfrak{K}} \Big|\sum_{\mathfrak{a} \in \mathfrak{K}} A(\mathfrak{a})\chi(\mathfrak{a})\Big|^2$$

$$\ll \frac{1}{h(\tilde{\mathfrak{m}})}(z\tau - z + z^{1-\frac{1}{n}} N\mathfrak{m}^{\frac{1}{n}} + N\mathfrak{m}) \sum_{\mathfrak{K}} \sum_{\chi \bmod \tilde{\mathfrak{m}}} \Big|\sum_{\mathfrak{a} \in \mathfrak{K}} A(\mathfrak{a})\chi(\mathfrak{a})\Big|^2$$

$$\ll (z\tau - z + z^{1-\frac{1}{n}} N\mathfrak{m}^{\frac{1}{n}} + N\mathfrak{m}) \sum_{z \leq N\mathfrak{a} \leq z\tau} |A(\mathfrak{a})|^2.$$

Substituting (4) for this result, we get

$$\sum_{\chi \bmod \tilde{\mathfrak{m}}} \int_{-T}^{T} |S(\chi, t)|^2 dt$$

$$\ll T^2 \int_{\tau^{-1}}^{\infty} \frac{z\tau - z + z^{1-\frac{1}{n}} N\mathfrak{m}^{\frac{1}{n}} + N\mathfrak{m}}{z} \sum_{z \leq N\mathfrak{a} \leq z\tau} |A(\mathfrak{a})|^2 dz$$

$$\ll \sum |A(\mathfrak{a})|^2 (N\mathfrak{a} + N\mathfrak{a}^{1-\frac{1}{n}} N\mathfrak{m}^{\frac{1}{n}} T + N\mathfrak{m} T),$$

which was the result obtained by Hirano in his master dissertation.

Theorem 4. If $\sum |A(\mathfrak{a})| < \infty$, then

$$\sum_{\chi \bmod \tilde{m}} \int_{-T}^{T} \left| \sum A(\mathfrak{a})\chi(\mathfrak{a}) N\mathfrak{a}^{it} \right|^2 dt \ll \sum \left(N\mathfrak{a} + N\mathfrak{a}^{1-\frac{1}{n}} N\mathfrak{m}^{\frac{1}{n}} T + N\mathfrak{m} T\right) |A(\mathfrak{a})|^2 .$$

Thirdly we shall introduce a new Lindelöf principle due to Fogels.

Theorem 5. Let $f_j(s)$ $(1 \leq j \leq m)$ be regular in the region D

$$\sigma_1 < \sigma < \sigma_2 , \quad -\infty < t < \infty ,$$

satisfying the conditions

$$F(s) = |f_1(s)|^p + \cdots + |f_m(s)|^p \leq c_0 e^{c_1 |t|}$$

and be continuous over the closure of D, p being some positive integer. If the conditions

$$F(s) \leq A(|t|+2)^a \log^c C(|t|+2) \qquad \text{on } \sigma = \sigma_1$$

$$F(s) \leq B(|t|+2)^b \log^c C(|t|+2) \qquad \text{on } \sigma = \sigma_2$$

are fulfilled, then there exists $c(\sigma_1, \sigma_2)$ such that

$$F(s) \leq c(\sigma_1, \sigma_2) A^{\frac{\sigma_2 - \sigma}{\sigma_2 - \sigma_1}} B^{\frac{\sigma - \sigma_1}{\sigma_2 - \sigma_1}} (|t|+2)^{a\frac{\sigma_2 - \sigma}{\sigma_2 - \sigma_1} + b\frac{\sigma - \sigma_1}{\sigma_2 - \sigma_1}} \{\log C(|t|+2)\}^c ,$$

uniformly in $\sigma_1 \leq \sigma \leq \sigma_2$, where a, b, c are given positive numbers and A, B, $C > 1$ are parameters not depending on t.

Proof. We know that

$$|\Gamma(s)| = \sqrt{2\pi}\, |t|^{\sigma - \frac{1}{2}} e^{-\frac{\pi}{2}|t|} \left\{1 + O\left(\frac{1}{|t|}\right)\right\}$$

uniformly in $\sigma_1 \leq \sigma \leq \sigma_2$, $|t| \geq 1$. Hence

$$\left| \frac{\Gamma\left(2 - \frac{s - \sigma_1}{2(\sigma_2 - \sigma_1)}\right)}{\Gamma\left(1 + \frac{s - \sigma_1}{2(\sigma_2 - \sigma_1)}\right)} \right| = (|t|+2)^{1 - \frac{\sigma - \sigma_1}{\sigma_2 - \sigma_1}} \left\{1 + O\left(\frac{1}{|t|+2}\right)\right\}, \tag{5}$$

and

$$\left| \frac{\Gamma\left(2 - \frac{\sigma_2 - s}{2(\sigma_2 - \sigma_1)}\right)}{\Gamma\left(1 + \frac{\sigma_2 - s}{2(\sigma_2 - \sigma_1)}\right)} \right| = (|t|+2)^{1 - \frac{\sigma_2 - \sigma}{\sigma_2 - \sigma_1}} \left\{1 + O\left(\frac{1}{|t|+2}\right)\right\}, \tag{6}$$

in the region $\sigma_1 \leqq \sigma \leqq \sigma_2$.

