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GENERAL BOUNDARY PROBLEMS FOR LINEAR DIFFERENTIAL EQUATIONS

by

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Contents

Preface

The aim of this paper is to study an existence theory of solutions of boundary problems for general linear differential equations. There are many methods to prove the existence of solutions. But they cannot be applied equally to all types of boundary problems, such as elliptic, evolutional, or mixed type problems. In this paper the author tries to lay the foundations of a method which can be applied to various types of boundary problems. Especially we have obtained existence theorems for elliptic boundary problems in non-compact manifolds, evolutional boundary problems with Cauchy data given on the characteristic boundary, and similar ones with respect to Schwartz' distributions. Moreover our techniques may be applied to mixed type problems for evolution or Tricomi equations.

Our approach is fairly different from a traditional one. We do not use completions of function spaces with respect to a norm. We deal with many local spaces of distributions directly, which are endowed with the structure of Frechet or more complex locally convex spaces. First we improve Treves' conditions [25] of surjectivity of a continuous linear mapping on a Frechet space to another, so that they can be applied more directly to closed linear operators appearing in our problems. Employing the calculations in Chapter two, we can immediately write a necessary and sufficient condition for the solvability of each suitably posed boundary problem. It consists of two kinds of conditions. One is on the semi-global existence of solutions, that is, the existence in any relatively compact open subset. An equivalent condition is given as a collection of inequalities for the dual operator with respect to some kinds of norms, such as Sobolev ones. The other condition is called T-convexity, which is a generalization of the classical P-convexity condition for linear differential equations with no boundary conditions (cf. Malgrange [17], Treves [25], and Hörmander [13]). This guarantees a possibility that a global solution can be constructed by approximations using semi-global solutions.

We explain the plan of this paper. In Chapter I we develop an existence theory for linear equations in locally convex spaces. In the next Chapter II we introduce the function space $\mathfrak{F}(\tilde{\Omega}, \omega_1; E)$. Roughly speaking, this consists of distribution sections of the vector bundle E, which can be extended through a part ω of the boundary of $\tilde{\Omega}$ to belong to the function space $\mathcal{F}$, and which vanishes outside the part ω_1 of ω. We prove some properties of these spaces in order to apply the main theorem of Chapter I. After that we can immediately obtain necessary and sufficient conditions for the solvability of many boundary problems. Using these results we study elliptic boundary problems in Chapter III, where we prove the existence of solutions in non-compact manifolds. Chapter IV is devoted to the study of boundary problems of evolution type. In these cases we recognize important roles played by differential operators on the boundary, which are induced by the original differential operator and a normal vector field on the boundary. When we want to reduce a Cauchy type condition to the property that functions belong to our function spaces, especially to $C^\infty(\tilde{\Omega}, \omega_1)$, we have to solve such differential equations on the boundary. We can solve many kinds of Cauchy problems with data given on characteristic boundaries. Especially the Goursat problem is solved. In Chapter V we study the existence of distribution solutions. Many difficulties arise from the complicated topological structures of the spaces of distributions. For a more detailed description of the contents we refer to the introductions of each chapter.

Finally the author would like to thank Professor K. Yosida for his stimulative advice in the early stage of this work. Thanks are also due to Professor H. Komatsu for many valuable suggestions.

Chapter I. Fundamental lemmas in locally convex spaces.

§1.0. Introduction.

This chapter is devoted to the study of abstract existence theory for linear equations. The results of this chapter form a basis of the subsequent three chapters. Combining Theorem 1.2.1. and the calculations in Chapter II, we can immediately obtain conditions for the solvability of linear equations. Our main theorem is a generalization of a result due to Treves[25] and Harvey[7], and in many cases their result is sufficient for our use. They gave conditions for a linear operator to be surjective. But we encounter many cases where the range of the operator has some kinds of compatibility conditions. In such instances our theorem can be used. Typical examples are overdetermined systems of linear differential equations with constant coefficients (see Ehrenpreis[5] and Hörmander[9]). We can explain the serious gap between determined systems and overdetermined ones from our point of view. In the overdetermined case we have to find a new element z' in the estimate (1.2.2) of Theorem 1.2.1. This causes a very hard problem.

Now in section 1.1. we make some definitions and preliminary propositions for the next chapter. We have to calculate the norm (1.1.1) in many concrete cases and this will be done in Chapter II.

In section 1.2. we state our main theorem and its proof will be given in section 1.3. Its essential part is contained in the proof of the open mapping theorem, where the step by step construction of a solution is done (see Ptak[21]). Our task is to reform the conditions to be more manageable.

§1.1. Preliminaries

Let E and F be (Hausdorff) locally convex spaces, and T a densely defined linear operator of E into F. Let E' be the dual space of E. We denote the value of $x' \in E'$ at $x \in E$ by $\langle x, x' \rangle$. The absolute value $|x'|$ of $x' \in E'$ is defined by the equality $|x'|(x) = |\langle x, x' \rangle|$, for $x \in E$. It is obvious that $|x'|$ is a continuous seminorm on E. Let D(T), R(T), ${}^tT$ represent the domain, the range, and the dual operator of T respectively. By Spec E we denote the set of all continuous seminorms on E. For every seminorm $p \in \mathrm{Spec}\,E$ and a constant $C > 0$, we define $C \cdot p$ by $(C \cdot p)(x) = C \cdot p(x)$ for $x \in E$. For $p, q \in \mathrm{Spec}\,E$, we write $p \leqq q$ if $p(x) \leqq q(x)$ for $x \in E$. We call $\mathcal{B}$ a basis of continuous seminorms on E if and only if $\mathcal{B}$ is a subset of Spec E and to each $p \in \mathrm{Spec}\,E$ there exist $q \in \mathcal{B}$ and a constant $C > 0$ such that $p \leqq C \cdot q$. For $x' \in E'$ and $p \in \mathrm{Spec}\,E$, we write

$$\|x'\|_p = \inf\{C > 0;\ |x'| \leqq C \cdot p\}. \qquad (1.1.1)$$

If there exists no such positive constant C, we set $\|x'\|_p = \infty$. For any seminorm $p \in \mathrm{Spec}\,E$, let E_p be the normed space E/Ker p with the norm induced by p. Here we denote by Ker p the kernel of p, or the set of all $x \in E$ such that $p(x) = 0$. Let $\hat{E}_p$ and E'_p be the completion and the dual space of E_p respectively. It is easy to verify that (1.1.1) is a norm in the Banach space of all $x' \in E'$ such that (1.1.1) is finite, which is isomorphic to the

Banach space E'_p. In the following we identify these two spaces.

Proposition 1.1.1. Let E, F, G be locally convex spaces such that F is a subspace of E and there exists a continuous open linear surjection ρ of F onto G. We denote the natural injection of F into E by ι. For every $p \in \operatorname{Spec} E$ we write

$$p^*(z) = \inf\{p(y);\quad y \in F \text{ and } \rho(y) = z\},\ z \in G. \tag{1.1.2}$$

Then p^* is a continuous seminorm on G, and the set of all such p^* ($p \in \operatorname{Spec} E$) is equal to Spec G. Moreover if $\mathcal{B}$ is a basis of Spec E, then p^* ($p \in \mathcal{B}$) form a basis of Spec G.

Take two seminorms p, $q \in \operatorname{Spec} E$ such that $q = p^* \circ \rho$ on F. Then ${}^t\rho$ induces an isomorphism of G'_{p^*} onto $F'_{p^* \circ \rho}$, and ${}^t\rho^{-1} \circ {}^t\iota$ induces an epimorphism, i.e. an open continuous linear surjection, of E'_q onto G'_{p^*}. Hence for any $z' \in G'_{p^*}$ we have

$$\|z'\|_{p^*} = \inf\{\|x'\|_q;\quad x' \in E' \text{ and } {}^t\iota(x') = {}^t\rho(z')\}. \tag{1.1.3}$$

Proof. Let $p \in \operatorname{Spec} E$. Then p^* is a seminorm on G. Since ρ is open, there exists a seminorm $r \in \operatorname{Spec} G$ such that the following holds. If $z \in G$ and $r(z) \leqq 1$, then there exists $y \in F$ such that $p(y) \leqq 1$ and $\rho(y) = z$. Hence it follows from (1.1.2) that $p^*(z) \leqq 1$. Thus we have proved that p^* is continuous.

Next take a seminorm $r \in \operatorname{Spec} G$. Then $r \circ \rho$ is an element of Spec F and hence $r \circ \rho$ is equal to $p \circ \iota$ for some $p \in \operatorname{Spec} E$. Therefore it follows that $r = p^*$. Moreover there exist $p_1 \in \mathcal{B}$ and a constant $C > 0$ such that $p \leqq C \cdot p_1$ in E, hence $r \circ \rho \leqq C \cdot p_1 \circ \iota$. Then we obtain

$$\begin{aligned} r(z) &\leqq C \cdot \inf\{p_1(y);\quad y \in F \text{ and } \rho(y) = z\} \\ &= C \cdot p_1^*(z), \quad z \in G. \end{aligned}$$

Therefore the former part of Proposition 1.1.1. is proved.

Let p, $q \in \operatorname{Spec} E$ and $q = p^* \circ \rho$ on F. Then for any $x' \in E'_q$ we have

$$\|{}^t\tau(x')\|_{q\circ\tau} = \|x'\circ\tau\|_{q\circ\tau} \leqq \|x'\|_q < \infty,$$

and hence it follows that ${}^t\tau(x') \in F'_{q\circ\tau} = F'_{p^*\circ\rho}$. Therefore ${}^t\tau$ induces a continuous linear operator of E'_q into $F'_{q\circ\tau}$. If $y' \in F'_{q\circ\tau}$, the inequality $|y'| \leqq \|y'\|_{q\circ\tau} \cdot q\circ\tau$ holds. Hence from the Hahn-Banach theorem there exists $x' \in E'$ such that $|x'| \leqq \|y'\|_{q\circ\tau} \cdot q$ on E and $x'\circ\tau = y'$. Then we have $\|x'\|_q \leqq \|y'\|_{q\circ\tau}$ and ${}^t\tau(x') = y'$. Therefore ${}^t\tau$ induces an epimorphism of E'_q onto $F'_{q\circ\tau}$.

Now take $z' \in G'_{p^*}$. Then for every $y \in F$ we obtain

$$\begin{aligned} |\langle y, {}^t\rho(z')\rangle| &= |\langle \rho(y), z'\rangle| \\ &\leqq \|z'\|_{p^*} \cdot p^*\circ\rho(y) \\ &= \|z'\|_{p^*} \cdot q(y). \end{aligned}$$

Hence it follows that ${}^t\rho(z') \in F'_{q\circ\tau}$ and $\|{}^t\rho(z')\|_{q\circ\tau} \leqq \|z'\|_{p^*}$. Therefore ${}^t\rho$ induces a continuous linear operator of G'_{p^*} into $F'_{q\circ\tau}$.

If $y' \in F'_{p^*\circ\rho}$, the inequality $|y'| \leqq \|y'\|_{p^*\circ\rho} \cdot p^*\circ\rho$ implies that $\operatorname{Ker}\rho \subset \operatorname{Ker} y'$. Then there exists $z' \in G'$ such that $z'\circ\rho = y'$. Hence it follows that $y' = {}^t\rho(z')$ and

$$\begin{aligned} \|z'\|_{p^*} &= \inf\{C > 0;\ |\langle z, z'\rangle| \leqq C\cdot p^*(z) \text{ for all } z \in G\} \\ &= \inf\{C > 0;\ |\langle y, y'\rangle| \leqq C\cdot p^*\circ\rho(y) \text{ for all } y \in F\} \\ &= \|y'\|_{p^*\circ\rho}. \end{aligned}$$

Thus we have proved that ${}^t\rho$ induces an isomorphism of G'_{p*} onto $F'_{q\cdot\tau}$, and this completes the proof.

Proposition 1.1.2. Let E_j, $j=1,2,\cdots,\ell$ be locally convex spaces, and F their product space. Let $\mathcal{B}_j$ be a basis of $\operatorname{Spec} E_j$ for each $j=1,2,\cdots,\ell$. Then the following seminorms in F

$$(x_j)\longmapsto q((x_j))=\sum_{j=1}^{\ell} p_j(x_j),\quad p_j\in\mathcal{B}_j,\ j=1,2,\cdots,\ell$$

form a basis of Spec F. Moreover we have the following isomorphism

$$F'_q\cong\prod_{j=1}^{\ell}(E_j)'_{p_j}.$$

The proof of this proposition is easy and then we omit it.

§1.2. The main theorem

Let T be a densely defined linear operator of E into F, and N a subset of F.

Definition 1.2.1. The pair (E, N) is called T-convex if for every seminorm $p\in\operatorname{Spec} E$ there exists a seminorm $q\in\operatorname{Spec} F$ such that the following holds. If $y'\in D({}^tT)$ and $\|{}^tT(y')\|_p$ is finite, then y' vanishes on $N\cap\operatorname{Ker} q$.

This definition is a generalization of the P-convexity condition found by Malgrange [17] in the theory of general partial differential equations and then generalized by Treves [25] in the theory of locally convex spaces. The following theorem is a generalization of their results. The essential part of the proof has been already well-known in the study of the open mapping theorem (cf. Ptàk [21]).

Theorem 1.2.1. Let E and F be Frechet spaces, T a densely defined closed linear operator of E into F, and N a closed subspace of F containing the range R(T) of T. Let $\mathcal{B}_E$ and $\mathcal{B}_F$ be bases of continuous seminorms on E and F respectively. Then $R(T) = N$ if and only if the following conditions (1) and (2) hold. Moreover (2) and (3) are equivalent.

(1) The pair (E, N) is T-convex.

(2) For every $y \in N$ and $q \in \mathcal{B}_F$ there exists $x \in D(T)$ such that $q(y - T(x)) = 0$.

(3) For every seminorms $p \in \mathcal{B}_E$ and $q \in \mathcal{B}_F$ there exist $r \in \mathcal{B}_F$ and a positive constant C such that the following is true. To every $y' \in D({}^tT)$, which vanishes in $\operatorname{Ker} q$, there exists $z' \in D({}^tT)$, which also vanishes in $\operatorname{Ker} q$, such that

$$< y, y' > = < y, z' > \quad \text{for all } y \in N, \qquad (1.2.1)$$

and

$$\| z' \|_r \leq C \| {}^tT(z') \|_p . \qquad (1.2.2)$$

Remark 1.2.2. If E is a B-complete space (cf. [21]) and N is a barrelled subspace of F containing R(T), then the conclusion of the theorem is also true.

Corollary 1.2.3. Let E and F be Frechet spaces, and T a densely defined closed linear operator of E into F. Let $\mathcal{B}_E$ and $\mathcal{B}_F$ be bases of continuous semi-norms on E and F respectively. Then the range of T is closed if and only if the following two conditions (1) and (2) hold. Under these conditions the range of T is equal to the polar of the kernel of ${}^tT$.

(1) For every $p \in \mathcal{B}_E$ there exists $q \in \mathcal{B}_F$ such that $y' \in D({}^tT)$ and $\|{}^tT(y')\|_p < \infty$ implies the existence of $z' \in D({}^tT)$, which satisfies ${}^tT(y') = {}^tT(z')$ and $z' = 0$ in $\operatorname{Ker} q$.

(2) For every seminorms $p \in \mathcal{B}_E$ and $q \in \mathcal{B}_F$ there exist a seminorm $r \in \mathcal{B}_F$ and a positive constant C such that the following is true. For every $y' \in D({}^tT)$, which vanishes in $\operatorname{Ker} q$, there exists another $z' \in D({}^tT)$, which also vanishes in $\operatorname{Ker} q$, such that ${}^tT(y') = {}^tT(z')$ and

$$\|z'\|_r \leqq C\|{}^tT(z')\|_p.$$

This corollary follows from the above theorem. Its simple proof may be left to the reader.

§1.3. <u>The proof of Theorem</u> 1.2.1.

(I) Conditions (1) and (2) imply $R(T) = N$.

Let U be a neighbourhood of 0 in E. Then there exists a seminorm $p \in \operatorname{Spec} E$ such that its closed unit ball $B_p = \{x \in E;\ p(x) \leqq 1\}$ is contained in U. Let B be a subset of N defined by

$$B = \{y \in N;\ |\langle y, y' \rangle| \leqq \|{}^tT(y')\|_p \text{ for any } y' \in D({}^tT)\}. \quad (1.3.1)$$

From the T-convexity of (E, N) there exists $q \in \operatorname{Spec} F$ corresponding to p.

Now take an element y in N. From the condition (2), there exists $x \in D(T)$ such that $q(y-T(x))=0$. Then $y-T(x)$ is contained in $N \cap \operatorname{Ker} q$. Suppose $y' \in D({}^tT)$ and $\|{}^tT(y')\|_p$ is finite, then $\langle y-T(x), y'\rangle = 0$. Hence we obtain

$$|\langle y, y'\rangle| = |\langle T(x), y'\rangle| = |\langle x, {}^tT(y')\rangle| \leqq p(x)\cdot \|{}^tT(y')\|_p. \qquad (1.3.2)$$

Taking $\lambda = p(x)^{-1}$ or $\lambda = 1$, according as $p(x) \neq 0$ or $= 0$, we have $\lambda y \in B$. Thus we have proved that B is absorbing in N. Therefore B is a barrel in N. Since N is a barrelled space, B is a neighbourhood of 0 in N.

Next we prove that B is contained in $[T(B_p)]_N$, the closure of $T(B_p)$ in N. Take an element $y \in N$ which does not belong to $[T(B_p)]_N$. Then by Mazur's theorem there exists $z' \in N'$ such that $|\langle y, z'\rangle| > 1$ and $|z'| \leqq 1$ $[T(B_p)]_N$. From the Hahn-Banach theorem z' is equal to the restriction of some $y' \in F'$ to N. Then $|\langle y, y'\rangle| > 1$ and $|y'| \leqq 1$ $[T(B_p)]_N$, which implies that $|\langle T(x), y'\rangle| \leqq p(x)$ for all $x \in D(T)$. Hence the linear functional $x \longmapsto \langle T(x), y'\rangle$ is continuous linear on D(T), and then there exists $x' \in E'$ such that $\langle T(x), y'\rangle = \langle x, x'\rangle$ for all $x \in D(T)$. Therefore y' belongs to $D({}^tT)$ and ${}^tT(y') = x'$. Since $|{}^tT(y')| \leqq p$, we have

$$\|{}^tT(y')\|_p \leqq 1 < |\langle y, y'\rangle|.$$

Then y does not belong to B.

We have proved that $B \subset [T(B_p)]_N \subset [T(U)]_N$, so that $[T(U)]_N$ is a neighbourhood of 0 in N. Therefore T is almost open as an operator of E into N. Since E is B-complete, we can conclude that $R(T) = N$ (cf. Pták [21] and Mochizuki [19]).

(II) The relation $R(T) = N$ implies the conditions (1) and (2).

The condition (2) is trivial. Let $p \in \operatorname{Spec} E$. Let B be the subset of N defined by (1.3.1). Then B is absorbing in N. In fact take an element $y \in N$. From the hypothesis there exists $x \in D(T)$ such that $y = T(x)$. Then we have the same inequality as (1.3.2) for every $y' \in D({}^tT)$. Hence λy is contained in B if we take $\lambda = p(x)^{-1}$ or $\lambda = 1$.

Since N is barrelled and B is a barrel in N, the set B is a neighbourhood of 0 in N. Let q_o be the seminorm on N defined by

$$q_o(y) = \inf\{C > 0;\ y \in C \cdot B\}, \quad y \in N.$$

Then q_o is a continuous seminorm on N and hence it is the restriction of some $q \in \operatorname{Spec} F$ to N.

Now let y be an element of $N \cap \operatorname{Ker} q$ and y' an element of $D({}^tT)$ such that $\|{}^tT(y')\|_p$ is finite. Since y belongs to the kernel of q_o, there exists a sequence of positive constants $C_n > 0$, $n = 1, 2, \cdots$, tending to 0 as n tends to infinity, such that $y \in C_n \cdot B$, $n = 1, 2, \cdots$. Then we have

$$|\langle y, y' \rangle| \leqq C_n \cdot \|{}^tT(y')\|_p$$

and the second term tends to 0 as n becomes large. Hence it follows that $\langle y, y' \rangle = 0$. We have thus proved that (E, N) is T-convex.

(III) The condition (2) implies (3).

Let $p \in \mathcal{B}_E$ and $q \in \mathcal{B}_F$. We denote the canonical epimorphism of F onto F/Ker q by ρ. Let $T_q = \rho \circ T$. Since ρ is open, the image $\rho(N)$ is also a barrelled space. From the condition (2) we have $R(T_q) = \rho(N)$. Then from the open mapping theorem T_q is an open mapping of E onto $\rho(N)$. Hence $T_q(B_p)$ is a neighbourhood of 0 in $\rho(N)$. We define the seminorm r_o on $\rho(N)$ by

$$r_o(y_q) = \inf \{ C > 0; \ \ y_q \in C \cdot T_q(B_p) \}, \ y_q \in \rho(N).$$

Then r_o is a continuous seminorm on $\rho(N)$. From Proposition 1.1.1. there exist $r \in \mathcal{B}_F$ and a constant $C_o > 0$ such that $r_o \leqq C_o \cdot r^*$ on $\rho(N)$, where r^* is defined by

$$r^*(y_q) = \inf \{ r(y); \ \ y \in F \text{ and } \rho(y) = y_q \}, \ \ y_q \in F/\text{Ker}\, q.$$

Let $y' \in D({}^tT)$ and $\langle y, y' \rangle = 0$ for all $y \in \text{Ker}\, q$. Let $y \in N$ and $\rho(y) \in C \cdot T_q(B_p)$. Then there exists $x \in D(T)$ such that $p(x) \leqq 1$ and $\rho(y) = C \cdot \rho \circ T(x)$. Then it follows that $y - C \cdot T(x) \in$ $\in \text{Ker}\, q$ and hence we have

$$|\langle y, y' \rangle| = C \cdot |\langle T(x), y' \rangle| = C\, |\langle x, {}^tT(y') \rangle| \leqq C \cdot \|{}^tT(y')\|_p.$$

Therefore we obtain

$$|\langle y, y' \rangle| \leqq r_o \circ \rho(y) \cdot \|{}^tT(y')\|_p \leqq C_o \cdot r^* \circ \rho(y) \cdot \|{}^tT(y')\|_p,$$

for all $y \in N$. From the Hahn-Banach theorem there exists $z' \in F'$

such that $z' = y'$ in N and

$$|\langle y, z' \rangle| \leqq C_o\, r^{*} \circ \rho(y) \cdot \|{}^tT(y')\|_p$$

for all $y \in F$. Then z' vanishes on $\operatorname{Ker} q$ and for every $x \in D(T)$ we have $\langle x, {}^tT(y') \rangle = \langle T(x), z' \rangle$. Hence it follows that $z' \in D({}^tT)$ and ${}^tT(z') = {}^tT(y')$. Moreover we have

$$\|z'\|_r \leqq C_o \|{}^tT(y')\|_p = C_o \|{}^tT(z')\|_p.$$

Therefore the condition (3) holds.

(IV) The condition (3) implies (2).

