Note on Proizvolov's example

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Let f be a map (= continuous map) of a topological

space X onto a topological space Y.   We say that f is a

compact-covering map if every compact subset of Y is the image

of some compact subset of X under f.   In the following cases,

every open map is compact-covering :

(1)   (E. Michael [4])   X is a metric space and Y is a T_2

space and, for some metric on X, $f^{-1}(y)$ is complete for

each $y \in Y$.

(2)   (A. Arhangel'skii [2])   X is a Čech-complete space and

Y is a T_2 space.

(3)   (K. Alster [1])   X is a metric space and Y is a countable

T_2 space.

(4)   (K. Nagami [5])   X is a p-space and Y is a T_3 space, and

$f^{-1}(y)$ is compact for each $y \in Y$.

On the other hand, V.V.Proizvolov [6] constructed an

example such that there exists an open, at most two-to-one

map from a Lindelöf, first countable T_3 space onto a compact

metric space, which is not compact-covering.

In this note, we would like to give an adding explanation

for his example, using the following lemma :

Lemma. Let $(X, \mathcal{I}_1)$ **and** $(X, \mathcal{I}_2)$ **be compact** T_2 **spaces**

with $\mathcal{I}_1 \subset \mathcal{I}_2$. **Then** $\mathcal{I}_1 = \mathcal{I}_2$ **holds.**

This is an immediate consequence from the fact that the

identity map from $(X, \mathcal{I}_2)$ to $(X, \mathcal{I}_1)$ is homeomorphic.

Proizvolov's example. Let P be the set $[0,1] \times [0,1]$ and

$P_0 = [0,1] \times \{\frac{1}{2}\}$, and define the topology of P as below :

If $p \in P - P_0$, p has a neighborhood base in the usual sense of

Euclidean plane. For any $p = (p_1, \frac{1}{2}) \in P_0$ and for any natural

numbers ℓ , m and n, let $U_{\ell mn}(p)$ be the subset of P which

consists of p and of all points satisfying one of the follow-

ing three conditions : (1) $p_2' \leq \frac{1}{2}$, and $(p_1 + \frac{1}{n} - p_1')^2 + (\frac{1}{2} - p_2')^2$

$< \frac{1}{n^2}$ or $(p_1 - \frac{1}{n} - p_1')^2 + (\frac{1}{2} - p_2')^2 < \frac{1}{n^2}$; (2) $p_1 - \frac{2}{n} < p_1' < p_1 + \frac{2}{n}$,

$\frac{1}{2} \leq p_2'$ and $p_2' - \frac{1}{2} < \frac{1}{m} |p_1' - p_1|$; (3) $\frac{1}{2} \leq p_2' < \frac{1}{2} + \frac{1}{\ell}$ and $p_2' - \frac{1}{2}$

$> m|p_1' - p_1|$, , and let $\{U_{\ell mn}(p)\}_{\ell,m,n=1}^{\infty}$ be the neighborhood base

at p. Then it is easily seen that P is a Lindelöf, first

countable T_3 space. Let $Y = [0,1] \times [0, \frac{1}{2}]$ be the subspace of

the Euclidean plane and f the map from P onto Y such that it

identifies the points which are symmetric with respect to P_0.

Then f is clearly an open, at most two-to-one map.   It remains

to show that f is not compact-covering.   On the contrary,

suppse f is compact-covering.   Then there exists a compact

subset K of P, whose image by f covers Y.   Let $\mathcal{T}_1$ be the

topology of K as the subspace of the Euclidean plane, and let

$\mathcal{T}_2$ be the topology of K as the subspace of P.   Then $\mathcal{T}_1 < \mathcal{T}_2$

holds.   Hence, by Lemma $\mathcal{T}_1 = \mathcal{T}_2$ holds.   On the other hand,

since K covers P_0, by the definition of $\mathcal{T}_2$ it contains no

countable base ; however, $\mathcal{T}_1$ contains a countable base.

This contradiction shows that f is not compact-covering.

Supplement.   A space X is called a space of countable

(resp. point-countable) type if every compact subset (resp.

point) of X is contained in some compact subset of X with a

countable neighborhood base (cf. [3]).

As for the research of K. Nagami [5], there was a question

whether every open compact map defined on a T_3 space of count-

able type is compact-covering, and it was informed that V. V.

Proizvolov [6] solved it in the negative.   However, in his

example mentioned above, P could not be of countable type.

Because, P_o is a compact G_δ-subset of P which has no countable

neighborhood base in P.   Hence, it seems that such question

remains still open.

Refernces

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