Take σ_0 such that $1-\sigma_0<\sigma_1$ and define

$$\log C(s+\sigma_0) = \int_1^{C(s+\sigma_0)} \frac{dz}{z},$$

the contour lying on the right half plane $\sigma \geqq 0$. Then,

$$\log C(s+\sigma_0) = \log C(|t|+2)\left\{1+O\left(\frac{1}{\log C(|t|+2)}\right)\right\}.$$

Now we define

$$G(s) = A^{\frac{\sigma_2-s}{\sigma_2-\sigma_1}} B^{\frac{s-\sigma_1}{\sigma_2-\sigma_1}} \left\{\frac{\Gamma\left(2-\frac{s-\sigma_1}{2(\sigma_2-\sigma_1)}\right)}{\Gamma\left(1+\frac{s-\sigma_1}{2(\sigma_2-\sigma_1)}\right)}\right\}^a \left\{\frac{\Gamma\left(2-\frac{\sigma_2-s}{2(\sigma_2-\sigma_1)}\right)}{\Gamma\left(1+\frac{\sigma_2-s}{2(\sigma_2-\sigma_1)}\right)}\right\}^b \{\log C(s+\sigma_0)\}^c.$$

Obviously $G(s)$ is regular in $\sigma_1 \leqq \sigma \leqq \sigma_2$ and there exists c_2 satisfying

$$|G(s)| \geqq c_2$$

in the region stated above. By (5) and (6),

$$|G(s)| = A(|t|+2)^a \log^c C(|t|+2)\left\{1+O\left(\frac{1}{\log C(|t|+2)}\right)\right\}$$

on $\sigma=\sigma_1$ and

$$= B(|t|+2)^b \log^c C(|t|+2)\left\{1+O\left(\frac{1}{\log C(|t|+2)}\right)\right\}$$

on $\sigma=\sigma_2$. Further, from the assumption of the theorem,

$$\frac{F(s)}{|G(s)|} \leqq c_3 e^{c_1|t|} \qquad \text{for} \quad \sigma_1 \leqq \sigma \leqq \sigma_2$$

$$\leqq c_4 \qquad \text{for } \sigma=\sigma_1 \text{ and } \sigma=\sigma_2$$

For any $\varepsilon>0$,

$$|e^{\varepsilon s^2}| \frac{F(s)}{|G(s)|}$$

is subharmonic in $\sigma_1<\sigma<\sigma_2$ and is continuous on $\sigma_1\leqq\sigma\leqq\sigma_2$. Applying the usual Lindelöf principle to this function and letting $\varepsilon\to 0$, we get the desired result.

From now on let χ be a primitive character mod $\tilde{m}$, $\tilde{m}$ being $m\mathfrak{p}_\infty^{(1)}\cdots\mathfrak{p}_\infty^{(q)}$ as before. We can find ρ_j ($1\leqq j\leqq s=N(m)$) in $(m\mathfrak{d})^{-1}$ which satisfy the following conditions:

$$\rho_j \equiv 1 \mod \mathfrak{p}_\infty^{(i)} \quad (1\leqq i\leqq q),$$

each pair of ρ_j is never congruent mod $\mathfrak{d}^{-1}$, moreover, if $\rho \in (\mathfrak{m}\mathfrak{d})^{-1}$, then

$$\rho \equiv \rho_j \quad \mathrm{mod}\ \mathfrak{d}^{-1}$$

for some ρ_j . The set of these ρ_j is called a complete residue system mod $\mathfrak{d}^{-1}$ in $(\mathfrak{m}\mathfrak{d})^{-1}$.

Let $\xi \neq 0$ and $\xi \in (\mathfrak{m}\mathfrak{d})^{-1}$. Put

$$G(\xi, \chi) = \sum_{\beta} \chi(\beta) e^{2\pi i\, S(\beta \xi)},$$

where β runs over a complete residue system mod $\tilde{\mathfrak{m}}$, satisfying

$$\beta \equiv 1 \quad \mathrm{mod}\ \mathfrak{p}_{\infty}^{(i)} \qquad (1 \leq i \leq q).$$

Further we define

$$G(\chi) = \sum_{\rho} \chi(\rho\, \mathfrak{m}\mathfrak{d}) e^{2\pi i\, S(\rho)},$$