Let $q \in \mathcal{B}_F$. Take a seminorm $p \in \mathcal{B}_E$ and define the set B' by

$$B' = \{ y_q \in \rho(N);\ |\langle y_q, y'_q \rangle| \leqq \|{}^tT_q(y'_q)\|_p \text{ for all } y'_q \in D({}^tT_q) \}.$$

Then we can easily obtain the fact that B' is equal to

$$\{ \rho(y);\ y \in N \text{ and } |\langle y, y' \rangle| \leqq \|{}^tT(y')\|_p \text{ if } y' \in D({}^tT) \text{ and } y' \text{ vanishes in } \operatorname{Ker} q \}.$$

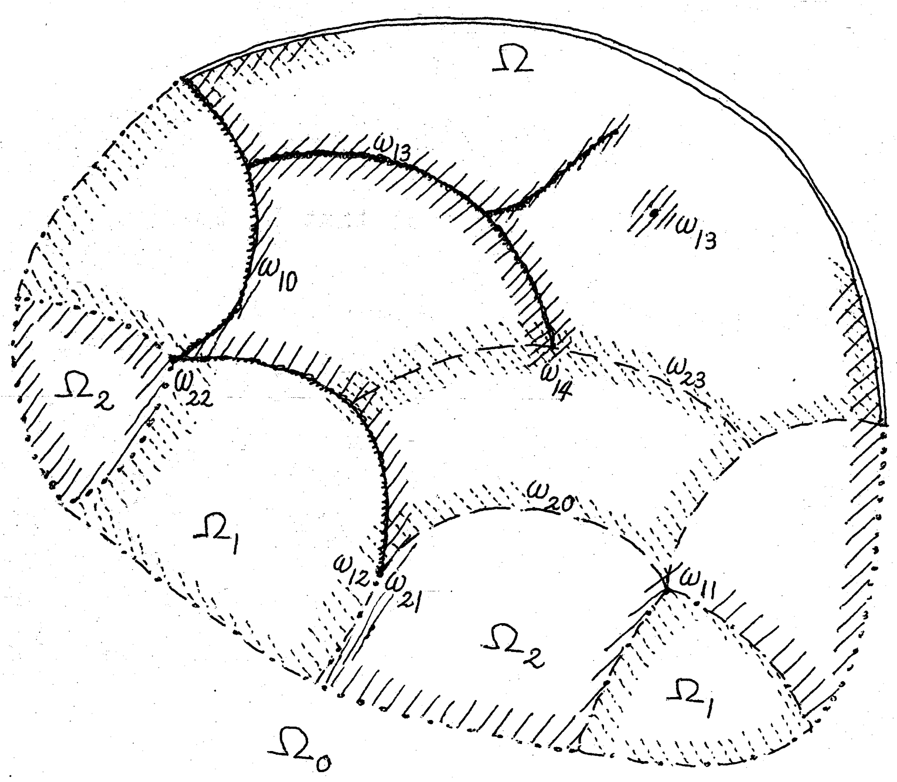
We can prove that B' is absorbing in $\rho(N)$. In fact let $y \in N$. From the condition (3) there exist $r \in \mathcal{B}_F$ and a constant $C > 0$ such that the conclusin of (3) holds. Then for any $y' \in D({}^tT)$, which is equal to zero on $\operatorname{Ker} q$, we obtain for some $z' \in D({}^tT)$

$$|\langle y, y' \rangle| = |\langle y, z' \rangle| \leqq r(y) \|z'\|_r \leqq$$
$$\leqq C \cdot r(y) \|{}^tT(z')\|_p = C \cdot r(y) \|{}^tT(y')\|_p.$$

Hence we have $\lambda \cdot \rho(y) \in B'$ for some λ.

We have proved that B' is a barrel in $\rho(N)$, and then B' is a neighbourhood of 0 in $\rho(N)$, Moreover as in the proof of (I) we can prove the inclusion $B' \subset [T_q(B_p)]_{\rho(N)}$. Therefore the set $[T_q(B_p)]_{\rho(N)}$ is a neighbourhood of 0 in $\rho(N)$. In other word T_q is almost open, and hence the range of T_q is equal to $\rho(N)$. Then the condition (2) is valid and the proof is complete.

<u>Figure</u> (see section 2.2.)



Chapter II. The spaces $\mathcal{F}(\tilde{\Omega}, \omega_1; E)$.

§2.0. Introduction.

In this chapter we define our basic spaces and prove some of their properties in order to apply Theorem 1.2.1. Roughly speaking, $\mathcal{F}(\tilde{\Omega}, \omega_1; E)$ is the space of all distribution sections of the vector bundle E, which can be extended from ω to be an element of $\mathcal{F}$ and vanishes outside the subset ω_1 of the boundary. This function space plays a central role in the following chapters, where a unified treatment of boundary problems for linear differential equations will be done. Because the topological structure of $\mathcal{F}(\tilde{\Omega}, \omega_1; E)$ is very complicated, we have to solve many delicate problems.

In the first section we state some elementary facts on the spaces of sections of vector bundles. Sometimes we will have to calculate on local coordinate patches. In section 2 we define $\mathcal{F}(\tilde{\Omega}, \omega_1; E)$ and show that our definition is independent of the choice of subsidiary sets. Section 3 is devoted to the study of $C^\infty(\tilde{\Omega}, \omega_1; E)$, which consists of C^∞ sections. In section 4 we study Sobolev spaces, but we have to impose some restrictions on the boundary, that is, we have to assume curve segment property. The author has not succeeded in proving Proposition 2.4.1. without this hypothesis. Section 5 is preliminary for the consideration of linear differential equations with constant coefficients in Chapter IV.

After having finished the calculations of this chapter, we can apply Theorem 1.2.1. and immediately obtain conditions for the solvability of boundary problems. We will meet many such applications in later chapters.

§2.1: Preliminaries.

Let M be a σ-compact C^∞ manifold (without boundary) of dimension n. Take a family $\mathcal{K}$ of C^∞ coordinate systems $\varkappa$ on M. Then $\varkappa$ is a diffeomorphism of an open subset $U_\varkappa$ of M onto $\varkappa(U_\varkappa) \subset \mathbb{R}^n$. We can choose them such that $\{U_\varkappa;\ \varkappa\in\mathcal{K}\}$ is a locally finite open covering of M. Fix a family of functions $\chi_\varkappa \in C_0^\infty(U_\varkappa)$ such that $0 \leqq \chi_\varkappa(x) \leqq 1$ and $\sum_{\varkappa\in\mathcal{K}} \chi_\varkappa^2(x) = 1$, $x\in M$.

In the following sections E denotes an N-dimensional complex C^∞ vector bundle over M. Let π be the projection of E onto M, and $\Phi_\varkappa$, $\varkappa\in\mathcal{K}$ a family of C^∞ coordinate charts over $\varkappa$. Then $\Phi_\varkappa$ is a diffeomorphism of $U_\varkappa \times \mathbb{C}^N$ onto $\pi^{-1}(U_\varkappa)$. If $g_{\varkappa\varkappa'}$ is the C^∞ transition function on $U_\varkappa \cap U_{\varkappa'}$, then we have

$$\Phi_{\varkappa'}(x,\ w) = \Phi_\varkappa(x,\ g_{\varkappa\varkappa'}(x)\cdot w) \qquad (2.1.1)$$

for all $x\in U_\varkappa \cap U_{\varkappa'}$ and $w\in\mathbb{C}^N$.

By $C^\infty(M;\ E)$ we denote the space of all C^∞ sections of E over M with the usual topology (cf. Schwartz [22]). Then it is isomorphic (as locally convex spaces) to the space of all families $(u_\varkappa)_{\varkappa\in\mathcal{K}} \in \prod_{\varkappa\in\mathcal{K}} C^\infty(\varkappa(U_\varkappa))^N$ which satisfy the following relations:

$$u_{\varkappa'} = (g_{\varkappa'\varkappa}\circ\varkappa'^{-1})\cdot(u_\varkappa\circ(\varkappa\varkappa'^{-1})) \text{ on } \varkappa'(U_\varkappa \cap U_{\varkappa'}). \qquad (2.1.2)$$

The latter space is endowed with the weakest locally convex topology such that the mapping $(u_\varkappa)_{\varkappa\in\mathcal{K}} \longmapsto u_\varkappa \in C^\infty(\varkappa(U_\varkappa))^N$ is continuous for all $\varkappa\in\mathcal{K}$. This correspondence is given by the following relations:

$$u(x) = \Phi_{\varkappa}(x,\ u_{\varkappa}\circ\varkappa(x)),\quad \text{for all } x \in U_{\varkappa}. \quad (2.1.3)$$

Proposition 2.1.1. The space $C^{\infty}(M;\ E)$ is Frechet-Schwartz, i. e. a limit of a compact (or completely continuous) projective sequence of locally convex spaces (cf. [14] and the references therein).

Proof. Take a sequence K_j, $j=1,2,\cdots$ of compact subsets of M such that each K_j is contained in the interior of K_{j+1} and the union of all K_j is equal to M. Let X_j be the space of all C^j sections of E over K_j such that they can be extended as C^j sections of E over M. The topology of X_j is induced by the restriction mapping of $C^j(M;\ E)$ onto X_j. Let φ_j be the restriction map of X_j into X_{j-1}. Then φ_j is a compact operator. In fact we can use local coordinates to reduce the problem to that in $\mathbb{R}^n$. Then the Ascoli-Arzela theorem can be used. We leave the details to the reader.

Then the sequence $X_1 \xleftarrow{\varphi_2} X_2 \xleftarrow{\varphi_3} X_3 \xleftarrow{\varphi_4} \cdots$ is compact and its limit space can be identified with $C^{\infty}(M;\ E)$.

By $C_0^{\infty}(M;\ E)$ we denote the space of all C^{∞} cross sections of E over M with compact support. This space has the usual Schwartz topology. Let Ω be the volume bundle of M, and E^* the dual bundle of E. Then the space $\mathcal{D}'(M;\ E)$ of all distribution sections of E over M is defined as the dual space of $C_0^{\infty}(M;\ E^*\otimes\Omega)$. By $\mathcal{E}'(M;\ E)$ we denote the space of all distribution sections in $\mathcal{D}'(M; E)$ with compact support. As locally convex spaces $\mathcal{D}'(M; E)$ is isomorphic to the dual space of $C_0^\infty(M; E)$. Similarly $\mathcal{E}'(M; E)$ is isomorphic to the dual space of $C^\infty(M; E)$ as locally convex spaces. In fact it is enough to remark that $C^\infty(M; E)$ and $C^\infty(M; E^*\otimes\Omega)$ are isomorphic. The space $\mathcal{D}'(M; E)$ is canonically isomorphic to the space of all families $(u_\varkappa)_{\varkappa\in\mathcal{X}} \in \prod_{\varkappa\in\mathcal{X}} \mathcal{D}'(\varkappa(U_\varkappa))^N$ satisfying the same relations (2.1.2). Then we can choose the duality bracket such that the following relation holds:

$$< \varphi , u > = \sum_{\varkappa\in\mathcal{X}} <(\chi_\varkappa\circ\varkappa^{-1})\varphi_\varkappa, (\chi_\varkappa\circ\varkappa^{-1})u_\varkappa> \qquad (2.1.4)$$

for all $\varphi\in C_0^\infty(M, E)$ and $u\in\mathcal{D}'(M; E)$.

Let $s\in\mathbb{R}$. We define the Sobolev norm of $u\in\mathcal{E}'(\mathbb{R}^n)$ by

$$\|u\|_{(s)} = (\int|\hat{u}(\xi)|^2(1+|\xi|^2)^s đ\xi)^{1/2} \qquad (2.1.5)$$

if the integral is finite, where $\hat{u}(\xi) = <e^{-ix\cdot\xi}, u>$, and $đ\xi = (2\pi)^{-n}d\xi$. If the integral (2.1.5) diverges, then we write $\|u\|_{(s)} = \infty$. We define the Sobolef norm of $u\in\mathcal{E}'(M; E)$ by

$$\| u\|_{(s)} = (\sum_{\varkappa\in\mathcal{X}} \| (\chi_\varkappa\circ\varkappa^{-1})u_\varkappa\|_{(s)}^2)^{1/2}. \qquad (2.1.6)$$

By $H_{(s)}^c(M; E)$ we denote the space of all $u\in\mathcal{E}'(M; E)$ such that its norm $\|u\|_{(s)}$ is finite. By $H_{(s)}^{loc}(M; E)$ we denote the space of all $u\in\mathcal{D}'(M; E)$ such that $\varphi\cdot u\in H_{(s)}^c(M; E)$ for every $\varphi\in C_0^\infty(M)$. This space is endowed with the family of seminorms $u\longmapsto\|\varphi\cdot u\|_{(s)}$, $\varphi\in C_0^\infty(M)$. Then this is a Frechet space and its dual space is $H_{(-s)}^c(M; E)$ with respect to the following duality bracket:

$$<u, v> = \sum_{\varkappa\in\mathcal{K}} <(\chi_\varkappa\circ\varkappa^{-1})u_\varkappa, (\chi_\varkappa\circ\varkappa^{-1})v_\varkappa> \qquad (2.1.4')$$

for any $u\in H^{loc}_{(s)}(M; E)$ and $v\in H^{c}_{(-s)}(M; E)$. Both spaces $H^{c}_{(s)}(M; E)$ and $H^{loc}_{(s)}(M; E)$ do not depend on the choice of $\mathcal{K}$ and $\{\chi_\varkappa; \varkappa\in\mathcal{K}\}$. From (2.1.4') and (2.1.6) we obtain

$$|<u, v>| \leqq \|u\|_{(s)}\cdot\|v\|_{(-s)} \qquad (2.1.7)$$

for every $u\in H^{c}_{(s)}(M; E)$ and $v\in H^{c}_{(-s)}(M; E)$. The set of seminorms on $C^\infty(M; E)$:

$$u\longmapsto\|\varphi\cdot u\|_{(s)}, \quad s\in\mathbb{R} \text{ and } \varphi\in C^\infty_0(M)$$

is a basis of continuous seminorms on $C^\infty(M; E)$.

§2.2. The definition of the spaces $\mathcal{F}(\tilde{\Omega}, \omega_1; E)$.

Let Ω be an open subset of M, and ω an open subset of the topological boundary $\partial\Omega$ of Ω in M. Take a subset ω_1 of ω. We make the following definitions:

$$\tilde{\Omega}=\Omega\cup\omega, \quad \omega_2=\omega\setminus\omega_1, \quad \omega_{j0}=\mathrm{Int}_\omega(\omega_j\setminus(\overline{\Omega})^\circ),$$

$$\omega_{jj}=\omega_j\setminus(\overline{\omega}_{j0}\cup(\overline{\Omega})^\circ), \quad \omega_{j3}=\mathrm{Int}_\omega(\omega_j\cap(\overline{\Omega})^\circ),$$

$$\omega_{j4}=(\omega_j\cap(\overline{\Omega})^\circ)\setminus\omega_{j3}, \quad \omega_{12}=\omega_1\cap\partial_\omega\omega_{10},$$

$$\omega_{21}=\omega_2\cap\partial_\omega\omega_{20}, \quad j=1,2. \qquad (2.2.1)$$

Here $^\circ$, $^{-}$, ∂ represent the interior, the closure, and the boundary in M respectively. On the other hand Int_ω, ∂_ω represent the interior and the boundary in ω. Then each ω_j is the disjoint union of five sets ω_{j0}, ω_{j1}, ω_{j2}, ω_{j3}, ω_{j4}.

If ω has the curve segment property at every point of $\omega_{11} \cup \omega_{22}$ (Definition 2.4.1.), then

Proposition 2.2.1. there exist three open subset Ω_0, Ω_1, Ω_2 of M, which satisfy the following conditions:

(See the figure on page 16)

(1) $\Omega \subset \Omega_j \subset \Omega_0$, $j = 1,2$.

(2) $\Omega = \Omega_1 \cap \Omega_2$.

(3) $\Omega_0 \subset \overline{\Omega}_1 \cup \overline{\Omega}_2$.

(4) $\Omega_0 \cap \partial\Omega = \omega$.

(5) $\Omega_j \cap \partial\Omega = \omega_{jo} \cup \omega_{j3}$, $j = 1,2$.

(6) $\omega_{jj} \subset (\overline{\Omega_j \setminus \Omega})$, $j = 1,2$.

(7) $\omega_{ko} \cap (\overline{\Omega_j \setminus \Omega}) = \phi$, $j \neq k$.

Since there is no difficulty in the proof, we leave it as an exercise to the reader. As will be shown in the following discussions, the choice of Ω_j does not affect the results.

By $\tilde{\Omega}_j$ we denote the union of Ω_j and its boundary in Ω_0. Fix three open sets Ω_0, Ω_1, Ω_2 which satisfy the conditions of Proposition 2.2.1. Let $\mathcal{F}(\Omega_0; E)$ be a subspace of $\mathcal{D}'(\Omega_0; E)$ with a locally convex topology. Let ρ be the restriction operator of $\mathcal{F}(\Omega_0; E)$ into $\mathcal{D}'(\Omega_1; E)$. To every $u \in \mathcal{F}(\Omega_0; E)$ its restriction $\rho(u)$ is also denoted by $u|_{\Omega_1}$.

Definition 2.2.1. By $\mathcal{F}(\tilde{\Omega}_1; E)$ we denote the range of ρ , i.e. the set of all $u \in \mathcal{D}'(\Omega_1; E)$ such that there exists $v \in \mathcal{F}(\Omega_0; E)$ and its restriction to Ω_1 is equal to u. This space is equipped with the strongest locally convex topology such that ρ is continuous.

Definition 2.2.2. If ω_{13} is void, the space $\mathcal{F}(\tilde{\Omega}, \omega_1; E)$ is defined as the set of all distribution sections in $\mathcal{F}(\tilde{\Omega}_1; E)$ with its support in $\tilde{\Omega}$. This space has the natural topology as a subspace of $\mathcal{F}(\tilde{\Omega}_1; E)$.

Definition 2.2.3. By $C^\infty(\tilde{\Omega}, \omega_1; E)$ we denote the space of all C^∞ functions u in $C^\infty(\tilde{\Omega}_1; E)$ with its support in $\tilde{\Omega}$ such that P(u) vanishes in ω_{13} for any linear differential operator P (with C^∞ coefficients) in $C^\infty(\Omega_1; E)$. This space has the natural topology as a subspace of $C^\infty(\tilde{\Omega}_1; E)$. If ω_2 is void, we use the notation $\mathcal{E}^\infty(\tilde{\Omega}; E)$ instead of $C^\infty(\tilde{\Omega}, \omega_1; E)$.

Definition 2.2.4. If ω_{13} is void, the space $\overset{\circ}{\mathcal{F}}(\tilde{\Omega}_2; E)$ is defined as the set of all distribution sections in $\mathcal{F}(\Omega_0; E)$ with its support in $\tilde{\Omega}_2$. This space is endowed with the topology as a subspace of $\mathcal{F}(\Omega_0; E)$.

From Definition 2.2.3. the space $\mathcal{E}^\infty(\tilde{\Omega}_2; E)$ is the collection of all functions u in $C^\infty(\Omega_0; E)$ such that its support is contained in $\tilde{\Omega}_2$ and P(u) is equal to zero in ω_{13} for any differential operator P in $C^\infty(\Omega_0; E)$.

Proposition 2.2.2. The restriction operator ρ induces an epimorphism of $\overset{\circ}{\mathcal{F}}(\tilde{\Omega}_2; E)$ onto $\mathcal{F}(\tilde{\Omega}, \omega_1; E)$.

Proof. In fact the set of all u in $\mathcal{F}(\Omega_0; E)$ whose restriction to Ω_1 belongs to $\mathcal{F}(\tilde{\Omega}, \omega_1; E)$ is equal to $\overset{\circ}{\mathcal{F}}(\tilde{\Omega}_2; E)$. Then the remaining part of the proof is obvious.

Proposition 2.2.3. If $\mathcal{F}(\Omega_0; E)$ induces a sheaf of C_0^∞ modules over Ω_0, then the space $\mathcal{F}(\tilde{\Omega}, \omega_1; E)$ is independent of the choice of $\Omega_0, \Omega_1, \Omega_2$ except in a neighbourhood of ω

Proof. It is enough to notice that every distribution section in $\mathcal{F}(\tilde{\Omega}, \omega_1; E)$ can be modified outside $\tilde{\Omega}$ and can be extended to an element of $\mathcal{F}(M\backslash(\partial\Omega\backslash\omega))$ which vanishes outside a neighbourhood of $\tilde{\Omega}$ in Ω_0. The details will be left to the reader.

Almost all spaces which we study in the following sections satisfy the above condition. Since ω_{14} and ω_{24} does not affect the definition of $\mathcal{F}(\tilde{\Omega}, \omega_1; E)$, we can assume them to be void.

§2.3. The spaces $C^\infty(\tilde{\Omega}, \omega_1; E)$.

We assume in this section that the sets ω_{14}, ω_{23}, ω_{24} are void, since they have no meaning in the following discussions. Then we have $\omega \cap (\overline{\Omega})^0 = \omega_{13}$.

Proposition 2.3.1. The space $C^\infty(\tilde{\Omega}, \omega_1; E)$ is isomorphic to the closure in $C^\infty(\tilde{\Omega}; E)$ of the set of all functions φ in $C_0^\infty(\tilde{\Omega}; E)$ such that the closure of supp φ in Ω_0 does not meet with ω_1.

Proof. Let ρ_1 be the restriction mapping of $C^\infty(\tilde{\Omega}_1;\ E)$ onto $C^\infty(\tilde{\Omega};\ E)$. Then the restriction of ρ_1 to $C^\infty(\tilde{\Omega}, \omega_1;\ E)$ is injective, and hence by the open mapping theorem this is an isomorphism of $C^\infty(\tilde{\Omega}, \omega_1;\ E)$ onto $\rho_1(C^\infty(\tilde{\Omega}, \omega_1;\ E))$. Therefore it is enough to prove that any function in $\rho_1(C^\infty(\tilde{\Omega}, \omega_1;\ E))$ can be approximated by functions φ in $C_0^\infty(\tilde{\Omega};\ E)$ such that the closure of supp φ in Ω_0 does not meet with ω_1.

Let u be a function in $\rho_1(C^\infty(\tilde{\Omega}, \omega_1;\ E))$. Then there exists $v \in \overset{\circ}{C}{}^\infty(\tilde{\Omega}_2;\ E)$ such that $v|_\Omega = u$. Using a partition of unity subordinate to a family of coordinate neighbourhoods, it is enough to make an approximation in each coordinate neighbourhood. Then we can use an approximation by multiplying certain C^∞ cutoff functions (e.g. Schwartz [22] p.93-94). Hence we have a sequence of functions $v^{(j)}$, $j = 1,2,\cdots$ in $C_0^\infty(\Omega_2;\ E)$ such that $v^{(j)}$ tends to v in $C^\infty(\Omega_0;\ E)$ as j tends to infinity. Then the restriction of $v^{(j)}$ to Ω gives the required approximation of u, and the proof is complete.

Theorem 2.3.2. The space $C^\infty(\tilde{\Omega}, \omega_1;\ E)$ is separable Frechet Montel and $\mathcal{E}'(\tilde{\Omega}, \omega_1;\ E)$ is separable complete bornological Montel. Moreover the dual space of $C^\infty(\tilde{\Omega}, \omega_1;\ E)$ is isomorphic to $\mathcal{E}'(\tilde{\Omega}, \omega_2;\ E)$.

Proof. From Proposition 2.1.1. the space $C^\infty(\Omega_0;\ E)$ is Frechet-Schwartz, (FS) for short. Since $\overset{\circ}{C}{}^\infty(\tilde{\Omega}_2;\ E)$ is a closed subspace of $C^{\infty}(\Omega_0;\, E)$, it is also (FS). From Proposition 2.2.2. the restriction mapping ρ induces an epimorphism $\tilde{\rho}$ of $\overset{\circ}{C}^{\infty}(\tilde{\Omega}_2;\, E)$ onto $C^{\infty}(\tilde{\Omega},\, \omega_1;\, E)$. Hence the space $C^{\infty}(\tilde{\Omega},\, \omega_1;\, E)$ is also (FS), and then it is separable Frechet Montel (see [14] and the references therein). Moreover its dual space $C^{\infty}(\tilde{\Omega},\, \omega_1;\, E)'$ is isomorphic to the polar of Ker $\tilde{\rho} = \{\varphi \in \overset{\circ}{C}^{\infty}(\tilde{\Omega}_2;\, E)\ ;\ \varphi|_{\Omega_1} = 0\}$ in $\overset{\circ}{C}^{\infty}(\tilde{\Omega}_2;\, E)'$.

The dual space of $\overset{\circ}{C}^{\infty}(\tilde{\Omega}_2;\, E)$ is isomorphic to the quotient space of $C^{\infty}(\Omega_0;\, E)' \cong \mathcal{E}'(\Omega_0;\, E)$ by the polar of $\overset{\circ}{C}^{\infty}(\tilde{\Omega}_2;\, E)$. A distribution section u in $\mathcal{E}'(\Omega_0;\, E)$ belongs to the polar of $\overset{\circ}{C}^{\infty}(\tilde{\Omega}_2;\, E)$ if and only if $<\varphi\, ,\, u> = 0$ for every $\varphi \in \overset{\circ}{C}^{\infty}(\tilde{\Omega}_2;\, E)$. But from Proposition 2.3.1. it is equivalent to say that u vanishes in Ω_2. In fact to every $\varphi \in \overset{\circ}{C}^{\infty}(\tilde{\Omega}_2;\, E)$ there exists a sequence of functions φ_j, $j = 1, 2, \cdots$ in $C_0^{\infty}(\Omega_2;\, E)$ such that φ_j converges to φ in $C^{\infty}(\Omega_0;\, E)$. If u is equal to zero in Ω_2, then it follows that $<\varphi_j,\, u> = 0$. Hence we have

$$<\varphi\, ,\, u> = \lim_{j\to\infty} <\varphi_j,\, u> = 0.$$

Therefore the dual space of $\overset{\circ}{C}^{\infty}(\tilde{\Omega}_2;\, E)$ is isomorphic to $\mathcal{E}'(\tilde{\Omega}_2;E)$.

Next the polar of Ker $\tilde{\rho}$ in $\overset{\circ}{C}^{\infty}(\tilde{\Omega}_2;E)' \cong \mathcal{E}'(\tilde{\Omega}_2;E)$ is the space of all $u \in \mathcal{E}'(\tilde{\Omega}_2;E)$ such that supp $u \subset \tilde{\Omega}$. In fact it is enough to use Proposition 2.3.1. with respect to $\overset{\circ}{C}^{\infty}(\Omega_0\setminus\Omega_1;E)$ as in the above proof. We have thus proved that the dual space of $C^{\infty}(\tilde{\Omega},\, \omega_1;E)$ is isomorphic to $\mathcal{E}'(\tilde{\Omega}, \omega_2;E)$. Since $\mathcal{E}'(\tilde{\Omega}, \omega_1;E)$ is the dual of a (FS) space, it is separable complete bornological Montel, and this completes the proof.

Let $\mathcal{E} = C^\infty(\Omega_0;E)$ and $\mathcal{G} = C^\infty(\tilde{\Omega}, \omega_1;E)$. If ι is the natural injection of $\overset{\circ}{C}{}^\infty(\tilde{\Omega}_2;E)$ into $\mathcal{E}$ and ρ is the restriction mapping of $\overset{\circ}{C}{}^\infty(\tilde{\Omega}_2;E)$ onto $\mathcal{G}$, then we are in the same situation as in Proposition 1.1.1. Let $s \in \mathbb{R}$ and $\chi \in C_0^\infty(\Omega_0)$. (We can assume that χ does not vanish in the interior of supp χ.) We define two seminorms on $\mathcal{E}$ by

$$p(\varphi) = \|\chi \cdot \varphi\|_{(s)}, \quad \text{for all } \varphi \in C^\infty(\Omega_0;E), \tag{2.3.1}$$

and

$$q(\varphi) = \inf\{\|\chi \cdot \psi\|_{(s)};\ \psi \in C^\infty(\Omega_0;E) \text{ and } \psi = \varphi \text{ in } \Omega_1\}, \tag{2.3.2}$$

for all $\varphi \in C^\infty(\Omega_0;E)$. Then $p^* \circ \rho(\varphi) = q(\varphi)$ for every $\varphi \in \overset{\circ}{C}{}^\infty(\tilde{\Omega}_2;E)$. Here p^* is defined by (1.1.2), i.e.

$$p^*(\varphi) = \inf\{p(\psi);\ \psi \in \overset{\circ}{C}{}^\infty(\tilde{\Omega}_2;E) \text{ and } \psi = \rho(\varphi)\}, \tag{2.3.3}$$

for all $\varphi \in C^\infty(\tilde{\Omega}, \omega_1;E)$. We have to study the Banach space $\mathcal{G}'_{p^*}$.

<u>Proposition</u> 2.3.3. Let K_1 be a compact set contained in the interior of $K = \text{supp}\,\chi$. Then the Banach space

$$\{u \in H^c_{(-s)}(\Omega_0;E);\ \text{supp}\,u \subset \tilde{\Omega}_1 \cap K_1\} \tag{2.3.4}$$

is continuously imbedded into $\mathcal{E}'_q$ and $\mathcal{E}'_q$ is continuously imbedded into the Banach space

$$\{u \in H^c_{(-s)}(\Omega_0;E);\ \text{supp}\,u \subset \tilde{\Omega}_1 \cap K\}, \tag{2.3.5}$$

Where both spaces (2.3.4) and (2.3.5) are endowed with the norm $\|\cdot\|_{(-s)}$.

<u>Proof</u>. Let $u \in H^c_{(-s)}(\Omega_0;E)$ and $\operatorname{supp} u \subset \widetilde{\Omega}_1 \cap K_1$. There exists $\chi_0 \in C_0^\infty(\Omega_0)$ such that $\chi_0 \cdot \chi = 1$ in a neighbourhood of K_1. Then $\chi_0 \cdot \chi = 1$ in a neighbourhood of $\operatorname{supp} u$. Let $\varphi, \psi \in C^\infty(\Omega_0;E)$ and $\psi = \varphi$ in Ω_1. Then we have

$$<\varphi, u> = <\psi, u> = <\chi_0 \cdot \chi \cdot \psi, u> = <\chi \cdot \psi, \chi_0 u>,$$

and hence with some constant $C > 0$,

$$|<\varphi, u>| \leqq \|\chi \cdot \psi\|_{(s)} \cdot \|\chi_0 \cdot u\|_{(-s)} \leqq C \|\chi \cdot \psi\|_{(s)} \cdot \|u\|_{(-s)}.$$

Therefore we obtain for every $\varphi \in C^\infty(\Omega_0;E)$,

$$|<\varphi, u>| \leqq C \cdot q(\varphi) \cdot \|u\|_{(-s)}.$$

Then it follows that $u \in \mathcal{E}'_q$ and $\|u\|_q \leqq C\|u\|_{(-s)}$.

Next suppose that $u \in \mathcal{E}'_q$. Then u belongs to $\mathcal{E}' \cong$ $\cong \mathcal{E}'(\Omega_0;E)$. In addition we have for every $\varphi \in C_0^\infty(\Omega_0;E)$,

$$\begin{aligned} |<\varphi, u>| &\leqq q(\varphi) \cdot \|u\|_q \\ &\leqq \|u\|_q \cdot \inf\{\|\chi \cdot \psi\|_{(s)};\ \psi \in C^\infty(\Omega_0;E) \text{ and } \psi = \varphi \text{ in } \Omega_1\} \\ &\leqq \|u\|_q \cdot \|\chi \cdot \varphi\|_{(s)} \\ &\leqq C \cdot \|u\|_q \cdot \|\varphi\|_{(s)}, \end{aligned}$$

where C is a positive constant. Hence it follows that $u \in H^c_{(-s)}(\Omega_0;$ $;E)$ and $\|u\|_{(-s)} \leqq C\|u\|_q$ with another constant C. Moreover it can be shown that the support of u is contained in $\widetilde{\Omega}_1 \cap K$. In fact let φ be a C^∞ section in $C_0^\infty(\Omega_0;E)$ which is equal to zero in an open neighbourhood U of $\tilde{\Omega}_1 \wedge K$. There exists a function α in $C^\infty(\Omega_0)$ such that $\alpha = 1$ in a neighbourhood of $\tilde{\Omega}_1$ and $\alpha = 0$ in a neighbourhood of $K \setminus U$. Let $\psi_0 = \alpha \cdot \varphi$. Then it follows that $\psi_0 = \varphi$ in a neighbourhood of $\tilde{\Omega}_1$ and $\psi_0 = 0$ in a neighbourhood of $K = \operatorname{supp} \chi$. Hence we obtain

$$\begin{aligned} &|\langle \varphi , u \rangle| \leqq \\ &\leqq \| u\|_q \cdot \inf\{ \|\chi \cdot \psi\|_{(s)};\ \psi \in C^\infty(\Omega_0;E) \text{ and } \psi = \varphi \text{ in } \Omega_1 \} \\ &\leqq \| u\|_q \cdot \| \chi \cdot \psi_0\|_{(s)} \\ &= 0. \end{aligned}$$

Therefore it follows that $\langle \varphi , u \rangle = 0$ and the required inclusion $\operatorname{supp} u \subset \tilde{\Omega}_1 \wedge K$ holds. This finishes the proof.

<u>Theorem</u> 2.3.4. Let K_1 be a compact set contained in the interior of $K = \operatorname{supp} \chi$. Then the Banach space

$$\{u \in \mathcal{E}'(\tilde{\Omega}, \omega_2;E);\ \operatorname{supp} u \subset \tilde{\Omega} \wedge K_1 \text{ and } u = v|_{\Omega_2} \text{ for some } v \in H^c_{(-s)}(\Omega_0;E)\} \tag{2.3.6}$$

is continuously imbedded in to $\mathcal{G}'_{p^*}$, where the space (2.3.6) is endowed with the norm

$$\inf\{\|v\|_{(-s)};\ v \in H^c_{(-s)}(\Omega_0;E) \text{ and } u = v|_{\Omega_2}\}. \tag{2.3.7}$$

Moreover the Banach space $\mathcal{G}'_{p^*}$ is continuously imbedded into

$$\{u \in \mathcal{E}'(\tilde{\Omega}, \omega_2;E);\ \operatorname{supp} u \subset \tilde{\Omega} \wedge K \text{ and } u = v|_{\Omega_2} .$$

$$\text{for some } v \in H^c_{(-s)}(\Omega_0;E)\}. \qquad (2.3.8)$$

with the same norm (2.3.7).

Proof. Let K_2 be a compact set such that its interior contains K_1 and it is contained in the interior of K. Take a function φ in $C_0^\infty(\Omega_\rho)$ such that $\operatorname{supp}\varphi \subset K_2$ and $\varphi = 1$ in a neighbourhood of K_1. Let E_3, G_1, G_4 be three Banach spaces defined by (2.3.5), (2.3.6), (2.3.8) respectively. In addition we define the following Banach spaces:

$$E_1 = \{u \in H^c_{(-s)}(\Omega_0;E);\ \Omega_2 \cap \operatorname{supp} u \subset \tilde{\Omega}_R \cap K_1\},$$

$$E_2 = \{u \in H^c_{(-s)}(\Omega_0;E);\ \operatorname{supp} u \subset \tilde{\Omega}_1 \cap K_2\},$$

$$E_4 = \{u \in H^c_{(-s)}(\Omega_0;E);\ \Omega_2 \cap \operatorname{supp} u \subset \tilde{\Omega}_R \cap K\},$$

$$G_2 = \{v \in \mathcal{E}'(\tilde{\Omega}, \omega_2;E);\ v = w|_{\Omega_2} \text{ and } \operatorname{supp} w \subset \tilde{\Omega}_1 \cap K_2$$
$$\text{for some } w \in H^c_{(-s)}(\Omega_0;E)\},$$

$$G_3 = \{v \in \mathcal{E}'(\tilde{\Omega}, \omega_2;E);\ v = w|_{\Omega_2} \text{ and } \operatorname{supp} w \subset \tilde{\Omega}_1 \cap K$$
$$\text{for some } w \in H^c_{(-s)}(\Omega_0;E)\},$$

where E_1, E_2, E_4 are endowed with the norm $\|\cdot\|_{(-s)}$, and G_2, G_3 are endowed with the norm (2.3.7). From Proposition 1.1.1. the restriction mapping ρ' induces an epimorphism of $\mathcal{E}'_q$ onto $\mathcal{G}'_{p^*}$. From the previous Proposition 2.3.3. the Banach space E_2 is continuously embedded into $\mathcal{E}'_q$ and $\mathcal{E}'_q$ is continuously embedded into E_3. By λ we denote the linear mapping of E_1 into E_2 defined by $\lambda(u) = \varphi \cdot u$, $u \in E_1$. Then λ is continuous linear and we have the following commutative diagram:

$$\begin{array}{ccccccccc}
E_1 & \xrightarrow{\lambda} & E_2 & \hookrightarrow & \mathcal{E}'_q & \hookrightarrow & E_3 & \hookrightarrow & E_4 \\
\rho' \downarrow & & \rho' \downarrow & & \rho' \downarrow & & \rho' \downarrow & & \rho' \downarrow \\
G_1 & \hookrightarrow & G_2 & \hookrightarrow & \mathcal{G}'_{p*} & \hookrightarrow & G_3 & \hookrightarrow & G_4
\end{array}$$

The restriction mapping ρ' induces five epimorphisms . Three upper inclusions and λ are continuous. Therefore the four lower inclusions are also continuous, and then the theorem is proved.

<u>Proposition</u> 2.3.5. Let K' be a compact subset of Ω_0 such that its interior contains $K = \operatorname{supp} \chi$. Then every C^∞ section in $C^\infty(\tilde{\Omega}, \omega_1; E)$ which is equal to zero in $\Omega \wedge K'$ is contained in $\operatorname{Ker} p^*$. Every element of $\operatorname{Ker} p^*$ is equal to zero in $\Omega \wedge K$. Moreover every distribution section in $\mathcal{E}'(\tilde{\Omega}, \omega_2; E)$ is equal to zero in $\operatorname{Ker} p^*$ if and only if its support is contained in $\tilde{\Omega} \wedge K$.

<u>Proof</u>. Let φ be a section in $C^\infty(\tilde{\Omega}, \omega_1; E)$ which vanishes in $\Omega \wedge K'$. Then there exists $\psi_0 \in \overset{\circ}{C}{}^\infty(\tilde{\Omega}_2; E)$ such that $\varphi = \psi_0 |_{\Omega_1}$ and $\psi_0 = 0$ in K. Hence we have

$$p^*(\varphi) \leq \| \chi \cdot \psi_0 \|_{(s)} = 0.$$

Next suppose that $\varphi \in C^\infty(\tilde{\Omega}, \omega_1; E)$ and $p^*(\varphi) = 0$. Take any C^∞ section α in $C_0^\infty(\Omega \wedge K^0; E)$, where K^0 is the interior of $K = \operatorname{supp} \chi$. Then K is equal to the closure of K^0. Since χ does not vanish in a neighbourhood of $\operatorname{supp} \alpha$, there exists $\beta \in C_0^\infty(\Omega \wedge K^0; E)$ such that $\alpha = \chi\cdot\beta$ in Ω. If $\psi \in \overset{\circ}{C^\infty}(\tilde{\Omega}_2;E)$ and $\varphi = \psi|\Omega_1$, then we obtain

$$|\langle\alpha, \varphi\rangle| = |\langle\chi\cdot\beta, \psi\rangle| = |\langle\beta, \chi\cdot\psi\rangle|$$
$$\leqq \|\beta\|_{(-s)}\cdot\|\chi\cdot\psi\|_{(s)}.$$

Then it follows that

$$|\langle\alpha, \varphi\rangle| \leqq \|\beta\|_{(-s)}\cdot p^*(\varphi) = 0.$$

Hence we have $\langle\alpha, \varphi\rangle = 0$ for any such α. Therefore φ is equal to zero in $\Omega \cap K^o$ and then in $\Omega \cap K$. The remaining part of the proposition is obvious.

§2.4. <u>The spaces $H^{loc}_{(s)}(\tilde{\Omega}, \omega_1;E)$</u>.

For the sake of simplicity we assume in this section that the intersection of ω and $(\overline{\Omega})^o$ is void. Take three open sets Ω_0, Ω_1, Ω_2 satisfying the conditions of Proposition 2.2.1. Since we cannot use the cut-off approximation as in §2.3., we make the following definition:

<u>Definition</u> 2.4.1. We say that ω has the <u>curve segment property</u> at $x \in \omega$ (with respect to Ω) if there exist an open neighbourhood V of x in Ω_0 and a real everywhere non-vanishing C^∞ vector field X on V such that any integral curve of X in V from any point of $\omega \cap V$ is contained in $\Omega \cap V$.

If ω has the curve segment property at x with respect to Ω, then it also has the same property at x with respect to $\Omega_0 \setminus \tilde{\Omega}$. In fact every integral curve of $-X$ in V from x is contained in $V \setminus \tilde{\Omega}$.

If ω has the curve segment property at $x \in \omega$, then we can choose a sufficiently small neighbourhood V of x in Ω_0 and a local chart $\varkappa : V \longrightarrow \varkappa(V) \subset \mathbb{R}^n$ such that to every point $y \in \varkappa(\omega \cap V)$ the set

$$\{z \in \varkappa(V);\ y' = z' \text{ and } y_n < z_n\}$$

is contained in $\varkappa(\Omega \cap V)$. Here we write $y = (y_1, y_2, \cdots, y_n)$ and $y' = (y_1, \cdots, y_{n-1})$. In order to prove this fact we only have to solve a simple ordinary differential equation, and we leave it to the reader.

<u>Proposition</u> 2.4.1. Let $s \in \mathbb{R}$. If ω has the curve segment property at ω_1, then the set

$$\{\varphi \in C_0^\infty(\tilde{\Omega}_1; E);\ \overline{\operatorname{supp} \varphi} \cap (\Omega_1 \setminus \tilde{\Omega}) = \phi\} \tag{2.4.1}$$

is dense in $H^{loc}_{(s)}(\tilde{\Omega}, \omega_1; E)$.

<u>Proof</u>. From the hypothesis there exist a locally finite open covering $(V_\alpha)_{\alpha \in A}$ of ω_1 in Ω_0 and a family of local charts $\varkappa_\alpha : V_\alpha \longrightarrow \varkappa_\alpha(V_\alpha) \subset \mathbb{R}^n$ such that to every $y \in \varkappa_\alpha(\omega \cap V_\alpha)$ the set

$$\{z \in \varkappa_\alpha(V_\alpha);\ y' = z' \text{ and } y_n < z_n\} \tag{2.4.2}$$

is contained in $\varkappa_\alpha(\Omega \cap V_\alpha)$. Then it is possible to choose Ω_1 such that the set $\{z \in \varkappa_\alpha(V_\alpha);\ y' = z' \text{ and } z_n < y_n\}$ is contained in $\varkappa_\alpha(\Omega_1 \cap V_\alpha \setminus \Omega)$ for all $y \in \omega_{11}$. Take a locally finite open covering $(V_\beta)_{\beta \in B}$ of

$\Omega \setminus (\bigcup_{\alpha \in A} V_\alpha)$ in Ω such that $\omega_1 \cap V_\beta = \phi$ for all $\beta \in B$. Then using a C^∞ partition of unity subordinate to $(V_\alpha)_{\alpha \in A \cup B}$, we can reduce the problem to that in each V_α, $\alpha \in A \cup B$. Therefore it is enough to make an approximation in each $V_\alpha \cong \varkappa_\alpha(V_\alpha)$, $\alpha \in A$. Then we can use an approximation by translation. After that we can use convolution by the usual C^∞ functions and then we obtain a required approximation (see the proof of Proposition 2.5.5.). We leave the details of the proof as an exercise for the reader.

Theorem 2.4.2. Let $s \in \mathbb{R}$. We assume that ω has the curve segment property and Ω_2 can be selected such that its boundary in Ω_0 also has the curve segment property at every their point. Then $H^{loc}_{(s)}(\tilde{\Omega}, \omega_1; E)$ is a Frechet space and its dual space is weak*-isomorphic to $H^c_{(-s)}(\tilde{\Omega}, \omega_2; E)$.

Proof. Since $H^{loc}_{(s)}(\Omega_0; E)$ is a Frechet space, $H^{loc}_{(s)}(\tilde{\Omega}, \omega_1; E)$ is also a Frechet space. Let $\imath$ be the natural injection of $\overset{0}{H}{}^{loc}_{(s)}(\tilde{\Omega}_2; E)$ into $H^{loc}_{(s)}(\Omega_0; E)$. Then its dual map ${}^t\imath$ is a weak*-homomorphism of $H^{loc}_{(s)}(\Omega_0; E)' \cong H^c_{(-s)}(\Omega_0; E)$ onto $\overset{0}{H}{}^{loc}_{(s)}(\tilde{\Omega}_2; E)'$. Let ρ_2 be the restriction map of $H^c_{(-s)}(\Omega_0; E)$ onto $H^c_{(-s)}(\tilde{\Omega}_2; E)$. Then for every $u \in H^c_{(-s)}(\Omega_0; E)$ we have ${}^t\imath(u) = 0$ if and only if $\rho_2(u) = 0$. In fact we have ${}^t\imath(u) = 0$ if and only if $\langle \varphi, u \rangle = 0$ for all $\varphi \in \overset{0}{H}{}^{loc}_{(s)}(\tilde{\Omega}_2; E)$, and the latter condition is equivalent with $\rho_2(u) = 0$ from Proposition 2.4.1. Therefore there exists a weak*-isomorphism of $\overset{0}{H}{}^{loc}_{(s)}(\tilde{\Omega}_2; E)'$ onto $H^c_{(-s)}(\tilde{\Omega}_2; E)$.

Now the dual of the restriction mapping ρ of $\overset{0}{H}{}^{loc}_{(s)}(\tilde{\Omega}_2;E)$ onto $H^{loc}_{(s)}(\tilde{\Omega}, \omega_1;E)$ gives a weak*-isomorphism of $H^{loc}_{(s)}(\tilde{\Omega}, \omega_1;E)'$ into $\overset{0}{H}{}^{loc}_{(s)}(\tilde{\Omega}_2;E)' \cong H^c_{(-s)}(\tilde{\Omega}_2;E)$. Because ρ is an open mapping, the range of ${}^t\rho$ is equal to the polar of $\operatorname{Ker}\rho$. Then u belongs to the range of ${}^t\rho$ if and only if $\langle \varphi, u\rangle = 0$ for every $\varphi \in \overset{0}{H}{}^{loc}_{(s)}(\tilde{\Omega}_2;E)$ which is equal to zero in Ω_1. From Proposition 2.4.1. the latter condition is equivalent with the inclusion relation $\operatorname{supp} u \subset \hat{\Omega}$. Therefore the range of ${}^t\rho$ is equal to $H^c_{(-s)}(\hat{\Omega}, \omega_2;E)$, and the proof is finished.