ρ being over a complete residue system mod $\mathfrak{d}^{-1}$ in $(\mathfrak{m}\mathfrak{d})^{-1}$. Take $\eta = \eta(\xi)$ such that

$$\eta \equiv 1 \quad \mathrm{mod}\ \mathfrak{m}, \qquad \eta \equiv \xi \quad \mathrm{mod}\ \mathfrak{p}_{\infty}^{(i)} \qquad (1 \leq i \leq q),$$

then we have

$$G(\xi, \chi) = \bar{\chi}(\eta(\xi)\xi\, \mathfrak{m}\mathfrak{d})\, G(\chi)$$

and

$$\chi(\eta(\xi)) = (\mathrm{sgn}\, \xi^{(1)}) \ldots (\mathrm{sgn}\, \xi^{(q)}).$$

If we define the normalized Gauss sum by

$$I(\chi) = (-i)^q\, G(\chi) \,/\, \sqrt{N(\mathfrak{m})},$$

then we have the following relation[7]

$$I(\chi)\, I(\bar{\chi}) = 1.$$

Let $\zeta(s, \chi)$ and $\zeta_k(s)$ be functions defined by

$$\sum \chi(\mathfrak{a})\, N\mathfrak{a}^{-s} \quad \text{and} \quad \sum N\mathfrak{a}^{-s}$$

for $\sigma > 1$, the summation being over all non-zero integral ideals in k. Define, for $\sigma > 1$,

$$\Gamma(s, \chi) = \int_0^{\infty}\!\!\cdots\!\!\int e^{-\sum_{p=1}^{n} z_p} \prod_{p=1}^{n} z_p^{\frac{z + a_p}{2}} \frac{dz_1 \cdots dz_{r+1}}{z_1 \cdots z_{r+1}},$$

where $r+1 = r_1 + r_2$ and

$$a_p = \begin{cases} 1 & (1 \leq p \leq q) \\ 0 & (q+1 \leq p \leq n) \end{cases}, \qquad z_p = z_{p+r_2} \quad (r_1 + 1 \leq p \leq r_1 + r_2)$$

which turns out to be

$$2^{-r_2 s}\, \Gamma\!\left(\frac{s+1}{2}\right)^q \Gamma\!\left(\frac{s}{2}\right)^{r_1 - q} \Gamma(s)^{r_2}.$$

Further define

$$\Phi(s,\chi) = \frac{(2\pi)^{r_2}}{\sqrt{|d|}} A(\chi)^{\frac{s}{2}} \Gamma(s,\chi)\zeta(s,\chi)$$

and

$$A(\chi) = \pi^{-n}|d|\, N\mathfrak{m}.$$

Similarly we define

$$\Gamma(s,\chi,b) = \int\!\!\cdot\!\!\int_{z_p>0,\ z_1\cdots z_n \geqq \frac{Nb^2}{A(\chi)}} e^{-\sum_{p=1}^{n} z_p} \prod_{p=1}^{n} z_p^{\frac{s+a_p}{2}} \frac{dz_1\cdots dz_{r+1}}{z_1\cdots z_{r+1}}$$

and

$$\Psi(s,\chi) = \frac{(2\pi)^{r_2}}{\sqrt{|d|}} A(\chi)^{\frac{s}{2}} \sum \chi(b)\Gamma(s,\chi,b)\, Nb^{-s}.$$

We know that $\Psi(s,\chi)$ is an integral function which satisfies the formula

$$\Phi(s,\chi) = -\frac{2^{r_1+r_2}\pi^{r_2}Rh}{w\sqrt{|d|}}\frac{E(\chi)}{s(1-s)} + \Psi(s,\chi) + I(\chi)\Psi(1-s,\bar\chi) \quad (7)$$

where

$$E(\chi) = \begin{cases} 1 & \text{if } \mathfrak{m}=\mathfrak{o},\ \chi \text{ principal} \\ 0 & \text{otherwise.} \end{cases}$$

The functional equation

$$\Phi(s,\chi) = I(\chi)\,\Phi(1-s,\bar\chi)$$

follows from (7) immediately [7].

The formula (7) can be transformed in the following form:

$$\Phi(s,\chi) = -\frac{2^{r_1+r_2}\pi^{r_2}Rh}{w\sqrt{|d|}}\frac{E(\chi)}{s(1-s)} + \frac{(2\pi)^{r_2}}{\sqrt{|d|}}\left(\pi|d|^{-\frac{1}{n}} N\mathfrak{m}^{-\frac{1}{n}}\right)^{\frac{r}{2}}$$

$$\sum_{\mathfrak{a}\neq\mathfrak{o}} \int\!\!\cdot\!\!\int_{\substack{t_p>0\\ N(t)\geqq 1}} \left(\chi(\mathfrak{a})N(t)^{\frac{s}{2}} + I(\chi)\bar\chi(\mathfrak{a})N(t)^{\frac{1-s}{2}}\right) N\mathfrak{a}^{\frac{g}{n}} e^{-\pi|d|^{-\frac{1}{n}} N\mathfrak{a}^{\frac{2}{n}} S(t)} \sqrt{t_1\cdots t_g}\ \frac{dt_1\cdots dt_{r+1}}{t_1\cdots t_{r+1}}, \quad (8)$$

with the abbreviations $N(t)=t_1\cdots t_n$ and $S(t)=t_1+\cdots+t_n$. The formula (8) is nothing but the Siegel formula in case $E(\chi)=1$ [6].