<u>Theorem</u> 2.4.3. Let $s,t \in \mathbb{R}$. If K is a compact subset of $\tilde{\Omega}$, and $s < t$, then the inclusion mapping of the Banach space

$$\{u \in H^c_{(t)}(\tilde{\Omega}, \omega_1;E);\ \operatorname{supp} u \subset K\} \qquad (2.4.3)$$

into the Banach space

$$\{u \in H^c_{(s)}(\tilde{\Omega}, \omega_1;E);\ \operatorname{supp} u \subset K\} \qquad (2.4.4)$$

is completely continuous. Here (2.4.3) is given the norm

$$\inf\{\|v\|_{(t)};\ v \in H^c_{(t)}(\Omega_0;E) \text{ and } u = v|_{\Omega_1}\}, \qquad (2.4.5)$$

and (2.4.4) is given the similar one.

Conversely if the inclusion mapping is completely continuous for one K with interior points, then it follows that $s < t$.

<u>Proof</u>. Let χ be a function in $C^\infty_0(\Omega_0)$ such that $\chi = 1$ in a neighbourhood of K. Let V be the closed unit ball in the

Banach space (2.4.3). Then to each $u \in V$ there exists $\tilde{u} \in H^c_{(t)}(\Omega_0; E)$ such that $\tilde{u}|_{\Omega_1} = u$ and $\|\tilde{u}\|_{(t)} \leqq 2$. By $\tilde{V}$ we denote the set of $\chi \cdot \tilde{u}$ for all $u \in V$. Then it follows that $\rho(\tilde{V}) = V$ and $\tilde{V}$ is a bounded set in the Banach space

$$\{v \in H^c_{(t)}(\Omega_0; E);\ \operatorname{supp} v \subset \operatorname{supp} \chi\},$$

which has the norm $\|v\|_{(t)}$. Hence $\tilde{V}$ is precompact in the Banach space

$$\{v \in H^c_{(s)}(\Omega_0; E);\ \operatorname{supp} v \subset \operatorname{supp} \chi\}.$$

Therefore $V = \rho(\tilde{V})$ is precompact in (2.4.4).

The latter part of the theorem is obvious.

Using the same argument as in Theorem 2.3.4. and Proposition 2.3.5. we can prove the following theorem:

Theorem 2.4.4. Suppose that ω has the curve segment property at every its point and we can choose Ω_2 such that its boundary in Ω_0 also has the same property at every its point. Let $s \in \mathbb{R}$ and $\chi \in C^\infty_0(\Omega_0)$. Take a compact set K_1 in the interior of $K = \operatorname{supp} \chi$. Let p^* be the seminorm in $\mathcal{G} = H^{loc}_{(s)}(\tilde{\Omega}, \omega_1; E)$

defined by

$$p^*(u) = \inf\{ \|\chi\cdot v\|_{(s)};\ v\in H^{loc}_{(s)}(\Omega_0;E) \text{ and } u = v|_{\Omega_1}\}, \tag{2.4.6}$$

for all $u\in H^{loc}_{(s)}(\tilde{\Omega}, \omega_1;E)$. Then the Banach space (2.3.6) is continuously embedded into $\mathcal{G}'_{p^*}$, and $\mathcal{G}'_{p^*}$ is continuously embedded into the Banach space (2.3.8). Moreover if $u\in H^c_{(-s)}(\tilde{\Omega}, \omega_2;E)$ and $\operatorname{supp} u\subset \tilde{\Omega}\cap K_1$, then u is equal to zero in $\operatorname{Ker} p^*$. If u is equal to zero in $\operatorname{Ker} p^*$, then it follows that $\operatorname{supp} u\subset\tilde{\Omega}\cap K$.

§2.5. The spaces $\mathcal{B}^{loc}_{p,k}(\tilde{\Omega}, \omega_1)$ when $M = \mathbb{R}^n$.

In this section we assume that $M = \mathbb{R}^n$ and E is the trivial line bundle $\mathbb{R}^n\times\mathbb{C}$. As in the previous section we also assume that $\omega\cap(\overline{\Omega})^o=\phi$. Take three open sets $\Omega_0, \Omega_1, \Omega_2$ satisfying the conditions of Proposition 2.2.1.

We use the notations of Hörmander [8]. Let $1\leqq p<\infty$ and $k\in\mathcal{K}(\mathbb{R}^n)$, the set of all temperate weight functions. Then $\mathcal{B}_{p,k}$ is defined as the space of all distributions $u\in\mathcal{S}'(\mathbb{R}^n)$ such that its Fourier transform $\hat{u}$ is a function and

$$\|u\|_{p,k} = (\int|k(\xi)\hat{u}(\xi)|^p d\xi)^{1/p}<\infty. \tag{2.5.1}$$

For the sake of simplicity we assume that ω has the segment property defined as follows:

Definition 2.5.1. We say that ω has the segment property at $x\in\omega$ if there exist a neighbourhood V of x and a vector $æ\in\mathbb{R}^n$ such that for every $y\in\omega\cap V$ and $0<\varepsilon<1$

$$y + \varepsilon\cdot e \in \Omega \wedge V.$$

We can repeat the same argument as in section 2.4. and obtain the following four propositions:

Proposition 2.5.1. If ω has the segment property at every point of ω_1, then the set

$$\{ \varphi \in C_0^\infty(\tilde{\Omega}_1);\ \overline{\operatorname{supp}\varphi} \wedge (\Omega_1 \setminus \Omega) = \phi \}$$

is dense in $\mathcal{B}_{p,k}^{loc}(\tilde{\Omega}, \omega_1)$.

Theorem 2.5.2. $\mathcal{B}_{p,k}^{loc}(\tilde{\Omega}, \omega_1)$ is a Frechet space. If ω has the segment property at every its point and we can choose Ω_2 such that its boundary in Ω_0 also has the same property at every its point, then the dual space of $\mathcal{B}_{p,k}^{loc}(\tilde{\Omega}, \omega_1)$ is weak*-isomorphic to $\mathcal{B}_{p',1/k}^{c}(\tilde{\Omega}, \omega_2)$, where $\frac{1}{p} + \frac{1}{p'} = 1$.

Theorem 2.5.3. Take two temperate weight functions $k_1, k_2 \in \mathcal{K}(\mathbb{R}^n)$. If

$$k_2(\xi)/k_1(\xi) \longrightarrow 0, \text{ as } \xi \longrightarrow \infty, \tag{2.5.2}$$

then for every compact subset K of $\tilde{\Omega}$ the inclusion mapping of the Banach space

$$\{ u \in \mathcal{B}_{p,k_1}^{c}(\tilde{\Omega}, \omega_1);\ \operatorname{supp} u \subset K \} \tag{2.5.3}$$

into the Banach space

$$\{ u \in \mathcal{B}_{p,k_2}^{c}(\tilde{\Omega}, \omega_1);\ \operatorname{supp} u \subset K \} \tag{2.5.4}$$

is completely continuous. Here the former space is endowed with the norm

$$\inf\{\|v\|_{p,k_1};\ v\in\mathcal{B}^c_{p,k_1}(\Omega_0) \text{ and } u=v|_{\Omega_1}\},$$

and the latter space is endowed with the similar one.

Conversely if the inclusion mapping is completely continuous for one set K with interior points, then the condition (2.5.2) holds.

Theorem 2.5.4. Suppose that ω has the segment property at every its point and we can choose Ω_2 such that its boundary in Ω_0 also has the same property at every its point. Let $1\leqq p<\infty$, $\frac{1}{p}+\frac{1}{p'}=1$, and $k\in\mathcal{K}(\mathbb{R}^n)$. Take a function χ in $C_0^\infty(\Omega_0)$ and a compact subset K_1 of the interior of $K=\operatorname{supp}\chi$. Define the seminorm p^* in $\mathcal{G}=\mathcal{B}^{loc}_{p,k}(\tilde{\Omega},\omega_1)$ by

$$p^*(u)=\inf\{\|\chi\cdot v\|_{p,k};\ v\in\mathcal{B}^{loc}_{p,k}(\Omega_0) \text{ and } u=v|_{\Omega_1}\}$$

for all $u\in\mathcal{B}^{loc}_{p,k}(\tilde{\Omega},\omega_1)$. Then the Banach space

$$\{u\in\mathcal{B}^c_{p',1/k}(\tilde{\Omega},\omega_2);\ \operatorname{supp}u\subset K_1\cap\tilde{\Omega} \text{ and } u=v|_{\Omega_2} \text{ for some } v\in\mathcal{B}^c_{p',1/k}(\Omega_0)\} \tag{2.5.5}$$

is continuously imbedded into $\mathcal{G}'_{p^*}$, and $\mathcal{G}'_{p^*}$ is continuously imbedded into the Banach space

$$\{u\in\mathcal{B}^c_{p',1/k}(\tilde{\Omega},\omega_2);\ \operatorname{supp}u\subset K\cap\tilde{\Omega} \text{ and } u=v|_{\Omega_2} \text{ for some } v\in\mathcal{B}^c_{p',1/k}(\Omega_0)\}. \tag{2.5.6}$$

Here (2.5.5) and (2.5.6) is endowed with the norm

$$\inf\{\|v\|_{p',1/k};\ v\in\mathcal{B}^c_{p',1/k}(\Omega_0) \text{ and } u=v|_{\Omega_2}\}.$$

Moreover if $u \in \mathcal{B}^c_{p',1/k}(\tilde{\Omega}, \omega_2)$ and $\operatorname{supp} u \subset \tilde{\Omega} \cap K_1$, then u vanishes in Ker p^*. If u vanishes in Ker p^*, then it follows that $\operatorname{supp} u \subset \tilde{\Omega} \cap K$.

Proposition 2.5.5. Suppose that we can choose Ω_1 such that the boundary of Ω_1 in Ω_0 has the segment property at every its point. Let K and K' be two compact subset of Ω_0 such that K is contained in the interior of K'. Then there is a positive constant C such that for every function u in $C^\infty_0(\tilde{\Omega}, \omega_1)$, whose support is contained in $\tilde{\Omega} \cap K$, the following inequalities hold:

$$\begin{aligned} &\inf\{\|v\|_{p,k};\ v \in \mathcal{B}^c_{p,k}(\Omega_0) \text{ and } u = v|_{\Omega_1}\} \\ \leqq\ &\inf\{\|v\|_{p,k};\ v \in C^\infty_0(\Omega_0),\ \operatorname{supp} v \subset K', \text{ and } u = v|_{\Omega_1}\} \\ \leqq\ &C \cdot \inf\{\|v\|_{p,k};\ v \in \mathcal{B}^c_{p,k}(\Omega_0) \text{ and } u = v|_{\Omega_1}\}. \qquad (2.5.7) \end{aligned}$$

Proof. It is possible to take any Ω_1, because the inequalities (2.5.7) are essentially independent of Ω_1. Then from the hypothesis we can take Ω_1 whose boundary in Ω_0 has the segment property at every its point. Choose two compact sets K_1 and K_2 such that each K, K_1, K_2 is contained in the interior of K_1, K_2, K' respectively. Let χ be a function in $C^\infty_0(\mathbb{R}^n)$ such that $\int \chi(x)dx = 1$.. Set $\chi_\varepsilon(x) = \varepsilon^{-n}\chi(x/\varepsilon)$ for $\varepsilon > 0$.

From the selection of Ω_1 there exist a finite number of open sets $V_1, V_2, \cdots, V_l$, which is contained in K', and the same number of vectors $æ_1, æ_2, \cdots, æ_l$ in $\mathbb{R}^n$ such that V_j's are a covering of

$K_2 \cap \partial_{\Omega_0}\Omega_1$ and

$$V_j \cap \partial_{\Omega_0}\Omega_1 + \varepsilon\cdot e_j \subset K'^{\circ}\setminus\tilde{\Omega}_1, \quad 0<\varepsilon<1 \text{ and } j=1,2,\cdots,l.$$

Then there exist functions $\varphi_1, \varphi_2, \cdots, \varphi_l$ in $C_0^{\infty}(\Omega_0)$ such that $\operatorname{supp}\varphi_j \subset V_j$, $\sum_{j=1}^{l}\varphi_j = 1$ in a neighbourhood of $K_1 \cap \partial_{\Omega_0}\Omega_1$, and

$$(\operatorname{supp}\varphi_j)\setminus\Omega_1 + \varepsilon\cdot e_j \subset K'^{\circ}\setminus\tilde{\Omega}_1, \quad 0<\varepsilon<1/2 \text{ and } j=1,2,\cdots,l. \tag{2.5.8}$$

Moreover take functions $\varphi_{l+1}, \cdots, \varphi_m$ in $C_0^{\infty}(\Omega_0)$ such that

$$\operatorname{supp}\varphi_j \cap \partial_{\Omega_0}\Omega_1 = \phi, \quad j=l+1,\cdots,m,$$

and

$$\sum_{j=1}^{m}\varphi_j = 1 \text{ in } K_1.$$

Since the first inequality in (2.5.7) is trivial, it remains to prove the second inequality in (2.5.7). Now let $u \in C_0^{\infty}(\tilde{\Omega}, \omega_1)$ and $\operatorname{supp} u \subset \tilde{\Omega} \cap K$. Then there exists a function u_0 in $C_0^{\infty}(\Omega_0)$ such that $u = u_0|_{\Omega_1}$ and $\operatorname{supp} u_0 \subset K_1$. Take any function v in $\mathcal{B}_{p,k}^{c}(\Omega_0)$ such that $u = v|_{\Omega_1}$. If we write

$$w_j = \varphi_j(v-u_0), \quad j=1,2,\cdots,m,$$

then it follows that $w_j \in \mathcal{B}_{p,k}^{c}(\Omega_0)$ and

$$\operatorname{supp} w_j \subset (\operatorname{supp}\varphi_j)\setminus\Omega_1, \quad j=1,2,\cdots,m. \tag{2.5.9}$$

Here we define translation operators by

$$\tau_{\varepsilon}^{(j)}(\varphi)(x) = \varphi(x-\varepsilon\cdot e_j), \quad \varphi \in C_0^{\infty}(\mathbb{R}^n) \text{ and } x \in \mathbb{R}^n,$$

for $\varepsilon \in \mathbb{R}$ and $j=1,2,\cdots,l$. For distributions $u \in \mathcal{D}'(\mathbb{R}^n)$ we define $\tau_{\varepsilon}^{(j)}(u)$ by

$$<\varphi,\ \tau_{\varepsilon}^{(j)}(u)> = <\tau_{-\varepsilon}^{(j)}(\varphi),\ u>,\quad \varphi\in C_0^{\infty}(\mathbb{R}^n).$$

From (2.5.8) and (2.5.9) we have

$$\operatorname{supp} w_j + \varepsilon\cdot æ_j \subset K'^{o} \setminus \tilde{\Omega}_1,\quad 0<\varepsilon<1/2 \text{ and } j=1,2,\cdots,\ell,$$

and hence it follows that

$$\operatorname{supp} \tau_{\varepsilon}^{(j)}(w_j) \subset K'^{o} \setminus \tilde{\Omega}_1,\quad 0<\varepsilon<1/2 \text{ and } j=1,2,\cdots,\ell.$$

Then there is a constant $\delta_0 > 0$ such that

$$\tau_{\varepsilon}^{(j)}(w_j) * \chi_\delta \in C_0^{\infty}(K'^{o}\ \tilde{\Omega}_1)$$

for every $0<\varepsilon<1/2$, $0<\delta<\delta_0$, $j=1,2,\cdots,\ell$, and moreover

$$w_j * \chi_\delta \in C_0^{\infty}(K'^{o}\ \tilde{\Omega}_1),\quad j=\ell+1,\cdots,m.$$

Now define the functions $v_{\varepsilon,\delta}$ by

$$v_{\varepsilon,\delta} = u_0 + \sum_{j=1}^{\ell} \tau_{\varepsilon}^{(j)}(w_j) * \chi_\delta + \sum_{j=\ell+1}^{m} w_j * \chi_\delta.$$

Then it follows that $v_{\varepsilon,\delta} \in C_0^{\infty}(\Omega_0)$, $\operatorname{supp} v_{\varepsilon,\delta} \subset K'$, $u = v_{\varepsilon,\delta}|_{\Omega_1}$, and in addition

$$\begin{aligned} \| v_{\varepsilon,\delta} \|_{p,k} &\leqq \\ &\leqq \| u_0 - \sum_{j=1}^{m} w_j \|_{p,k} + \sum_{j=1}^{\ell} \| w_j - \tau_{\varepsilon}^{(j)}(w_j) * \chi_\delta \|_{p,k} \\ &\quad + \sum_{j=\ell+1}^{m} \| w_j - w_j * \chi_\delta \|_{p,k} \\ &= \| (\sum_{j=1}^{m} \varphi_j) v \|_{p,k} + \cdots \\ &\leqq C\| v \|_{p,k} + \sum_{j=1}^{\ell} \| w_j - \tau_{\varepsilon}^{(j)}(w_j) * \chi_\delta \|_{p,k} + \sum_{j=\ell+1}^{m} \| w_j - w_j * \chi_\delta \|_{p,k}, \end{aligned} \tag{2.5.10}$$

where the positive constant C is independent of u and v. Since the last two term of (2.5.10) tends to zero as $\varepsilon, \delta \longrightarrow 0$, we obtain

$$\overline{\lim_{\substack{\varepsilon \to 0 \\ \delta \to 0}}} \| v_{\varepsilon,\delta} \|_{p,k} \leqq C \| v \|_{p,k},$$

and then the proof is complete.

Remark. For many reasons the following definition of $\mathcal{F}(\tilde{\Omega}, \omega_1; E)$ is better than that in section 2.2. We need not assume that ω_{13} is void.

Definition 2.2.4'. By $\mathring{\mathcal{F}}(\tilde{\Omega}; E)$ we denote the closure of $C_0^\infty(\Omega; E)$ in $\mathcal{F}(\Omega_0; E)$. This space has the natural topology as a closed subspace of $\mathcal{F}(\Omega_0; E)$.

If $\mathcal{F}(M; E)$ induces a sheaf of C_0^∞ modules over M, the definition of $\mathring{\mathcal{F}}(\tilde{\Omega}; E)$ (and $\mathcal{F}(\tilde{\Omega}; E)$ also) does not depend on the choice of Ω_0.

Definition 2.2.2'. By $\mathcal{F}(\tilde{\Omega}, \omega_1; E)$ we denote the space $\rho(\mathring{\mathcal{F}}(\tilde{\Omega}_2; E))$, that is, the space of all distribution sections $u \in \mathcal{D}'(\tilde{\Omega}_1; E)$ which is the restriction of some $v \in \mathring{\mathcal{F}}(\tilde{\Omega}_2; E)$ to Ω_1.

Using Whitney's theorem [28] (also [18]), we can easily prove that $C^\infty(\tilde{\Omega}, \omega_1; E)$ does not depend on the choice of Ω_0, Ω_1, and Ω_2.

Chapter III. Elliptic boundary problems.

§3.0. Introduction.

This chapter deals with boundary problems mainly for elliptic differential equations. Some results are not restricted to elliptic operators. If we combine the results of Chapter I and II, we immediately obtain necessary and sufficient conditions for solvability of equations (Lemma 3.2.2., 3.3.2., etc.). They are constructed by two types of conditions. One is on semi-global solvability, that is, whether the equation is solvable in any compact subset. We can write necessary estimates using Sobolev norms. The other condition is on the relation between the boundary and the characteristics of the differential operator. They are represented using support of distribution sections. These facts are essentially well-known if the boundary is void (e.g. [8]).

In section one we prove some preliminary propositions. Reduction of the problem to the usual form is done. In the next section we treat differential equations in the space $C^\infty(\tilde{\Omega};E)$. This is an almost trivial generalization of the classical theory of differential equations without boundary conditions. We have to study differential equations in the boundary which is induced by the boundary conditions. In section 3 we state and explain our main theorem of this chapter (Theorem 3.3.1.). We can prove the existence of solutions for elliptic boundary problems in non-compact manifolds. Unlike the case of compact manifolds, solutions always exist. The last section is devoted to the proof. We can shorten the proof using parametricies for elliptic boundary operators.

§3.1. Preliminaries.

Let M be a σ-compact C^∞ manifold of dimension n. Let Ω be an open subset of M, ω an open subset of the boundary of Ω in M. We assume in this chapter that ω is smooth, that is, ω is an n—1 dimensional submanifold of M. Write $\tilde{\Omega} = \Omega \cup \omega$ as before. Now take an open subset Ω_0 of M such that $\tilde{\Omega}$ is contained in Ω_0 and the intersection of Ω_0 and the boundary of Ω in M is equal to ω. Let E and E' be two complex C^∞ vector bundles on M. As in section 2.1. fix the duality between $C_0^\infty(M;E)$ and $\mathcal{D}'(M;E)$, etc. Take a nontangential real C^∞ vector field ν in a neighbourhood of ω. Fix connections for each E and E'. By D_ν and D'_ν we denote differentiations in the direction ν of C^∞ sections of E and E' respectively. Thus D_ν is a first order linear differential operator from E to E in a neighbourhood U of ω and its principal symbol is $\langle \nu(x), \xi \rangle$, $x \in U$ and $\xi \in T_x^*(M)$. By R we denotes the trace operator of $C^\infty(\tilde{\Omega};E)$ onto $C^\infty(\omega;E)$, the space of all C^∞ sections of $E|_\omega$ over ω. Here $E|_\omega$ denotes the bundle obtained by restricting E to ω. By R' we denote the trace operator of $C^\infty(\tilde{\Omega};E')$ onto $C^\infty(\omega;E')$. Then the composition $R \circ D_\nu$ is a continuous linear operator of $C^\infty(\tilde{\Omega};E)$ onto $C^\infty(\omega;E)$, and similar for $R' \circ D'_\nu$. For any $s > 3/2$ we can extend these operators as continuous operators of $H_{(s)}^{loc}(\tilde{\Omega};E)$ onto $H_{(s-3/2)}^{loc}(\omega;E)$ and of $H_{(s)}^{loc}(\tilde{\Omega};E')$ onto $H_{(s-3/2)}^{loc}(\omega;E')$ respectively.

Now consider a linear differential operator P from E to E', i.e. a continuous linear operator of $C^\infty(M;E)$ into $C^\infty(M;E')$ such that $\operatorname{supp} P(u) \subset \operatorname{supp} u$ for all $u \in C^\infty(M;E)$. Then to every local chart $\varkappa: U_\varkappa \longrightarrow \varkappa(U_\varkappa) \subset \mathbb{R}^n$ there exists a matrix ${}^\varkappa P$ of usual differential operators with C^∞ coefficients in $\varkappa(U_\varkappa)$ such that

$$(P(u))_\varkappa = {}^\varkappa P(u_\varkappa) \text{ in } \varkappa(U_\varkappa), \quad u \in C^\infty(M;E), \qquad (3.1.1)$$

where $u_\varkappa$ and $(P(u))_\varkappa$ are defined by (2.1.3) with respect to some local charts of E and E' over $\varkappa$ respectively. Then we have the following proposition. We omit the proof, because we will use a similar argument in the proof of Proposition 4.1.1.

<u>Proposition</u> 3.1.1. If P is of finite order in a neighbourhood of ω, there exists a positive integer ℓ and linear differential operators with C^∞ coefficients $A_0, A_1, \cdots, A_\ell$ from $E|_\omega$ to $E'|_\omega$ such that A_ℓ is not identically equal to zero and

$$R' \circ P(u) = \sum_{j=0}^{\ell} A_j \circ R \circ D_\nu^j(u), \quad u \in C^\infty(\widetilde{\Omega};E). \qquad (3.1.2)$$

This representation is unique.

Suppose that $N = N'$ and P is an elliptic operator of order m in Ω_0, that is, its principal symbol $P_m(x, \xi)$ is a bijective mapping of E_x into E'_x for every $x \in \Omega_0$ and every non-zero $\xi \in T_x^*(\Omega_0)$. Here E_x is the space of all fibers at x. Then it follows that $\ell = m$ and A_m is a zeroth order isomorphism of $C^\infty(\omega;E)$ onto $C^\infty(\omega;E')$.

Consider another complex C^∞ vector bundle E'' and a linear differential operator B from E to E'' which is of finite order in a neighbourhood of ω. Let R'' be the trace operator of $C^\infty(\widetilde{\Omega};E'')$ onto $C^\infty(\omega;E'')$. Write

$$R^m = \underbrace{\begin{pmatrix} R & & 0 \\ & \ddots & \\ 0 & & R \end{pmatrix}}_{m} \quad \text{and } B^m(D_\nu) = \begin{pmatrix} \text{id.} \\ D_\nu \\ D_\nu^2 \\ \\ D_\nu^{m-1} \end{pmatrix}.$$

<u>Proposition</u> 3.1.2. Under the above conditions there exist two differential operators B_ω from $E|_\omega$ to $E''|_\omega$ and $\tilde{B}$ from $E'|_\omega$ to $E''|_\omega$, such that the operator

$$(P, R''\circ B):\ C^\infty(\tilde{\Omega};E) \longrightarrow C^\infty(\tilde{\Omega};E') \times C^\infty(\omega;E'')$$

is decomposed as the composition of three operators:

$$R''\circ B = T_3\circ T_2\circ T_1. \tag{3.1.3}$$

Here we have

$$T_1 = (P,\ R^m\circ B^m(D_\nu)):\ C^\infty(\tilde{\Omega};E) \longrightarrow C^\infty(\tilde{\Omega};E') \times C^\infty(\omega;E)^m$$

$$T_2 = \begin{pmatrix} \text{id.} & 0 \\ 0 & B_\omega \end{pmatrix}:\ C^\infty(\tilde{\Omega};E') \times C^\infty(\omega;E)^m \longrightarrow C^\infty(\tilde{\Omega};E') \times C^\infty(\omega;E'')$$

$$T_3 = \begin{pmatrix} 1 & 0 \\ \underset{A_m^{-1}\circ}{R'\circ\tilde{B}} & 1 \end{pmatrix}:\ C^\infty(\tilde{\Omega};E') \times C^\infty(\omega;E'') \longrightarrow C^\infty(\tilde{\Omega};E') \times C^\infty(\omega;E'').$$

<u>Proof.</u> From Proposition 3.1.1. we have two decompositions

$$R'\circ P = \sum_{j=0}^{m} A_j\circ R\circ D_\nu^j$$

and

$$R''\circ B = \sum_{k=0}^{l} B_k\circ R\circ D_\nu^k.$$

Because A_m is a bijective operator of order zero, we can write

$$R''\circ B = A_m^{-1}\circ \underset{\circ B^0}{R'\circ P} + \sum_{k=0}^{m-1} B_k^0\circ R\circ D_\nu^k$$

with some differential operators B^o, B_k^o, $k=0,1,2,\cdots,m-1$ from $E|_\omega$ to $E''|_\omega$. Then it is enough to set $B_\omega = (B_0^o, B_1^o, \cdots, B_{m-1}^o)$ and $\tilde{B} = B^o$. This finishes the proof.

§3.2. Differential equations without boundary conditions.

First we consider the following differential equation

$$P(u) = f \tag{3.2.1}$$

for $u \in C^\infty(\tilde{\Omega};E)$ and $f \in C^\infty(\tilde{\Omega};E')$. To study this equation we need not assume that ω is of C^∞ class.

Theorem 3.2.1. Suppose that P is elliptic. Then the equation (3.2.1) has a solution $u \in C^\infty(\tilde{\Omega};E)$ for each $f \in C^\infty(\tilde{\Omega};E')$, which satisfies $\langle f, \varphi\rangle = 0$ when $\varphi \in \mathring{\mathcal{E}}'(\tilde{\Omega};E)$ and ${}^tP(\varphi) = 0$, if and only if the following condition holds:

(A) To every real number s and every compact set $K \subset \tilde{\Omega}$, there exists another compact set $K' \subset \tilde{\Omega}$ such that $\varphi \in \mathring{\mathcal{E}}'(\tilde{\Omega};E')$, ${}^tP(\varphi) \in \mathring{H}^c_{(s)}(\tilde{\Omega};E)$, and $\operatorname{supp} {}^tP(\varphi) \subset K$ implies the existence of another $\psi \in \mathring{\mathcal{E}}'(\tilde{\Omega};E')$, which satisfies ${}^tP(\varphi) = {}^tP(\psi)$ and $\operatorname{supp} \psi \subset K'$.

Proof. From Corollary 1.2.3., it is enough to prove the following statement:

(B) To every real number s and every compact set $K \subset \tilde{\Omega}$, there exist a real number t and a positive constant C such that $\varphi \in \mathring{\mathcal{E}}'(\tilde{\Omega};E')$, $\operatorname{supp} \varphi \subset K$, and ${}^tP(\varphi) \in \mathring{H}^c_{(s)}(\tilde{\Omega};E)$ implies the existence of another $\psi \in \mathring{\mathcal{E}}'(\tilde{\Omega};E')$ such that $\operatorname{supp} \psi \subset K$, ${}^tP(\psi) = {}^tP(\varphi)$, and $\|\psi\|_{(t)} \leq C \cdot \|{}^tP(\psi)\|_{(s)}$.

(e.g. [24])
Since P is elliptic, there exists a positive constant C, which depends on K and s, such that $\varphi \in \mathring{\mathcal{E}}'(\tilde{\Omega};E')$, ${}^tP(\varphi) \in \mathring{H}^c_{(s)}(\tilde{\Omega};E)$ and supp $\varphi \subset K$ implies

$$\|\varphi\|_{(s+m)} \leqq C \cdot \|{}^tP(\varphi)\|_{(s)} + C \cdot \|\varphi\|_{(s+m-1)}. \quad (3.2.2)$$

Set $t = s+m$. Now assume that the conclusion of (B) does not hold. Then there exists a sequence $\varphi_n \in \mathring{\mathcal{E}}'(\tilde{\Omega};E')$, $n = 1,2,\cdots$ such that

(1) supp $\varphi_n \subset K$, ${}^tP(\varphi_n) \in \mathring{H}^c_{(s)}(\tilde{\Omega};E)$, and $\|{}^tP(\varphi_n)\|_{(s)} \longrightarrow 0$ as $n \longrightarrow \infty$,

(2) there exists $\psi_n \in \mathring{\mathcal{E}}'(\tilde{\Omega};E')$ which satisfies supp $\psi_n \subset K$, ${}^tP(\psi_n) = {}^tP(\varphi_n)$, and $\|\psi_n\|_{(s+m)} = 1$, and finally

(3) $\psi \in \mathring{\mathcal{E}}'(\tilde{\Omega};E')$, supp $\psi \subset K$, and ${}^tP(\varphi_n) = {}^tP(\psi)$ implies $\|\psi\|_{(s+m)} \geqq 1$.

From Rellich's theorem the sequence ψ_n, $n = 1,2,\cdots$ has a subsequence which converges to some ψ_o with respect to the norm $\|\cdot\|_{(s+m-1)}$. We write the subsequence by the same letter. Then ${}^tP(\psi_n) = {}^tP(\varphi_n)$ converges to ${}^tP(\varphi_o) = 0$. Set $\psi'_n = \psi_n - \psi_o$, $n = 1,2,\cdots$. Then it follows that $\psi'_n \in \mathring{\mathcal{E}}'(\tilde{\Omega};E')$, supp $\psi'_n \subset K$, ${}^tP(\varphi_n) = {}^tP(\psi'_n)$, and $\|\psi'_n\|_{(s+m-1)} \longrightarrow 0$ as $n \longrightarrow \infty$. From (3) we have $\|\psi'_n\|_{(s+m)} \geqq 1$. But this contradicts with (3.2.2), and the statement (B) is proved. This finishes the proof of Theorem 3.2.1.

Corollary 3.2.2. Suppose that P is elliptic and $\tilde{\Omega}$ is compact. Then the equation (3.2.1) has a solution $u \in C^\infty(\tilde{\Omega};E)$ for $f \in C^\infty(\tilde{\Omega};E')$, if $\varphi \in \mathring{\mathcal{E}}'(\tilde{\Omega};E)$ and ${}^tP(\varphi) = 0$ implies $\langle f, \varphi \rangle = 0$.

Corollary 3.2.3. Suppose that for every relatively compact open subset U of Ω_o, the union of all compact connected components

components of $\Omega\setminus U$ is relatively compact, and $\tilde{\Omega}$ has no relatively compact component which is open in Ω_0. If in addition Ω_0 is a real analytic manifold and P is an elliptic operator with real analytic coefficients, then the equation (3.2.1) has a solution $u \in C^\infty(\tilde{\Omega};E)$ for every $f \in C^\infty(\tilde{\Omega};E')$.

Proof. It is enough to prove that $\tilde{\Omega}$ is P-convex. Let K be a compact set in $\tilde{\Omega}$. If $\varphi \in \mathring{\mathcal{E}}'(\tilde{\Omega};E)$ and $\operatorname{supp} {}^tP(\varphi) \subset K$, then we have ${}^tP(\varphi) = 0$ in $\Omega_0\setminus K$. Therefore φ is real analytic in $\Omega_0\setminus K$. Let U be an open relatively compact neighbourhood of K in Ω_0. We define K' as the closure of the union of $\overline{U}\cap\tilde{\Omega}$ and all compact components of $\Omega\setminus U$. Then K' is compact. If K is empty, then K' is also empty. We have $\operatorname{supp} \varphi \subset K'$ and the proof is finished.

Without the hypothesis of real analyticity there arise difficult problems (e.g. Hörmander [8] or Harvey [7]). For subelliptic operators we can prove a similar theorem, using the results of Hörmander, Egorov, and Treves [26]. Moreover we can immediately obtain theorems on the existence of solutions in Sobolev spaces $H^{loc}_{(s)}(\tilde{\Omega};E)$. We leave the details as an exercise for the reader.

Now consider a linear differential operator B_ω of $C^\infty(\omega;E)^m$ into $C^\infty(\omega;E'')$. We will prove two propositions on sufficient conditions for B_ω-convexity.

Proposition 3.2.4. If B_ω is a differential operator of order zero and its symbol $B_\omega(x)$ is a surjection of E_x^m onto E''_x, $x \in \omega$, then ω is B_ω-convex, that is, for every compact set $K \subset \omega$ there exists another compact set $K' \subset \omega$, which can be taken as empty if K is empty, such that $\varphi \in \mathcal{E}'(\omega;E'')$ and $\operatorname{supp} {}^tB_\omega(\varphi) \subset K$

implies supp $\varphi \subset K'$.

Proof. Since $B_\omega(x)$ is surjective, its transpose ${}^tB_\omega(x)$ is injective. Therefore if $\varphi \in \mathcal{E}'(\omega ;E'')$ and ${}^tB_\omega(\varphi) = 0$ in an open subset U of ω, then it follows that $\varphi = 0$ in U. Hence the proposition follows.

Proposition 3.2.5. Let ω be an open convex subset of $\mathbb{R}^{n-1}$, and B_ω be a differential operator with constant coefficients. Assume that $N'' \leqq m\cdot N$ and the rank of $B_\omega(\xi_0)$ is N'' for some $\xi_0 \in \mathbb{R}^n$. Then ω is B_ω-convex.

Proof. We can write $B_\omega(\xi) = (B_1(\xi), B_2(\xi))$, where $B_1(\xi)$ is an $N'' \times N''$ matrix and $\det B_1(\xi_0) \neq 0$. Then ω is $\det B_1(D)$-convex. Let K be a compact convex subset of ω. If $\varphi \in \mathcal{E}'(\omega ;E'') \cong \mathcal{E}'(\omega)^{N''}$ and $\operatorname{supp} {}^tB_\omega(D)\varphi \subset K$, then it follows that $\operatorname{supp}(\det {}^tB_1(D)\varphi) \subset K$. Hence we obtain $\operatorname{supp} \varphi \subset K$, and the proof is complete.

§3.3. Elliptic boundary problems in non-compact manifolds.

In this section we assume that Ω_0 is a real analytic manifold and P is an elliptic operator with real analytic coefficients in Ω_0. Without these assumptions we have to deal with some kinds of pseudo-convexity conditions on the boundary of $\tilde{\Omega}$. The order of P is $m = 2\cdot l$ and the dimensions of E and E' are the same, that is, we consider only determined systems. Let E_j, $j = 1,2,\cdots,l$ be N-dimensional complex C^∞ vector bundles on Ω_0 and let their direct sum be denoted by E''. By R_j we denotes the trace operator of $C^\infty(\tilde{\Omega};E_j)$ onto $C^\infty(\omega;E_j)$. Let B_j be a differential operator from E to E_j and its order is m_j in a neighbourhood of ω, $j = 1,2,\cdots,l$. Set $B = (B_1, B_2, \cdots, B_l)$. Then we have the decomposition (3.1.3). Since T_3 is an isomorphism, it is enough for us to consider only $T_2 \circ T_1$. Write

$$B_\omega \circ R^m \circ B^m(D_\nu) = (p_1, p_2, \cdots, p_l) : C^\infty(\tilde{\Omega}; E) \longrightarrow \bigoplus_{j=1}^{l} C^\infty(\omega; E_j).$$

Then each p_j is of order m_j. By B_j^o and p_j^o we denote the principal part of B_j and p_j of order m_j respectively. We say that $(P, R'' \circ B)$ is an elliptic boundary system if and only if the system $(P, p_1, p_2, \cdots, p_l)$ is elliptic in the usual sense, that is, to every local chart $\varkappa : \tilde{U}_\varkappa \longrightarrow \varkappa(\tilde{U}_\varkappa) \subset \overline{\mathbb{R}^n_+}$ and every point $x \in \omega \cap \tilde{U}_\varkappa$ the boundary problem for a differential equation

$$\begin{cases} {}^\varkappa P_m(\varkappa(x), D_z) u(z) = 0 \text{ in } \mathbb{R}^n \\ {}^\varkappa p_j^o(\varkappa(x), D_z) u(z) = 0, \text{ when } z_n = 0, \end{cases}$$

has no solution of the form $u(z) = e^{i\langle z', \xi' \rangle} w(z_n)$ such that $\xi' \in \mathbb{R}^{n-1}$, $\xi' \neq 0$, and $w(z_n)$ is an N-vector of nonzero bounded functions for $z_n \geqq 0$. Cf. [8, 10, 24].

<u>Theorem</u> 3.3.1. Let $s \geqq m$. Assume that $\tilde{\Omega}$ has no compact component and to every relatively compact open subset U of Ω_0 the union of all compact connected components of $\tilde{\Omega} \setminus U$ is also relatively compact. If $(P, R'' \circ B)$ is an elliptic boundary system, then the following three state-ments are equivalent:

(i) To every compact set $K \subset \omega$ there exists another compact set $K' \subset \omega$, which can be chosen as empty set if K is empty, such that $\varphi \in \bigoplus_{j=1}^{l} H^{c}_{(-m+m_j+\frac{1}{2})}(\omega ;E_j)$ and supp ${}^{t}B_{\omega}(\varphi) \subset K$ implies supp $\varphi \subset K'$.

(ii) The equation

$$\begin{cases} P(u) = F \\ R_j \circ B_j(u) = f_j, \quad j = 1,2,\cdots,l, \end{cases} \tag{3.3.1}$$

has a solution $u \in C^{\infty}(\tilde{\Omega};E)$ for every $F \in C^{\infty}(\tilde{\Omega};E')$ and $f_j \in C^{\infty}(\omega ;E_j)$, $j = 1,2,\cdots,l$.

(iii) The equation (3.3.1) has a solution $u \in H^{loc}_{(s)}(\tilde{\Omega};E)$ for every $F \in H^{loc}_{(s-m)}(\tilde{\Omega};E')$ and $f_j \in H^{loc}_{(s-m_j-\frac{1}{2})}(\omega ;E_j)$, $j = 1,2,\cdots,l$.

If ω is B_{ω}-convex, then the above condition (i) is valid. Therefore we obtain sufficient conditions for (i) from Propositions 3.2.4. and 3.2.5. The well-known Dirichlet condition satisfies the hypothesis of Proposition 3.2.4. Therefore under the assumption of the previous theorem the Dirichlet boundary problem always has a solution. The following lemma is an immediate consequence of Theorems 1.2.1. and 2.4.4.

Lemma 3.3.2. The equation (3.3.1) has a solution $u \in H^{loc}_{(m)}(\tilde{\Omega};E)$ for every $F \in H^{loc}_{(0)}(\tilde{\Omega};E')$ and $f_j \in H^{loc}_{(m-m_j-\frac{1}{2})}(\omega;E_j)$, $j=1,2,\cdots,l$ if and only if the following two conditions are valid:

(1) To every compact set $K \subset \tilde{\Omega}$ there exists another compact set $K' \subset \tilde{\Omega}$ such that $\Phi \in \overset{\circ}{H}{}^{c}_{(0)}(\tilde{\Omega};E')$, $\varphi_j \in H^{c}_{(-m+m_j+\frac{1}{2})}(\omega ;E_j)$, $j = 1,2,\cdots,l$ and supp $({}^{t}P(\Phi) + \sum_{j=1}^{l} {}^{t}p_j(\varphi_j)) \subset K$ implies

supp $\Phi \subset K'$ and supp $\varphi_j \subset \omega_\cap K'$, $j = 1,2,\cdots,l$.

(3) To every compact set $K \subset \tilde{\Omega}$ there exists a positive constant C such that $\Phi \in \overset{\circ}{H}{}^c_{(o)}(\tilde{\Omega};E')$, $\varphi_j \in H^c_{(-m+m_j+\frac{1}{2})}(\omega;E_j)$, supp $\Phi \subset K$, and supp $\varphi_j \subset \omega_\cap K$, $j = 1,2,\cdots,l$ implies

$$\|\Phi\|_{(o)} + \sum_{j=1}^{l} \|\varphi_j\|_{(-m+m_j+\frac{1}{2})} \leq C \| {}^tP(\Phi) + \sum_{j=1}^{l} {}^tp_j(\varphi_j)\|_{(-m)}. \tag{3.3.2}$$

§3.4. The proof of Theorem 3.3.1.

If (ii) is true, there exists a solution $u \in C^\infty(\omega;E)^m$ of the equation $B_\omega(u) = f$ for every $f \in C^\infty(\omega;E'')$. Let K be a compact subset of ω, and $s < \min_{j=1,\cdots,l}(-m+m_j+\frac{1}{2})$. Then from Lemma 3.2.2. there exists a compact subset K' of ω such that $\varphi \in H^c_{(s)}(\omega;E'')$ and supp ${}^tB_\omega(\varphi) \subset K$ implies supp $\varphi \subset K'$. Hence (i) holds.

If (iii) is true, we obtain (ii) from the regularity theorem for elliptic boundary problems. Then we have to prove (iii) in the case when $s = m$. Therefore it is enough to check two conditions (1) and (3) of Lemma 3.3.2. under the assumption that (i) is true.

(I) Proof of (1) in Lemma 3.3.2. Let K be a compact set in $\tilde{\Omega}$. Choose a relatively compact open subset U of Ω_o which contains K. Let K_1 denotes the closure of the union of $\overline{U}_\cap \tilde{\Omega}$ and all compact connected components of $\tilde{\Omega} \backslash U$. From an assumption of the theorem K_1 is a compact set in $\tilde{\Omega}$. Since we have assumed that (i) is true there exists a compact set $K_2 \subset \omega$ such that (i) holds if K and K' is replaced by $K_1 \cap \omega$ and K_2 respectively. Set $K' = K_1 \cup K_2$. If K is empty, we can take K' to be empty.

Write $X = \bigoplus_{j=1}^{l} H^c_{(-m+m_j+\frac{1}{2})}(\omega; E_j)$. Let $\Phi \in \overset{\circ}{H}{}^c_{(o)}(\tilde{\Omega}; E')$, $\varphi = (\varphi_1, \varphi_2, \cdots, \varphi_l) \in X$, and $\operatorname{supp}({}^tP(\Phi) + \sum_{j=1}^{l} {}^tp_j(\varphi_j)) \subset K$.

Let Φ_1 denotes the restriction of Φ to Ω. Then it follows that $\Phi_1 \in \mathcal{E}'(\tilde{\Omega}; E')$ and $\operatorname{supp} {}^tP(\Phi_1) \subset K \cap \Omega$. Since ${}^tP$ is elliptic with real analytic coefficients, Φ_1 is real analytic in $\Omega \backslash K$. Therefore we have $\operatorname{supp} \Phi_1 \subset K_1 \cap \Omega$.

Now let Φ_2 denotes the restriction of Φ to the complement of K_1 in Ω_0. Then Φ_2 is a distribution section of order zero and its support is contained in ω. Hence there exists $\psi \in \mathcal{D}'(\omega \backslash K_1; E')$ such that $\Phi_2 = {}^tR'(\psi)$. Cf. Schwartz [22]. Then we have

$$0 = {}^tP(\Phi_2) + \sum_{j=1}^{l} {}^tp_j(\varphi_j)|\omega \backslash K_1$$

$$= {}^tP \circ {}^tR'(\psi) + \sum_{j=1}^{l} {}^t(B_\omega \circ R^m \circ B^m(D_\nu))_j(\varphi_j|\omega \backslash K_1)$$

$$= \sum_{k=0}^{m} {}^tD_\nu^k \circ {}^tR \circ {}^tA_k(\psi) + \sum_{k=0}^{m-1} {}^tD_\nu^k \circ {}^tR \circ {}^tB_k^o(\varphi|\omega \backslash K_1). \quad (3.4.1)$$

Here we have used the representation (3.1.2), and wrote $B_\omega = (B_0^o, B_1^o, \cdots, B_{m-1}^o)$. (3.4.1) implies that ${}^tA_m(\psi) = 0$. Since ${}^tA_m$ is bijective, we can conclude that $\psi = 0$, that is, $\Phi_2 = 0$.

Then (3.4.1) implies that ${}^{t}B_{\omega}(\varphi|\omega\backslash K_1)=0$, that is, supp ${}^{t}B_{\omega}(\varphi)\subset$ $\subset K_1\cap\omega$. From the hypothesis (i) it follows that supp $\varphi\subset K_2$. We have thus proved that supp $\Phi\subset K'$ and supp $\varphi\subset K'\cap\omega$, and the proof of (1) is complete.