If we take σ_0 such that

$$1-\sigma_0 \leqq \sigma_1 \leqq \sigma_2 \leqq \sigma_0$$

for given $\sigma_1 \leqq \sigma_2$, then it follows from (8) that

$$\Phi(s, \chi) + \frac{2^{r_1+r_2}\pi^{r_2} R h}{w\sqrt{|d|}} \frac{E(\chi)}{s(1-s)} \ll c(\sigma_0)$$

and

$$(s-1)^{E(\chi)} \zeta(s, \chi) \ll c(\sigma_0)\, e^{c|t|}\, N\mathfrak{m}^{\frac{\sigma_0 - 1}{2}} \qquad (9)$$

for $\sigma_1 \leqq \sigma \leqq \sigma_2$.

Let $0 < \delta < 1$. Obviously

$$|\zeta(1+\delta+it, \chi)| \leqq \zeta_K(1+\delta) \ll \frac{1}{\delta} \qquad (10)$$

Put

$$f(s, \chi) = I(\chi)\, A(\chi)^{\frac{1}{2}-s}\, 2^{r_2(2s-1)} \left\{\frac{\Gamma(1-\frac{s}{2})}{\Gamma(\frac{1}{2}+\frac{s}{2})}\right\}^{q} \left\{\frac{\Gamma(\frac{1}{2}-\frac{s}{2})}{\Gamma(\frac{s}{2})}\right\}^{r_1 - q} \left\{\frac{\Gamma(1-s)}{\Gamma(s)}\right\}^{r_2}$$

Then the functional equation can be expressed as

$$\zeta(s, \chi) = f(s, \chi)\, \zeta(1-s, \bar{\chi}). \qquad (11)$$

With the aids of (10) and (11), we get

$$\zeta(-\delta+it, \chi) \ll c(\sigma_1, \sigma_2)\, N\mathfrak{m}^{\frac{1}{2}+\delta} |t|^{(\frac{1}{2}+\delta)n} \left\{1 + O\left(\frac{1}{|t|}\right)\right\} \qquad (12)$$

for $\sigma_1 \leqq \sigma \leqq \sigma_2$, $|t| \geqq 1$, since

$$f(s, \chi) \ll c(\sigma_1, \sigma_2)\, N\mathfrak{m}^{\frac{1}{2}-\sigma} |t|^{(\frac{1}{2}-\sigma)n} \left\{1 + O\left(\frac{1}{|t|}\right)\right\}.$$

Hence, from Theorem 5, we have

$$\zeta(s, \chi) \ll N\mathfrak{m}^{\frac{1}{2}(1-\sigma)} (|t|+2)^{\frac{n}{2}(1-\sigma)} \log N\mathfrak{m}(|t|+2)$$

for $0 \leqq \sigma \leqq 1$, provided $E(\chi)=0$. In applying the theorem we use (9) and (10), (12), putting

$$\delta = \frac{1}{\log N\mathfrak{m}(|t|+2)}.$$

More precisely we shall get the following result.

Theorem 6. In case $E(\chi)=0$, we have

$$\zeta(s, \chi) \ll \begin{cases} c(\sigma)\{N\mathfrak{m}(|t|+2)^n\}^{\frac{1}{2}-\sigma} \log N\mathfrak{m}(|t|+2) & \sigma \leqq 0 \\ \{N\mathfrak{m}(|t|+2)^n\}^{\frac{1-\sigma}{2}} \log N\mathfrak{m}(|t|+2) & 0 \leqq \sigma \leqq 1 \\ \log N\mathfrak{m}(|t|+2) & 1 \leqq \sigma \end{cases}$$

In case $E(\chi)=1$, we can also obtain similar results by using

$$\frac{s(1-s)}{(s+\sigma_0)(1-s+\sigma_0)}\zeta_K(s)$$

in place of $\zeta_K(s)$, σ_0 being determined satisfying

$$2-\sigma_0 \leqq \sigma_1 \leqq \sigma_2 \leqq \sigma_0 .$$

In particular, we have

$$\zeta(\tfrac{1}{2}+it, \chi) \ll \{N\mathfrak{m}(|t|+2)^n\}^{\frac{1}{4}} \log N\mathfrak{m}(|t|+2).$$

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