(II) Proof of (3) in Lemma 3.3.2. Let K be a compact set in $\tilde{\Omega}$. Take any $\Phi\in\overset{\circ}{H}{}^{c}_{(o)}(\tilde{\Omega};E')$ and $\varphi=(\varphi_1,\varphi_2,\cdots,\varphi_l)\in X$ such that supp $\Phi\subset K$ and supp $\varphi\subset K\cap\omega$. Write $T=B_{\omega}\circ R^m\circ B^m(D_\nu)=$ $=(p_1,p_2,\cdots,p_l)$, and

$$\|\varphi\|_{(s)}=\sum_{j=1}^{l}\|\varphi_j\|_{(s+m_j+\frac{1}{2})}.$$

In the following C or C' represent generic constants which does not depend on the choice of Φ and φ_j. We have proved the following estimate in the proof of Theorem 3.2.1.:

$$\|\Phi\|_{(o)}\leqq C\cdot\|{}^{t}P(\Phi)\|_{(-m)}. \tag{3.4.2}$$

This implies

$$\|\Phi\|_{(o)}\leqq$$

$$\leqq C\cdot\|{}^{t}P(\Phi)+{}^{t}T(\varphi)\|_{(-m)}+C\|{}^{t}T(\varphi)\|_{(-m)}$$

$$\leqq C\cdot\|{}^{t}P(\Phi)+{}^{t}T(\varphi)\|_{(-m)}+C'\|\varphi\|_{(-m)}. \tag{3.4.3}$$

Let χ be a function in $C_0^\infty(\omega)$ such that $\chi=1$ in a neighbourhood of $K\cap\omega$. Because the system (P, T) is elliptic, we can prove the existence of a continuous linear operator S of $Y=\bigoplus_{j=1}^{l}H^{loc}_{(m-m_j-\frac{1}{2})}(\omega;E_j)$ into $H^{loc}_{(m)}(\tilde{\Omega};E)$ and a pseudo-differential operator $Q_{-\infty}$ of degree $-\infty$ from E'' to E'' such that $P \circ S(\chi \cdot u) = 0$ and $T \circ S(\chi \cdot u) = \chi \cdot u + Q_{-\infty}(\chi \cdot u)$, $u \in Y$. We can modify the proof in the case of compact manifolds to show the above fact. See [8,10, 23,24]. Then for every $u \in Y$ we have

$$\begin{aligned} &|<u, \varphi>| = \\ &= |<\chi \cdot u, \varphi>| \\ &= |<T \circ S(\chi \cdot u) - Q_{-\infty}(\chi \cdot u), \varphi> + <P \circ S(\chi \cdot u), \Phi>| \\ &\leqq |<S(\chi \cdot u), {}^tT(\varphi) + {}^tP(\Phi)>| + |<\chi u, {}^tQ_{-\infty}(\varphi)>| \\ &\leqq C \cdot |||\chi \cdot u||| \cdot \|{}^tP(\Phi) + {}^tT(\varphi)\|_{(-m)} + C\,|||\chi \cdot u||| \cdot \|\varphi\|'_{(t)}. \end{aligned}$$

Here t is any real number and $|||\chi \cdot \;|||$ denotes a seminorm in Y. Then we obtain

$$\|\varphi\|'_{(-m)} \leqq C \cdot \|{}^tP(\Phi) + {}^tT(\varphi)\|_{(-m)} + C \cdot \|\varphi\|'_{(-m-1)}. \tag{3.4.4}$$

From (3.4.3) and (3.4.4) we have the following estimate.

$$\|\Phi\|_{(0)} + \|\varphi\|'_{(-m)} \leqq C\|{}^tP(\Phi) + {}^tT(\varphi)\|_{(-m)} + C \cdot \|\varphi\|'_{(-m-1)}.$$

This kind of estimate appears in Peetre [20].

Now prove (3.3.2). From (3.4.3) it is enough to prove the following estimate for every Φ:

$$\|\varphi\|'_{(-m)} \leqq C \cdot \|{}^tP(\Phi) + {}^tT(\varphi)\|_{(-m)}, \quad \varphi \in X \text{ and } \operatorname{supp} \varphi \subset K \cap \omega. \tag{3.4.5}$$

Assume that this estimate does not hold. Then there exists a sequence of distribution sections $\varphi_\mu \in X$, and $\Phi_\mu \in \overset{\circ}{H}{}^c_{(0)}(\tilde{\Omega}; E')$, $\mu = 1, 2, \cdots$ such that $\operatorname{supp} \varphi_\mu \subset K \cap \omega$, $\|\varphi_\mu\|'_{(-m)} = 1$, $\operatorname{supp} \Phi_\mu \subset K$, and $\|{}^tP(\Phi_\mu) + {}^tT(\varphi_\mu)\|_{(-m)} \longrightarrow 0$ as $\mu \longrightarrow \infty$. From (3.4.3) there

there exists a constant C_0 such that $\|\Phi_\mu\|_{(0)} \leq C_0$, $\mu = 1, 2, \cdots$. From Theorem 2.4.3. there exists subsequences of φ_μ and Φ_μ, which we denote by the same letters, such that φ_μ converges to some φ_0 with respect to the norm $\|\cdot\|'_{(m-1)}$, and Φ_μ converges to some Φ_0 with respect to the norm $\|\cdot\|_{(0)}$. Then ${}^tP(\Phi_\mu) + {}^tT(\varphi_\mu)$ converges weakly to ${}^tP(\Phi_0) + {}^tT(\varphi_0) = 0$. Hence we have ${}^tP(\Phi_0) = 0$ in Ω. Since $\tilde{\Omega}$ is P-convex from Corollary 3.2.3., we obtain $\Phi_0|_\Omega = 0$. But Φ_0 is assumed to be an L^2 section of E', and hence we can conclude that $\Phi_0 = 0$. It follows then ${}^tT(\varphi_0) = 0$. Hence we have $\sum_{k=1}^{m} {}^tD^k_\nu \circ {}^tR \circ {}^tB^o_k(\varphi_0) = 0$, which implies ${}^tB_\omega(\varphi_0) = 0$. From (i) we can conclude that $\varphi_0 = 0$. But this contradict with (3.4.4), since $\|\varphi_\mu\|'_{(-m)} = 1$. This contradiction completes the proof of (3.4.5), and then the proof of Theorem 3.3.1. is finished.

Chapter IV. Evolutional boundary problems.

§4.0. Introduction.

This chapter is devoted to the survey of boundary problems for evolution operators such as hyperbolic, parabolic, and some other such operators. If we use the results of Chapter I and II, we can immediately obtain such necessary and sufficient conditions of solvability for many boundary problems as Theorem 4.2.1., Lemma 4.3.1., 4.3.2., etc. If we want to solve equations in $\mathfrak{F}(\hat{\Omega}, \omega_1)$ instead of imposing such boundary conditions as Cauchy data, then we do not meet with any essential difficulty except in calculations. But if we consider boundary conditions of Cauchy type on a boundary which is not normal (Definition 4.1.1.), we have to solve a collection of differential equations on the boundary, which are induced by the original differential operator. In some simple cases we can solve such a family of differential equations. In section 4.1. we explain such a family of differential operators induced on the boundary. In section 4.2. the reduction of boundary conditions to our function spaces is done. We give a necessary and sufficient conditions of solvability for a mixed type problem in Theorem 4.2.1. They consist of two types of conditions, which have been explained in Preface. Almost all our results are concerned with C^∞ solutions, because the investigation of the regularity properties of solutions in Sobolev spaces leads us to such a complicated situation as in Theorem 4.2.4. Finally we comment on the equivalence of solvability of the equation in $\mathfrak{F}(\hat{\Omega}, \omega_1)$ and the extension property of solutions (Proposition 4.2.5.).

In section 4.3. we state two basic lemmas obtained from the results of Chapter I and II, and then give remarks on elliptic and strictly hyperbolic equations. In section 4.4. we study a special case where the differential operator has constant coefficients. In this case we can obtain many good results. Especially we can find a necessary and sufficient condition for differential operators in Theorem 4.4.2., using a result due to Hörmander [11]. If we apply the theory of overdetermined systems of differential equations with constant coefficients (Ehrenpreis [5]) to boundary systems induces by the boudary condition, then we obtain compatibility conditions on the boudary data (Corollary 4.4.3.). If the boundary condition is determined in the sense of Corollary 4.4.4., we can find a more refined condition on the boundary. If the differential operator is hyperbolic, then a complete geometric characterization of P(D)-convexity is obtained (Theorem 4.4.5.). Section 4.5. is devoted to the study of the case where the differential operator is not normal with respect to both parts of the boundary. Then we have to solve a system of boundary problems for the induced differential equations on the boundary. We can solve many Cauchy problems for wave equations with their data given on characteristic boundaries. The Goursat problem is solved there. For the sake of simplicity, we restrict our considerations to single differential equations. An extension to determined systems is rather easy.

§4.1. Preliminaries.

Let M be a σ-compact C^∞ manifold of dimension n. Let Ω be an open subset of M, ω an open subset of the boundary of Ω in M, and ω_1 a subset of ω. Take three open subset Ω_0, Ω_1, Ω_2 of M which satisfy the conditions of Proposition 2.2.1. For other notations we refer the reader to sections 2.1. and 2.2. In the following we use the letter M to denote the order of the differential operator P, since this will cause no confusion for the reader.

Now assume in this section that ω is of C^∞ class. Let ν be a real C^∞ vector field in a neighbourhood of ω, which is not tangential to ω. By D_ν we denote the first order linear differential operator with the symbol $\langle \nu(x), \xi \rangle$, $\xi \in T_x^*(\Omega_0)$. Let P be a linear differential operator (with C^∞ coefficients) in Ω_0 such that its order is $M < \infty$. Then we have the following proposition:

Proposition 4.1.1. There exist an integer $m \leq M$ and a unique family of differential operators in the boundary ω;

$$A_{i,s}, \quad i = 0,1,2,\cdots,M \text{ and } s = s_i, s_i+1, \cdots$$

where $s_i = \max(0, i-m)$, such that there exists a non-zero operator $A_{i,s}$ with $i = s+m$ and the following relation holds:

$$R \circ D_\nu^j \circ P = \sum_{k=0}^{j+m} A_k^{(j)} \circ R \circ D_\nu^k, \quad j = 0,1,2,\cdots. \qquad (4.1.1)$$

Here $A_k^{(j)}$ is defined by

$$A_k^{(j)} = \sum_{i=\max(0,k-j)}^{\min(k,M)} \binom{j}{i+j-k} A_{i,i+j-k}, \quad \text{if } 0 \leqq k \leqq j+m$$
$$A_k^{(j)} = 0, \quad \text{if } j+m < k. \tag{4.1.2}$$

Especially we have

$$A_{j+m}^{(j)} = \sum_{i=\max(0,m)}^{\min(j+m,M)} \binom{j}{i-m} A_{i,i-m}, \quad \text{if } j+m \geqq 0. \tag{4.1.3}$$

Proof. To every local patch $\tilde{U}$ in $\tilde{\Omega}$, we can choose a local chart $\varkappa : \tilde{U} \longrightarrow \varkappa(\tilde{U}) \subset \overline{\mathbb{R}^n_+} = \{x = (x', x_n) \in \mathbb{R}^n;\ x_n \geqq 0\}$ such that $\varkappa(\tilde{U} \cap \omega) = \{x \in \varkappa(\tilde{U});\ x_n = 0\}$ and the vector field ν is transformed by $\varkappa$ to a unit vector parallel to the x_n axis at every point of $\varkappa(\tilde{U} \cap \omega)$. Then the symbol of P is written as

$$P(x, \xi) = \sum_{i=0}^{M} A_i(x, \xi') \cdot \xi_n^i, \quad x \in \varkappa(\tilde{U}) \text{ and } \xi = (\xi', \xi_n) \in \mathbb{R}^n.$$

If we define

$$A_{i,s}(x, \xi') = \left(\frac{\partial}{\partial x_n}\right)^s A_i(x, \xi'), \quad x \in \varkappa(\tilde{U} \cap \omega) \text{ and } \xi' \in \mathbb{R}^{n-1},$$

Then the above proposition follows. The details may be left to the reader.

In the following discussions $A_{j+m}^{(j)}$, $j+m \geqq 0$, play a central role. We have to calculate $A_{i,s}$ for $i = s+m$, if we want to apply the following results to concrete differential equations. See Example 4.4.6., 4.5.2., and Theorem 4.5.3.

Definition 4.1.1. We say that P is semi-normal of degree $m \geqq 0$ with respect to ω if

$$A_{i,s} = 0, \quad m+s \leqq i \leqq M \text{ and } i \neq m, \tag{4.1.4}$$

in Proposition 4.1.1. for some vector field ν. If in addition $A_{m,0}$ is a multiplication operator by an everywhere non-zero function, then we say that P is normal of degree m with respect to ω.

This definition does not depend on the choice of ν. We leave its proof as an exercise for the reader. If p is semi-normal of degree m with respect to ω, then we have

$$A^{(j)}_{j+m} = A_{m,0}, \quad j = 0,1,2,\cdots$$

and

$$R\circ D^j_\nu \circ P = A_{m,0}\circ R\circ D_\nu^{j+m} + \sum_{k=0}^{j+m-1} A^{(j)}_k \circ R\circ D^k_\nu, \quad j = 0,1,2,\cdots. \quad (4.1.5)$$

§4.2. Reduction of the problem.

We refer to section 2.2. for our notations. Set $\tilde{\omega}_j = \omega_j \cup \omega_{21}$, $j = 1,2$, and suppose that ω_{12} is empty. Moreover for the sake of simplicity we assume that ω_{11}, ω_{13}, ω_{22} are void, that is, we assume that ω is the union of ω_{10}, ω_{20}, and ω_{21}. We can generalize some results to the case when the above sets are not empty. Assume the existence of two n-1 dimensional C^∞ submanifolds ω_{j5}, $j=1,2$ of Ω_0 such that each $\tilde{\omega}_j$ is contained in ω_{j5}, and the intersection of ω_{j5} and the boundary of ω_{j0} in ω_{j5} is equal to ω_{21}. Moreover assume that ω_{j5} does not meet with Ω. By R_j we denote the trace operator of $C^\infty(\tilde{\Omega})$ onto $C^\infty(\tilde{\omega}_j)$, the space of all C^∞ functions in ω_{j0} which can be extended to a C^∞ function in ω_{j5}. Take a real C^∞ vector field ν_j in a neighbourhood of ω_{j5}, which is not tangential to ω_{j5}, $j=1,2$. By D_{ν_j} we denote the first order linear differential operator with the symbol $\langle \nu_j(x), \xi \rangle$, $\xi \in T^*_x(\Omega_0)$. In the following all linear differential operators which we condider have C^∞ coefficients. Let P be a linear differential operator of order M in Ω_0, and $B=(B_1, B_2, \cdots, B_m)$ a linear differential operator of $C^\infty(\tilde{\Omega})$ into $C^\infty(\tilde{\Omega})^m$. As in section 2.1. we fix the duality of $C_0^\infty(\Omega_0)$ and $\mathcal{D}'(\Omega_0)$, Sobolev norms, etc. Then we have the following theorem, which can be applied to mixed problems for parabolic or hyperbolic equations.

<u>Theorem</u> 4.2.1. If P is normal of degree l with respect to ω_{10}, then the following three statements are equivalent:

(i) Let $F \in C^\infty(\tilde{\Omega})$, $f_j \in C^\infty(\tilde{\omega}_1)$, $j=1,2,\cdots,l$ and $g_k \in C^\infty(\tilde{\omega}_2)$, $k=1,2,\cdots,m$ satisfy the following compatibility conditions: Take a function Φ in $C^\infty(\tilde{\Omega})$ such that

$$\begin{cases} R_1 \circ P(\Phi) = R_1(F) & \text{in } \omega_{10} \\ R_1 \circ D_{\nu_1}^{j-1}(\Phi) = f_j & \text{in } \omega_{10},\ j=1,2,\cdots,l. \end{cases} \tag{4.2.1}$$

Then to any differential operator Q in $\tilde{\omega}_2$ the trace of $Q(g_k - R_2 \circ$ $\circ B_k(\Phi))$ to ω_{21} is equal to zero for $k = 1,2,\cdots,m$. Under the above condition the boundary problem

$$\begin{cases} P(u) = F & \text{in } \Omega \\ R_1 \circ D_{\nu_1}^{j-1}(u) = f_j & \text{in } \omega_{10},\ j=1,2,\cdots,l \\ R_2 \circ B_k(u) = g_k & \text{in } \omega_{20},\ k=1,2,\cdots,m \end{cases} \tag{4.2.2}$$

has a solution $u \in C^\infty(\tilde{\Omega})$.

(ii) The boundary problem

$$\begin{cases} P(v) = G & \text{in } \Omega_0 \\ R_2 \circ B_k(v) = h_k & \text{in } \omega_{25},\ k=1,2,\cdots,m \end{cases} \tag{4.2.3}$$

has a solution $v \in C^\infty(\tilde{\Omega}, \omega_1)$ for any $G \in C^\infty(\tilde{\Omega}, \omega_1)$ and any $h_k \in$ $\in \overset{\circ}{C}^\infty(\tilde{\omega}_2)$, $k=1,2,\cdots,m$.

(iii) The following two conditions hold:

(iii-1) To every real number s and every compact set $K \subset \tilde{\Omega}$ there exists another compact set $K' \subset \tilde{\Omega}$ such that $\Phi \in \mathcal{E}'(\tilde{\Omega}, \omega_2)$, $\varphi_1, \varphi_2, \cdots, \varphi_m \in \mathcal{E}'(\tilde{\omega}_2)$, ${}^tP(\Phi) + \sum_{k=1}^{m} {}^tB_k \circ {}^tR_2(\varphi_k) \in$ $\in H^c_{(s)}(\tilde{\Omega}, \omega_2)$, and $\operatorname{supp} {}^tP(\Phi) + \sum_{k=1}^{m} {}^tB_k \circ {}^tR_2(\varphi_k) \subset K$ implies $\operatorname{supp} \Phi \subset K'$ and $\operatorname{supp} \varphi_k \subset K' \cap \omega_{20}$, $k=1,2,\cdots,m$.

(iii-2) To every real number s and every compact set $K \subset \tilde{\Omega}$ there exist another real number t and a positive constant C such that $\Phi \in \mathcal{E}'(\tilde{\Omega}, \omega_2)$, $\varphi_1, \varphi_2, \cdots, \varphi_m \in \mathcal{E}'(\tilde{\omega}_2)$, $\mathrm{supp}\,\Phi \subset K$, and $\mathrm{supp}\,\varphi_k \subset K \wedge \omega_{20}$, $k=1,2,\cdots,m$ implies the following estimate:

$$\inf\{\|\tilde{\Phi}\|_{(t)};\ \tilde{\Phi} \in H^c_{(t)}(\Omega_0) \text{ and } \Phi = \tilde{\Phi}|_{\omega_2}\}$$
$$+\sum_{k=1}^{m} \inf\{\|\tilde{\varphi}\|_{(t)};\ \tilde{\varphi} \in H^c_{(t)}(\omega_{25}) \text{ and } \varphi_k = \tilde{\varphi}|\omega_{20}\}$$
$$\leqq C\cdot\inf\{\|\Psi\|_{(s)};\ \Psi \in H^c_{(s)}(\Omega_0) \text{ and } {}^tP(\Phi)+\sum_{k=1}^{m} {}^tB_k \circ {}^tR_2(\varphi_k) = \Psi|_{\Omega_2}\}. \tag{4.2.4}$$

<u>Proof.</u> From Theorem 1.2.1., 2.3.4. and Proposition 2.3.5. the equivalence of (ii) and (iii) follows. Assume that (i) is valid. Take any $G \in C^\infty(\tilde{\Omega}, \omega_1)$ and $h_k \in \overset{\circ}{C}{}^\infty(\tilde{\omega}_2)$, $k=1,2,\cdots,m$. Write $F = G|_\Omega$, $f_j \equiv 0$, $j=1,2,\cdots,l$, $g_k = h_k|\omega_{20}$, $k=1,2,\cdots,m$. Then the compatibility conditions in (i) are satisfied by F, (f_j), and (g_k). In fact we can take $\Phi \equiv 0$ for a solution of (4.2.1). Hence we have a solution $u \in C^\infty(\tilde{\Omega})$ of the equation (4.2.2) with the prescribed conditions.

Because P is normal of degree l with respect to ω_{10}, we have the following decomposition from (4.1.5.) with a family of differential operators A_l and $A_k^{(j)}$ in ω_{10}:

$$R_1 \circ D^j_{\nu_1} \circ P = A_l \circ R_1 \circ D^{j+l}_{\nu_1} + \sum_{k=0}^{j+l-1} A_k^{(j)} \circ R_1 \circ D^k_{\nu_1}, \quad j=0,1,2,\cdots. \tag{4.2.5}$$

From (4.2.2) and (4.2.5) we obtain, by induction on j,

$$R_1 \circ D^{j+l}_{\nu_1}(u) = 0, \quad j=0,1,2,\cdots.$$

Here we used the fact that A_l is bijective. Define a function v, which is equal to u in Ω and is zero outside Ω in Ω_1. Then v is a C^∞ function in $C^\infty(\tilde{\Omega}, \omega_1)$ and satisfies (4.2.3). Therefore (i) implies (ii).

Next we assume that (ii) is valid. Take $F \in C^\infty(\tilde{\Omega})$, $f_j \in C^\infty(\tilde{\omega}_1)$, $j=1,2,\cdots,l$ and $g_k \in C^\infty(\tilde{\omega}_2)$, $k=1,2,\cdots,m$, which satisfy the compatibility condition of (i). Define $f_{j+l} \in C^\infty(\tilde{\omega}_1)$, $j=1,2,\cdots$ by induction on j and

$$f_{j+l+1} = A_l^{-1}\Big(R_1 \circ D^j_{\nu_1}(F) - \sum_{k=0}^{j+l-1} A_k^{(j)}(f_{k+1})\Big). \qquad (4.2.6)$$

There exists a function $u_0 \in C^\infty(\tilde{\Omega})$ such that $R_1 \circ D^{j-1}_{\nu_1}(u_0) = f_j$, $j=1,2,\cdots$ (cf. Whitney [28]). By G we denote a function in Ω_1 which is equal to $F-P(u_0)$ in Ω and is zero outside Ω. Then G is function in $C^\infty(\tilde{\Omega}, \omega_1)$. In fact we have $R_1 \circ D^j_{\nu_1}(F-P(u_0)) = 0$, $j=0,1,2,\cdots$ from (4.2.5) and (4.2.6). Moreover $u_0|_\Omega$ satisfies the equation (4.2.1). Define a function h_k in ω_{25}, which is equal to $g_k - R_2 \circ B_k(u_0)$ in ω_{20} and is zero outside ω_{20}. Then it follows that $h_k \in \mathring{C}^\infty(\tilde{\omega}_2)$, $k=1,2,\cdots,m$. Hence there exists a solution $v \in C^\infty(\tilde{\Omega}, \omega_1)$ of the equation (4.2.3). Write $u = u_0 + v|_\Omega$. Then u is a solution of (4.2.2) and the statement (i) is proved. The proof of Theorem 4.2.1. is thus complete.

The compatibility condition in (i) does not depend on the selection of Φ. In fact from F and f_j, $j=1,2,\cdots,l$ we have constructed a family of infinite number of f_j, $j=1,2,\cdots$ at (4.2.6).

This family determines a C^{∞} jet in $\tilde{\omega}_1$. Then the compatibility condition is expressed that the above C^{∞} jet and g_k, $k=1,2,\cdots,m$ are compatible at every point of ω_{21}.

If, in addition, the boundary condition in ω_{20} is also of Cauchy type, then we can make a further reduction. But if P is not normal with respect to ω_{20}, then a difficult problem arises. The next theorem gives an answerin the simplest case.

<u>Theorem</u> 4.2.2. Suppose that $\tilde{\omega}_1$ and $\tilde{\omega}_2$ are regularly situated, that is, to every compact sets $K_j \subset \tilde{\omega}_j$, $j=1,2$ there exist positive constants C and α such that $d(x,K_2) \geqq C\cdot d(x, \omega_{21})^{\alpha}$, $x \in K_1$, where d is a metric in Ω_0 compatible with its topology (Lojasiewicz [16], and also [18]). We assume that P is normal of degree l with respect to ω_{10}, but semi-normal of degree m with respect to ω_{25}. Then we can write

$$R_2\circ D^k_{\nu_2}\circ P = B_m\circ R_2\circ D^{k-m}_{\nu_2} + \sum_{\mu=0}^{k+m-1} B^{(k)}_{\mu}\circ R_2\circ D^{\mu}_{\nu_2}, \quad k=0,1,2,\cdots \tag{4.2.7}$$

with a family of differential operators B_m and $B^{(k)}_{\mu}$ in ω_{25}. Moreover if the equation

$$P(v) = G \tag{4.2.8}$$

has a solution $v \in \overset{\circ}{C}{}^{\infty}(\tilde{\Omega})$ for every $G \in \overset{\circ}{C}{}^{\infty}(\tilde{\Omega})$, then the following two statements are equivalent:

(i) Let $F \in C^{\infty}(\tilde{\Omega})$, $f_j \in C^{\infty}(\tilde{\omega}_1)$, $j=1,2,\cdots,l$ and

$g_k \in C^\infty(\tilde{\omega}_2)$, $k=1,2,\cdots,m$ satisfy the compatibility condition of Theorem 4.2.1. (i), where B_k is replaced by $D_{\nu_2}^{k-1}$. Then the equation

$$\begin{cases} P(u) = F & \text{in } \Omega \\ R_1 \circ D_{\nu_1}^{j-1}(u) = f_j & \text{in } \omega_{10},\ j=1,2,\cdots,l \\ R_2 \circ D_{\nu_2}^{k-1}(u) = g_k & \text{in } \omega_{20},\ k=1,2,\cdots,m \end{cases} \tag{4.2.9}$$

has a solution $u \in C^\infty(\tilde{\Omega})$.

(ii) The equation

$$B_m(w) = g \tag{4.2.10}$$

has a solution $w \in \overset{\circ}{C}{}^\infty(\tilde{\omega}_2)$ for every $g \in \overset{\circ}{C}{}^\infty(\tilde{\omega}_2)$.

<u>Proof.</u> Suppose that (i) is true. Let $g \in \overset{\circ}{C}{}^\infty(\tilde{\omega}_2)$. There exists $F \in C^\infty(\tilde{\Omega})$ such that $R_2(F) = g$, $R_2 \circ D_{\nu_2}^k(F) = 0$, $k=1,2,\cdots$ and $R_1 \circ D_{\nu_1}^j(F) = 0$, $j=0,1,2,\cdots$ (Lojasiewicz [16], and also [18]). Then (i) implies the existence of $u \in C^\infty(\tilde{\Omega})$ such that $P(u) = F$ in Ω, $R_1^l \circ B^l(D_{\nu_1})(u) = 0$, and $R_2^m \circ B^m(D_{\nu_2})(u) = 0$. Define w to be the trace of $D_{\nu_2}^m(u)$ in ω_{20} and zero outside ω_{20} in ω_{25}. Then it follows that $w \in \overset{\circ}{C}{}^\infty(\tilde{\omega}_2)$ and (4.2.10) is satisfied.

Next assume that (ii) is valid. Let F, f_j, $j=1,2,\cdots,l$ and g_k, $k=1,2,\cdots,m$ satisfy the assumption of (i). Define f_{j+l}, $j=1,2,\cdots$ by (4.2.6). Choose a function Φ in $C^\infty(\tilde{\Omega})$ such that $R_1 \circ D_{\nu_1}^{j-1}(\Phi) = f_j$, $j=1,2,\cdots$, and set $h_k = R_2 \circ D_{\nu_2}^{k-1}(\Phi)$, $k=1,2,\cdots$.

Then the hypothesis implies that $\tilde{g}_k = g_k - h_k \in \overset{\circ}{C}{}^{\infty}(\tilde{\omega}_2)$, $k=1,2,\cdots,m$. Hence, by induction on k, the assumption (ii) implies the existence of solutions $\tilde{g}_{k+m} \in \overset{\circ}{C}{}^{\infty}(\tilde{\omega}_2)$, $k=1,2,\cdots$ of the equation

$$B_m(\tilde{g}_{k+m+1}) = R_2 \circ D_{\nu_2}^k(F) + \sum_{\mu=0}^{k+m-1} B_\mu^{(k)}(\tilde{g}_{\mu+1} + h_{\mu+1}) - B_m(h_{k+m+1}). \tag{4.2.11}$$

In fact the right hand side is equal to $R_2 \circ D_{\nu_2}^k(F - P(\Phi)) - \sum_{\mu=0}^{k+m-1} B_\mu^{(k)}(\tilde{g}_{\mu+1}) \in \overset{\circ}{C}{}^{\infty}(\tilde{\omega}_2)$. Now write $g_k = \tilde{g}_k|_{\omega_{20}} + h_k$, $k = m-1, m-2,\cdots$. Take a function Ψ in $C^\infty(\tilde{\Omega}, \omega_1)$ such that $R_2 \circ D_{\nu_2}^{k-1}(\Psi) = \tilde{g}_k|_{\omega_{20}}$, $k=1,2,\cdots$. Then we have

$$R_j \circ D_{\nu_j}^k (F - P(\Phi + \Psi)) = 0, \quad j=1,2 \text{ and } k=0,1,2,\cdots.$$

Let G be a function in Ω_0 defined to be equal to $F - P(\Phi + \Psi)$ in Ω and zero outside it. Since G belongs to $\overset{\circ}{C}{}^{\infty}(\tilde{\Omega})$, there exists $v \in \overset{\circ}{C}{}^{\infty}(\tilde{\Omega})$ such that (4.2.8) is valid. Set $u = \Phi + \Psi + v|_\Omega$. Then u is a solution of (4.2.9), and the theorem is proved.

Even if (4.2.7) does not hold, we can use the argument of the above proof. The essential part of the proof is the equation (4.2.11). Therefore the following theorem follows immediately.

<u>Theorem</u> 4.2.3. Assume that $\tilde{\omega}_1$ and $\tilde{\omega}_2$ are regularly situated, and P is normal of degree 1 with respect to ω_{10}. With respect to ω_{25} we suppose to have the decomposition

$$R_2 \circ D^k_{\nu_2} \circ P = B_{m_k} \circ R_2 \circ D^{m_k}_{\nu_2} + \sum_{\mu=0}^{m_k-1} B^{(k)}_{\mu} \circ R_2 \circ D^{\mu}_{\nu_2}, \quad k=0,1,2,\cdots \tag{4.2.12}$$

where m_k, $k=0,1,2,\cdots$ are assumed to be distinct integers larger than $m_0 = m$ or equal to zero. If the equation (4.2.8) has a solution $v \in \overset{\circ}{C}{}^{\infty}(\tilde{\Omega})$ for any $G \in \overset{\circ}{C}{}^{\infty}(\tilde{\Omega})$, and the equation

$$B_{m_k}(w) = g \tag{4.2.13}$$

has a solution $w \in \overset{\circ}{C}{}^{\infty}(\tilde{\omega}_2)$ for every $g \in \overset{\circ}{C}{}^{\infty}(\tilde{\omega}_2)$ and every $k=0,1,\cdots$, then the statement (i) of Theorem 4.2.2. is true.

If P is not normal with respect to both ω_{10} and ω_{20}, then we encounter with very difficult problems. We will investigate this situation in section 4.5., where the Goursat problem is our main concern.

Up to this point we have studied only C^{∞} solutions. We can study the regularity of solutions. But the situation becomes very complicated. As an example we state the next theorem:

<u>Theorem</u> 4.2.4. Let ω be equal to ω_{20}, and assume (4.2.7). Let p be a positive integer, s and t be real numbers such that $m \leqq s \leqq m+p$ and $M-m \leqq p(M-m-t)$. Set $\lambda = (p-1)(M-m) + p(1-t)$. Assume that the equation (4.2.8) has a solution $v \in \overset{0}{H}{}^{loc}_{(s)}(\tilde{\Omega})$ for any $G \in \overset{0}{H}{}^{loc}_{(p)}(\tilde{\Omega})$, and the equation (4.2.10) has a solution $w \in H^{loc}_{(r)}(\omega)$ for any $g \in H^{loc}_{(r-t)}(\omega)$, if $r \geqq \frac{1}{2}$. Then for every $F \in H^{loc}_{(\lambda)}(\tilde{\Omega})$ and every $g_k \in H^{loc}_{(\lambda+M-k-\frac{1}{2})}(\omega)$, $k=1,2,\cdots,m$

there exists a solution u in $H^{loc}_{(s)}(\tilde{\Omega})$ such that $P(u)=F$ in Ω and $R_2\circ D^k_{\nu_2}(u)=g_k$ in ω, $k=1,2,\cdots,m$.

<u>Proof</u>. From (4.2.7) the order of B_m is smaller than $M-m$ and the order of $B^{(k)}_{\mu}$ is smaller than $M-\max(0,\mu-k)$. Let $F\in H^{loc}_{(\lambda)}(\tilde{\Omega})$ and $g_k\in H^{loc}_{(\lambda+M-k-\frac{1}{2})}(\omega)$, $k=1,2,\cdots,m$, then the equation (4.2.11) has solutions $g_{k+m}\in H^{loc}_{(\lambda+t-k(M-m-t+1)-\frac{1}{2})}(\omega)$, $k=1,2,\cdots$, where we can set $\tilde{g}_k=g_k$. Then it follows that $u\in H^{loc}_{(s)}(\tilde{\Omega})$. We leave the details for the reader.

Now we have shown that in order to solve boundary problems (4.2.9) we should solve differential equations (4.2.8) and (4.2.10). This kind of equation will be treated in the subsequent two sections. Before going to the next section we comment on the extension of solutions.

Let $\mathcal{F}(\Omega_0)$ and $\mathcal{G}(\Omega_0)$ be suitable subspaces of $\mathcal{D}'(\Omega_0)$, and P a differential operator of $\mathcal{F}(\Omega_0)$ into $\mathcal{G}(\Omega_0)$. Then we obtain the next proposition, whose proof may be left for the reader.

<u>Proposition</u> 4.2.5. The equation

$$P(u)=f$$

has a solution $u\in\mathcal{F}(\tilde{\Omega},\omega_1)$ for any $f\in\mathcal{G}(\tilde{\Omega},\omega_1)$ if and only if $v\in\mathcal{F}(\tilde{\Omega}_1\backslash\Omega)$, $g\in\mathcal{G}(\tilde{\Omega}_1)$, and $P(v)=g$ in $\Omega_1\backslash\tilde{\Omega}$ implies the existence of $v_0\in\mathcal{F}(\tilde{\Omega}_1)$ such that $v=v_0$ in $\Omega_1\backslash\tilde{\Omega}$ and $P(v_0)=g$ in Ω_1.

§4.3. Differential equations in $\mathcal{F}(\tilde{\Omega}, \omega_1)$.

In this and the next sections ω_{10} and ω_{20} are not necessarily assumed to be smooth. From Theorem 1.2.1. and the theorems in section 2.3. or 2.4. we obtain the following two lemmas:

Lemma 4.3.1. The differential equation

$$P(u) = f \tag{4.3.1}$$

has a solution $u \in C^\infty(\tilde{\Omega}, \omega_1)$ for every $f \in C^\infty(\tilde{\Omega}, \omega_1)$ if and only if the following two conditions (1) and (2) are valid. Moreover (2) is equivalent to (3).

(1) To every compact set $K \subset \tilde{\Omega}$ and every real number s there exists another compact set $K' \subset \tilde{\Omega}$ such that $\varphi \in \mathcal{E}'(\tilde{\Omega}, \omega_2)$, ${}^tP(\varphi) \in H^c_{(s)}(\tilde{\Omega}, \omega_2)$, and $\operatorname{supp} {}^tP(\varphi) \subset K$ implies $\operatorname{supp} \varphi \subset K'$.

(2) To every compact set $K \subset \tilde{\Omega}$ and every $f \in C^\infty(\tilde{\Omega}, \omega_1)$ there exists $u \in C^\infty(\tilde{\Omega}, \omega_1)$ such that (4.3.1) holds in K°.

(3) To every compact set $K \subset \tilde{\Omega}$ and every real number $s \in \mathbb{R}$ there exist another real number t and a positive constant C such that $\varphi \in \mathcal{E}'(\tilde{\Omega}, \omega_2)$ and $\operatorname{supp} \varphi \subset K$ implies the following estimate:

$$\begin{aligned} &\inf\{\|\tilde{\varphi}\|_{(t)};\ \tilde{\varphi} \in H^c_{(t)}(\Omega_0) \text{ and } \varphi = \tilde{\varphi}|\Omega_2\} \\ &\leqq C \cdot \inf\{\|\psi\|_{(s)};\ \psi \in H^c_{(s)}(\Omega_0) \text{ and } {}^tP(\varphi) = \psi|\Omega_2\}. \end{aligned} \tag{4.3.2}$$

Lemma 4.3.2. Let $s, t \in \mathbb{R}$. Suppose that ω has the curve segment property at every its point and we can choose Ω_2 such that its boundary in Ω_0 has the same property. Then the equation

(4.3.1) has a solution $u \in H^{loc}_{(s)}(\tilde{\Omega}, \omega_1)$ for any $f \in H^{loc}_{(t)}(\tilde{\Omega}, \omega_1)$ if and only if the following two conditions (1) and (2) are true. Moreover (2) and (3) are equivalent.

(1) To every compact set $K \subset \tilde{\Omega}$ there exists another compact set $K' \subset \tilde{\Omega}$ such that $\varphi \in H^c_{(-t)}(\tilde{\Omega}, \omega_2)$ and supp ${}^tP(\varphi) \subset K$ implies supp $\varphi \subset K'$.

(2) To every compact set $K \subset \tilde{\Omega}_1$ and every $f \in H^{loc}_{(t)}(\tilde{\Omega}, \omega_1)$ there exists $u \in H^{loc}_{(s)}(\tilde{\Omega}, \omega_1)$ such that (4.3.1) holds in the interior of K.

(3) To every compact set $K \subset \tilde{\Omega}$ there exists a positive constant C such that $\varphi \in H^c_{(-t)}(\tilde{\Omega}, \omega_2)$ and supp $\varphi \subset K$ implies

$$\begin{aligned} &\inf\{\|\tilde{\varphi}\|_{(-t)};\ \tilde{\varphi} \in H^c_{(-t)}(\Omega_0) \text{ and } \varphi = \tilde{\varphi}|_{\Omega_2}\} \\ &\leqq C \cdot \inf\{\|\psi\|_{(-s)};\ \psi \in H^c_{(-s)}(\Omega_0) \text{ and } {}^tP(\varphi) = \psi|_{\Omega_2}\}. \end{aligned} \tag{4.3.2'}$$

Definition 4.3.1. The pair $(\tilde{\Omega}, \omega_1)$ is said to be P-convex (with respect to support) if and only if to every compact set $K \subset \tilde{\Omega}$ there exists another compact set $K' \subset \tilde{\Omega}$, which can be chosen to be empty if K is empty, such that $\varphi \in \mathcal{E}'(\tilde{\Omega}, \omega_2)$ and supp ${}^tP(\varphi) \subset K$ implies supp $\varphi \subset K'$.

If $(\tilde{\Omega}, \omega_1)$ is P-convex, then the conditions (1) of Lemma 4.3.1. and 4.3.2. are valid. If the boundary ω is void, this definition is essentially the same as the well-known P-convexity condition [13].

If the equation (4.3.1) has a solution $u \in C^\infty(\tilde{\Omega}, \omega_1)$

for any $f \in C^\infty(\tilde{\Omega}, \omega_1)$, then $\varphi \in \mathcal{E}'(\tilde{\Omega}, \omega_2)$ and ${}^tP(\varphi) = 0$ implies $\varphi = 0$. Hence the above difinition is reasonable.

We can obtain the following statement using the method of proof in sections 3.2. and 3.4.

Remark 4.3.3. Suppose that Ω_0 is a real analytic manifold and P is an elliptic operator with real analytic coefficients in it. Moreover assume that $\tilde{\Omega}$ has no compact connected component which is open in Ω_1, and to every relatively compact open subset U of Ω_0, the union of all compact components of $(\Omega \cup \omega_1) \setminus U$ is also relatively compact. Then it follows that $(\tilde{\Omega}, \omega_1)$ is P-convex.

Next consider strictly hyperbolic differential operators. Under the geometric conditions due to Leray [15], we can say that $(\tilde{\Omega}, \omega)$ is P-convex. Remark that the estimation (4.3.2) was investigated by Hörmander [8], and was applied to prove semi-global existence of solutions. We can combine their results and our previous theorems to obtain some theorems, but we leave it as an exercise for the reader.

§4.4. Evolution operators with constant coefficients.

In this section we consider linear differential equations with constant coefficients in $\mathbb{R}^n$, that is, $P = P(D)$ has the symbol $P(\xi)$, which is a polynomial of degree M. Then we obtain

Theorem 4.4.1. Assume the existence of a closed cone Γ with its vertex at the origin in $\mathbb{R}^n$ such that P(D) has a regular

For every compact set $K \subset \tilde{\Omega}$ there exists $æ \in \mathbb{R}^n$ such that $K + \varepsilon \cdot æ \subset \tilde{\Omega}$, $0 < \varepsilon < 1$.

fundamental solution E with support in Γ, i.e. $E \in \mathcal{B}^{loc}_{\infty,\tilde{P}}(\mathbb{R}^n)$, $\operatorname{supp} E \subset \Gamma$ and $P(D)E = \delta$, and moreover the following conditions hold: There exists a closed neighbourhood U of $\tilde{\Omega}$ in $\tilde{\Omega}_2$ and for every $x \in \omega_1$ there exists a neighbourhood V of x such that $U + \Gamma = \{y+z;\ y \in U \text{ and } z \in \Gamma\}$ does not meet with $V \setminus U$. Let $1 \leq p \leq \infty$ and $k \in \mathcal{K}(\mathbb{R}^n)$. Then the following six statements (i) to (vi) are equivalent. If in addition ω_{10} is of C^∞ class and P(D) is normal of degree m with respect to ω_{10}, then all seven statements (i) to (vii) are equivalent.

(i) The pair $(\tilde{\Omega}, \omega_1)$ is P(D)-convex.

(ii) The equation

$$P(D)u = f \tag{4.4.1}$$

has a solution $u \in C^\infty(\tilde{\Omega}, \omega_1)$ for every $f \in C^\infty(\tilde{\Omega}, \omega_1)$.

(iii) The equation (4.4.1) has a solution $u \in \mathcal{B}^{loc}_{p,k\tilde{P}}(\tilde{\Omega}, \omega_1)$ for every $f \in \mathcal{B}^{loc}_{p,k}(\tilde{\Omega}, \omega_1)$.

(iv) The equation (4.4.1) has a solution $u \in \mathcal{D}'(\tilde{\Omega}, \omega_1)$ for every $f \in C^\infty(\tilde{\Omega}, \omega_1)$.

(v) If $f \in C^\infty(\tilde{\Omega}_1)$, $u \in C^\infty(\tilde{\Omega}_1 \setminus \Omega)$, and $P(D)u = f$ in $\Omega_1 \setminus \tilde{\Omega}$, then there exists $\tilde{u} \in C^\infty(\tilde{\Omega}_1)$ such that $u = \tilde{u}|_{\Omega_1 \setminus \tilde{\Omega}}$ and $P(D)\tilde{u} = f$ in Ω_1.

(vi) If $f \in \mathcal{B}^{loc}_{p,k}(\tilde{\Omega}_1)$, $u \in \mathcal{B}^{loc}_{p,k\tilde{P}}(\tilde{\Omega}_1 \setminus \Omega)$, and $P(D)u = f$ in $\Omega_1 \setminus \tilde{\Omega}$, then there exists $\tilde{u} \in \mathcal{B}^{loc}_{p,k\tilde{P}}(\tilde{\Omega}_1)$ such that $u = \tilde{u}|_{\Omega_1 \setminus \tilde{\Omega}}$ and $P(D)\tilde{u} = f$ in Ω_1.

(vii) The equation

$$\begin{cases} P(D)u = F \quad \text{in } \Omega \\ R_1 \circ D_{\nu_1}^{j-1}(u) = f_j \quad \text{in } \omega_{10},\ j=1,2,\cdots,m, \end{cases} \tag{4.4.2}$$

has a solution $u \in C^\infty(\tilde{\Omega})$ for every $F \in C^\infty(\tilde{\Omega})$ and $f_j \in C^\infty(\tilde{\omega}_1)$, $j=1, 2,\cdots,m$.

<u>Proof.</u> The equivalence of (ii), (v), (vii) and that of (iii) and (vi) follows from the results of the previous section. It is trivial that (ii) implies (iv). Theorem 1.2.1. and 2.5.4. impliy an analog of Lemma 4.3.1. for $\mathcal{B}_{p,k}^{loc}$.

Let $f \in \mathcal{B}_{p,k}^{loc}(\tilde{\Omega}, \omega_1)$ and K be a compact set contained in $\tilde{\Omega}$. There exists $\tilde{f} \in \mathcal{B}_{p,k}^{c}(\Omega_o)$ which coincide with f in a neighbourhood V of $K \cap \Omega$ in Ω_1, and $\operatorname{supp} \tilde{f} \subset \overline{U}$. Set $u = E * \tilde{f} / \Omega_1$. Then we have $u \in \mathcal{B}_{p,k\tilde{P}}^{loc}(\tilde{\Omega}, \omega_1)$ and $P(D)u = f$ in V. If f is a C^∞ function, then u is also of C^∞ class. Then we have proved the semi-global solvability condition (2) in Lemm 4.3.1. and the analog for $\mathcal{B}_{p,k}^{loc}$.

Next prove that (i) is equivalent with the following condition:

(i') To every compact set $K \subset \tilde{\Omega}$ there exists another compact set $K' \subset \tilde{\Omega}$, which can be chosen to be void if K is void, such that $\varphi \in C_o^\infty(\tilde{\Omega}, \omega_2)$ and $\operatorname{supp} {}^tP(\varphi) \subset K$ implies $\operatorname{supp} \varphi \subset K'$.

In fact it is enough to make an approximation using translations and convolutions. Take a compact set $K \subset \tilde{\Omega}$. From the hypothesis of the theorem there exists a vector $æ \in \mathbb{R}^n$ such that $K+\varepsilon \text{æ} \subset \tilde{\Omega}$, $0<\varepsilon<1$. If we use translations in the direction æ and use convolution such as in the proof of Proposition 2.5.5., then the statement follows. Details may be left to the reader.

Finally the implication of (i) from (iv) can be proved using the same method as in the proof of Theorem 3.5.4., [8]. Then the proof is complete.

<u>Theorem</u> 4.4.2. Let $1 \leqq p < \infty$ and $k \in \mathcal{K}(\mathbb{R}^n)$. If $\tilde{\Omega}$ is contained in the closed half space H and ω_{10} is non-void and is contained in its boundary. Then the following statement (i″) and the five statements (ii) to (vi) in Theorem 4.4.1. are equivalent.

(i″) The pair $(\tilde{\Omega}, \omega_1)$ is P(D)-convex and P(D) is evolutional with respect to H, that is, there exists a fundamental solution in $\overset{\circ}{\mathcal{B}}{}^{loc}_{\infty,\check{p}}(H)$.

<u>Proof.</u> We have to prove that (ii) or (iii) implies (i″). P(D)-convexity follows from Lemma 4.3.1. or its analog for $\mathcal{B}^{loc}_{p,k}$. Now assume that (ii) is true. Let $x_0 \in \omega_{10}$ and $s \in \mathbb{R}$. Choose a compact neighbourhood K of x_0 and a compact set $K_0 \subset \Omega_1$ such that K is contained in the interior of K_0. Then from Lemma 4.3.1. and Proposition 2.5.5. there exist a real number t and a positive constant C such that $\varphi \in C_0^\infty(\Omega_1)$ and $\Omega \cap \operatorname{supp} \varphi \subset K$ implies

$$\begin{aligned} &\inf\{\|\tilde{\varphi}\|_{(t)};\ \tilde{\varphi} \in \overset{\circ}{C}{}_0^\infty(K_0) \text{ and } \tilde{\varphi} = \varphi \text{ in } \Omega\} \\ &\leqq C \cdot \inf\{\|\psi\|_{(s)};\ \psi \in \overset{\circ}{C}{}_0^\infty(K_0) \text{ and } \psi = P(-D)\varphi \text{ in } \Omega\}. \end{aligned} \tag{4.4.3}$$

Now take two positive integers μ and ν such that $\frac{n}{2} < t+2\nu$ and $s \leqq 2\mu$. Then from (4.4.3) we obtain with another positive constant C',

$$|\varphi(x_0)| \leqq$$
$$\leqq C' \cdot \inf\{\|\tilde{\varphi}\|_{(t+2\nu)};\ \tilde{\varphi} \in \overset{o}{C}{}_0^{\infty}(K_0) \text{ and } \tilde{\varphi} = \varphi \text{ in } \Omega\}$$
$$\leqq C' \cdot \inf\{\|\tilde{\varphi}\|_{(t)};\ \tilde{\varphi} \in \overset{o}{C}{}_0^{\infty}(K_0) \text{ and } \tilde{\varphi} = (1-\Delta)^{\nu}\varphi \text{ in } \Omega\}$$
$$\leqq C \cdot C' \cdot \inf\{\|\psi\|_{(s)};\ \psi \in \overset{o}{C}{}_0^{\infty}(K_0) \text{ and } \psi = P(-D)(1-\Delta)^{\nu}\varphi \text{ in } \Omega\} \tag{4.4.4}$$
$$\leqq C \cdot C' \cdot \inf\{\|\psi\|_{(2\mu)};\ \psi \in \overset{o}{C}{}_0^{\infty}(K_0) \text{ and } \psi = P(-D)(1-\Delta)^{\nu}\varphi \text{ in } \Omega\}$$

for any $\varphi \in C_0^{\infty}(\mathbb{R}^n)$ such that $\operatorname{supp}\varphi \subset K$. Write $\varphi' = (1-\Delta)^{\nu}\varphi$.

From Proposition 2.5.5. and (4.4.4) we have with another positive constant C,

$$|\varphi(x_0)| \leqq$$
$$\leqq C \cdot \inf\{\|\psi\|_{(2\mu)};\ \psi \in \overset{o}{H}{}^c_{(2\mu)}(K_0) \text{ and } \psi = P(-D)\varphi' \text{ in } \Omega\}.$$

Since $\overset{o}{C}{}_0^{2\mu}(K_0)$ is continuously embedded in $H_{(2\mu)}$, we obtain with another positive constant C,

$$|\varphi(x_0)| \leqq$$
$$\leqq C \cdot \inf\Big\{\sum_{|\alpha| \leqq 2\mu} \cdot \sup_{x \in K_0} |D^{\alpha}\psi(x)|;\ \psi \in \overset{o}{C}{}_0^{2\mu}(K_0) \text{ and } \psi = P(-D)\varphi' \text{ in } \Omega\Big\}.$$

We finally obtain with another positive constant C,

$$|\varphi(x_0)| \leqq C \sum_{|\alpha| \leqq 4\mu} \cdot \sup_{x \in H} |D^{\alpha}(1-\Delta)^{\nu} P(-D)\varphi(x)|, \tag{4.4.5}$$

for all $\varphi \in C_0^{\infty}(\mathbb{R}^n)$ such that $\operatorname{supp}\varphi \subset K$.

In fact let ρ be a C^∞ function such that supp $\rho \subset K_0$ and $\rho = 1$ in a neighbourhood of K. Define the function ψ_0 by

$$\psi_0(x) = \begin{cases} P(-D)\varphi'(x), \text{ if } x \in H. \\ \rho(x) \sum_{j \leq 2\mu} (\frac{\partial}{\partial \vartheta})^j P(-D)\varphi'(x-\beta\cdot\vartheta)(\langle x,\vartheta\rangle - \alpha)^j \\ \qquad \text{if } x \notin H. \end{cases}$$

where ϑ is the inner normal to H and $\frac{\partial}{\partial \vartheta}$ is the normal derivative and $\beta = \frac{\langle x,\vartheta\rangle - \alpha}{\langle \vartheta,\vartheta\rangle}$.

This implies that $\psi_0 \in \overset{\circ}{C}{}^{2\mu}_0(K_0)$, $\psi_0 = P(-D)\varphi'$ in H, and with a positive constant C',

$$\sum_{|\alpha| \leq 2\mu} \sup_{x \in K_0} |D^\alpha \psi_0(x)| \leq C' \sum_{|\alpha| \leq 4\mu} \sup_{x \in H} |D^\alpha P(-D)\varphi'(x)|.$$

Hence (4.4.5) follows.

Therefore (4.4.5) and a result due to Hörmander [11] implies that P(D) is evolutional with respect to H. If we assume (iii), then we can use a similar argument to prove (i''). This completes the proof of Theorem 4.4.2.

Corollary 4.4.3. Assume that $\tilde{\Omega}$ is contained in a closed half space $H \subset \mathbb{R}^n$ with the inner normal ϑ and ω_{10} $(= \omega)$ is contained in its boundary. Let $B_j(\xi)$, $j=1,2,\cdots,m$ be polynomials in $\xi \in \mathbb{R}^n$. We consider the equation

$$\begin{cases} P(D)u = F \quad \text{in } \Omega \\ R_1 \circ B_j(D)u = f_j \quad \text{in } \omega_{10}, \ j=1,2,\cdots,m. \end{cases} \tag{4.4.6}$$

If P(D) is normal with respect to ω_{10} and evolutional with respect to H, the pair $(\tilde{\Omega}, \omega_1)$ is P(D)-convex, and ω_{10} is convex, then the equation (4.4.6) has a solution $u \in C^\infty(\tilde{\Omega})$ if and only if $F \in C^\infty(\tilde{\Omega})$ and $f_j \in C^\infty(\tilde{\omega}_1)$, $j=1,2,\cdots,m$ satisfy the following compatibility condition:

Let $Q(\xi)$, $Q_j(\xi)$, $j=1,2,\cdots,m$ be polynomials in $\xi \in \mathbb{R}^n$ such that $Q_j(\xi + \tau\nu)$ is independent of $\tau \in \mathbb{C}$ and

$$Q(\xi)\cdot P(\xi) + \sum_{j=1}^{m} Q_j(\xi)\cdot B_j(\xi) \equiv 0.$$

Then we have

$$R_1 \circ Q(D)(F) + \sum_{j=1}^{m} Q_j(D)\cdot f_j = 0 \quad \text{in } \omega_{10}.$$

<u>Proof.</u> Since P(D) is normal (of degree 1) with respect to ω_{10}, we have the decomposition similar as (3.1.3). From the hypothesis $R_1^1 \circ B^1(D_{\nu_1})$ is surjective. Then it is enough to apply the result of Ehrenpreis [5], Theorem 6.1. to boundary system B_{ω_1}, which is a system of differential equations with constant coefficients. Details may be left for the reader.

If the induced boundary system B_{ω_1} is determined, we can obtain a more refined result. For the sake of simplicity we assume that $\tilde{\Omega}$ is contained in $\overline{\mathbb{R}^n_+}$ and $\omega_{10} = \tilde{\Omega} \cap \mathbb{R}^n_0$. Moreover assume that $P(\xi) = \sum_{j=0}^{m} a_j(\xi')\xi_n^j$ and $B_j(\xi) = \tilde{B}_j(\xi)\cdot P(\xi) + \sum_{k=0}^{m-1} b_{jk}(\xi')\xi_n^k$, $j=1,2,\cdots,m$. Set $B_{\omega_1}(\xi') = (b_{jk}(\xi'))$, which is an $m \times m$ matrix of polynomials in $\xi' \in \mathbb{R}^{n-1}$. Then the next theorem holds:

<u>Corollary</u> 4.4.4. Assume that P(D) is evolutional with respect to H and $(\tilde{\Omega}, \omega_1)$ is P(D)-convex. Moreover assume that $\det B_{\omega_1}(\xi') \not\equiv 0$, and $\tilde{\omega}_1$ is $a_m(D')$-convex. Then the equation (4.4.6) has a solution $u \in C^\infty(\tilde{\Omega})$ for every $F \in C^\infty(\tilde{\Omega})$ and $f_j \in C^\infty(\tilde{\omega}_1)$, $j=1,2,\cdots,m$ if and only if ω_{10} is $B_{\omega_1}(D')$-convex.

<u>Proof.</u> Theorem 4.2.2. implies that the equation

$$\begin{cases} P(D)(u) = F & \text{in } \Omega \\ R_1 \circ D_n^{j-1}(u) = f_j & \text{in } \omega_{10},\ j=1,2,\cdots,m \end{cases} \tag{4.4.7}$$

has a solution $u \in C^\infty(\tilde{\Omega})$ for every $F \in C^\infty(\tilde{\Omega})$ and $f_j \in C^\infty(\tilde{\omega}_1)$, $j=1, 2,\cdots,m$, if and only if $\tilde{\omega}_1$ is $a_m(D')$-convex. Then we have to obtain a condition for $B_{\omega_1}(D')$ to be surjective. Hence the theorem follows.

Next consider the geometric meaning of P(D)-convexity. We can obtain some delicate conditions using results on the uniqueness of solutions (see Hörmander [12] and the references therein). Here we give a necessary and sufficient condition for hyperbolic equations:

<u>Theorem</u> 4.4.5. Let $\tilde{\Omega}$ be connected and $\omega = \omega_{10}$ be non-empty. Let $\vartheta \in \mathbb{R}^n$ satisfies $x + \varepsilon \cdot \vartheta \in \Omega$, $0 < \varepsilon < \varepsilon_x$ for some $\varepsilon_x > 0$ and $x \in \omega$. Suppose that P(D) is hyperbolic with respect to a vector ϑ and let $\Gamma^* = \Gamma^*(P, \vartheta)$ be its forward propagation cone. Then $(\tilde{\Omega}, \omega)$ is P(D)-convex if and only if to every point $x \in \Omega$ the union of $x - \Gamma^*$ and ω is compact and in addition $x - \Gamma^*$ does not meet with $\partial\Omega \setminus \omega$, where $\partial\Omega$ is the boundary of Ω in $\mathbb{R}^n$.

Proof. First assume that the above geometric condition is valid. Let K be a compact set contained in $\tilde{\Omega}$. By U we denote the union of $x+\Gamma^*$ where $x\in\tilde{\Omega}$ and $x+\Gamma^*$ does not meet with K. Let K' be the closure of the complement of U in $\tilde{\Omega}$. From the hypothesis K' becomes compact. In addition $\varphi\in\mathcal{E}'(\tilde{\Omega},\omega_2)$ and $\operatorname{supp} P(-D)\varphi\subset K$ implies $\operatorname{supp}\varphi\subset K'$. Then $(\tilde{\Omega},\omega_1)$ is P(D)-convex.

Now let E be the fundamental solution of P(D) with its support in Γ^*. From Atiyah-Bott-Gärding [1], Theorem 8.9. there exists an integer k such that $\operatorname{supp}\Phi=-\Gamma^*$, if we set $\Phi=\underbrace{(-E)*(-E)*\cdots*(-E)}_{k}$. Moreover we have $\operatorname{supp} P(D)\Phi\subset\Gamma^*$, since $P(D)\Phi=P(D)^k\Phi*\underbrace{E*\cdots*E}_{k-1}=\delta*\underbrace{E*\cdots*E}_{k-1}$.

If there exists $x\in\Omega$ such that $x-\Gamma^*$ meets with $\partial\Omega\setminus\omega$, then translating this cone we can find $y\in\partial\Omega\setminus\omega$ and $y_0\in\Omega$ such that $y\in y_0-\Gamma^*$ and in a neighbourhood U of y, the intersection of $y_0-\Gamma^*$ and $U\setminus\Omega_0$ is void with a suitable Ω_0. Then using Φ we can prove that $(\tilde{\Omega},\omega_1)$ is not P(D)-convex by the standard argument ([8, 13]). This completes the proof.

Example 4.4.6. Let $n=3$ and $P=D_1^2+D_2^2-D_3^2+\sum_{|\alpha|\leq 1}a_\alpha(x)\cdot D^\alpha$ with $a_\alpha\in C^\infty(\mathbb{R}^3)$. Set $\Gamma^*=\{x\in\mathbb{R}^3;\ x_3\geqq 0 \text{ and } x_3^2\geqq x_1^2+x_2^2\}$. Let $\tilde{\Omega}=\Omega\cup\omega$ be a subset of $\Gamma^*\setminus\{0\}$ such that $\omega=\omega_1\cup\omega_2$, $\omega_2\subset\partial\Gamma^*$, and ω_1 is a space-like C^∞ surface. Let ν be a real C^∞ vector field which is not tangential to ω_1. Suppose that to every $x\in\Omega$ the intersection of $x-\Gamma^*$ and $\tilde{\Omega}$ is compact. Then the equation

$$\begin{cases} P(u) = F & \text{in } \Omega \\ R_1(u) = f_1 & \text{in } \omega_1 \\ R_1 \circ D_\nu(u) = f_2 & \text{in } \omega_1 \\ R_2(u) = g & \text{in } \omega_2 \end{cases}$$

has a unique solution $u \in C^\infty(\tilde{\Omega})$, if $F \in C^\infty(\tilde{\Omega})$, $f_1, f_2 \in C^\infty(\tilde{\omega}_1)$ and $g \in C^\infty(\tilde{\omega}_2)$ satisfy the following compatibility condition:

$$R_{21}(f_1) = R_{21}(g) \quad \text{in } \omega_{21}.$$

In fact we can write P using the polar coordinate as follows:

$$P = \frac{-1+2\cos^2\theta}{r^2} D_\theta^2 + \left\{\left(\frac{1}{r}+\frac{1}{\sqrt{2}}\right)D_r + i\frac{1-2r}{r^2}\right\} D_\theta - \cdots.$$

Then we obtain

$$A_{2,0} = 0, \quad A_{2,1} = -\frac{2}{r^2}, \cdots, \ldots$$

$$A_{1,0} = \left(\frac{1}{r}+\frac{1}{\sqrt{2}}\right)D_r + i\frac{1-2r}{r^2}, \ldots$$

Hence it follows that $m = 1$ and $A_{j+1}^{(j)} = A_{1,0} + jA_{2,1} = \left(\frac{1}{r}+\frac{1}{\sqrt{2}}\right)D_r - \cdots$.

Therefore we can apply Theorem 4.2.3. and a result due to Leray [15].

§4.5. The Goursat problem.

In Theorem 4.2.2., 4.2.3. and 4.2.4. we made an assumption that P is normal with respect to ω_{10}. In this section we drop this hypothesis. We use the same notation as in section 4.2.

Theorem 4.5.1. Assume that P is semi-normal of degree

1 and m with respect to both ω_{10} and ω_{20} respectively. Then we can write

$$
\begin{aligned}
R_1\circ D^j_{\nu_1}\circ P &= A_l\circ R_1\circ D^{j+l}_{\nu_1}+\sum_{k=0}^{j+l-1} A^{(j)}_k\circ R_1\circ D^k_{\nu_1} \\
R_2\circ D^j_{\nu_2}\circ P &= B_m\circ R_2\circ D^{j+m}_{\nu_2}+\sum_{k=0}^{j+m-1} B^{(j)}_k\circ R_2\circ D^k_{\nu_2},\quad j=0,1,2,\cdots.
\end{aligned}
\tag{4.5.1}
$$

In addition assume that $\tilde{\omega}_1$ and $\tilde{\omega}_2$ are regularly situated and ω_{21} is an n-2 dimensional C^∞ submanifold of Ω_0. Trace operator to ω_{21} is denoted by R_{21}. Two vector fields ν_1 and ν_2 are assumed to be tangential to $\tilde{\omega}_2$ and $\tilde{\omega}_1$ respectively at any point of ω_{21}. Assume that A_l is normal of degree m with respect to ω_{21}, and B_m is normal of degree l with respect to ω_{21}. Finally assume that the equation

$$P(u)=F \quad \text{in } \Omega \tag{4.5.2}$$

has a solution $u\in \mathring{C}^\infty(\tilde{\Omega})$ for every $F\in \mathring{C}^\infty(\tilde{\Omega})$, the equation

$$A_l(v)=f \quad \text{in } \omega_{10} \tag{4.5.3}$$

has a solution $v\in \mathring{C}^\infty(\tilde{\omega}_1)$ for every $f\in \mathring{C}^\infty(\tilde{\omega}_1)$, and the equation

$$B_m(w)=g \quad \text{in } \omega_{20} \tag{4.5.4}$$

has a solution $w\in \mathring{C}^\infty(\tilde{\omega}_2)$ for every $g\in \mathring{C}^\infty(\tilde{\omega}_2)$. Then the equation

$$
\begin{cases}
P(u)=F & \text{in } \Omega \\
R_1\circ D^{j-1}_{\nu_1}(u)=f_j & \text{in } \omega_{10},\ j=1,2,\cdots,l \\
R_2\circ D^{k-1}_{\nu_2}(u)=g_k & \text{in } \omega_{20},\ k=1,2,\cdots,m
\end{cases}
\tag{4.5.5}
$$

has a solution $u \in C^\infty(\tilde{\Omega})$ for every $F \in C^\infty(\tilde{\Omega})$, $f_j \in C^\infty(\tilde{\omega}_1)$, $j=1, 2,\cdots,l$ and $g_k \in C^\infty(\tilde{\omega}_2)$, $k=1,2,\cdots,m$ if and only if the following compatibility conditions are satisfied:

$$R_{21}\circ D^{k-1}_{\nu_2}(f_j) = R_{21}\circ D^{j-1}_{\nu_1}(g_k) \quad \text{in } \omega_{21} \qquad (4.5.6)$$

$$j=1,2,\cdots,l \text{ and } k=1,2,\cdots,m.$$

Proof. First we solve the following two boundary problems:

$$\begin{cases} A_l(f_{l+1}) = R_1(F) - \sum_{j=0}^{l-1} A_j^{(o)}(f_{j+1}) & \text{in } \omega_{10} \\ R_{21}(f_{l+1}) = R_{21}\circ D^l_{\nu_1}(g_1) & \text{in } \omega_{21} \\ R_{21}\circ D_{\nu_2}(f_{l+1}) = R_{21}\circ D^l_{\nu_1}(g_2) & \text{in } \omega_{21} \\ \vdots & \\ R_{21}\circ D^{m-1}_{\nu_2}(f_{l+1}) = R_{21}\circ D^l_{\nu_1}(g_m) & \text{in } \omega_{21} \end{cases} \qquad (4.5.7)$$

and

$$\begin{cases} B_m(g_{m+1}) = R_2(F) - \sum_{k=0}^{m-1} B_k^{(o)}(g_{k+1}) & \text{in } \omega_{20} \\ R_{21}(g_{m+1}) = R_{21}\circ D^m_{\nu_2}(f_1) & \text{in } \omega_{21} \\ R_{21}\circ D_{\nu_1}(g_{m+1}) = R_{21}\circ D^m_{\nu_2}(f_2) & \text{in } \omega_{21} \\ \vdots & \\ R_{21}\circ D^{l-1}_{\nu_1}(g_{m+1}) = R_{21}\circ D^m_{\nu_2}(f_l) & \text{in } \omega_{21} \end{cases} \qquad (4.5.8)$$

These problems have solutions $f_{l+1} \in C^\infty(\tilde{\omega}_1)$ and $g_{m+1} \in C^\infty(\tilde{\omega}_2)$, because A_l is normal of degree m with respect to ω_{21} and (4.5.3) is always solvable and similar for B_m. These f_{l+1} and g_{m+1} are compatible in the sense that $D^m_{\nu_2}(f_{l+1}) = D^l_{\nu_1}(g_{m+1})$ in ω_{21}.

Next we solve two equations for f_{l+2} and g_{m+2}, and then these two functions are compatible in ω_{21}. Repeating this process we obtain $f_j \in C^\infty(\tilde{\omega}_1)$, $j=1,2,\cdots$ and $g_k \in C^\infty(\tilde{\omega}_2)$, $k=1,2,\cdots$ such that

$$R_{21} \circ D^{k-1}_{\nu_2}(f_j) = R_{21} \circ D^{j-1}_{\nu_1}(g_k), \quad j,k=1,2,\cdots \tag{4.5.9}$$

and satisfies the equations

$$\begin{cases} R_1 \circ D^j_{\nu_1}(F) = A_l(f_{j+l+1}) + \sum_{k=0}^{j+l-1} A^{(j)}_k(f_{k+1}) \\ R_2 \circ D^k_{\nu_2}(F) = B_m(g_{k+m+1}) + \sum_{j=0}^{k+m-1} B^{(k)}_j(g_{j+1}). \end{cases} \tag{4.5.10}$$

From (4.5.9) there exists $\Phi \in C^\infty(\tilde{\Omega})$ such that

$$\begin{cases} R_1 \circ D^{j-1}_{\nu_1}(\Phi) = f_j, \quad j=1,2,\cdots \\ R_2 \circ D^{k-1}_{\nu_2}(\Phi) = g_k, \quad k=1,2,\cdots \end{cases} \tag{4.5.11}$$

Then (4.5.1), (4.5.10), and (4.5.11) implies

$$\begin{cases} R_1 \circ D^j_{\nu_1}(F - P(\Phi)) = 0, \quad j=0,1,2,\cdots \\ R_2 \circ D^k_{\nu_2}(F - P(\Phi)) = 0, \quad k=0,1,2,\cdots \end{cases} \tag{4.5.12}$$

If we define a function G, which is equal to $F - P(\Phi)$ in Ω and is zero outside it, then (4.5.12) implies that $G \in \overset{\circ}{C}{}^\infty(\tilde{\Omega})$. Since we can solve (4.5.2), there exists $v \in \overset{\circ}{C}{}^\infty(\tilde{\Omega})$ such that $P(v) = G$. Set $u = v|_\Omega + \Phi$. Then u is contained in $C^\infty(\tilde{\Omega})$ and satisfies (4.5.5). This completes the proof.

Example 4.5.2. Let $P(D) = D_1^2 + D_2^2 - D_3^2 + \sum_{|\alpha| \leqq 1} a_\alpha(x) D^\alpha$ with $a_\alpha \in C^\infty(\mathbb{R}^3)$, $\tilde{\Omega} \subset \{x \in \mathbb{R}^3;\ x_3 \geqq 0 \text{ and } x_3 = |x_2|\}$,

$$\omega_1 = \tilde{\Omega} \wedge \{x \in \mathbb{R}^3;\ x_2 = x_3 \geqq 0\},$$

$$\omega_2 = \tilde{\Omega} \wedge \{x \in \mathbb{R}^3;\ x_3 = -x_2 \geqq 0\},$$

$$\text{and} \quad \omega_{21} = \tilde{\Omega} \wedge \{x \in \mathbb{R}^3;\ x_3 = 0\}.$$

Assume that $\tilde{\Omega}$ is connected and for every $x \in \Omega$ the intersection of $x - \Gamma^*$ and $\tilde{\Omega}$ is compact, where Γ^* is the forward light cone defined in Example 4.4.6. Then the equation

$$\begin{cases} P(D)u = F & \text{in } \Omega \\ R_1(u) = f & \text{in } \omega_1 \\ R_2(u) = g & \text{in } \omega_2 \end{cases}$$

has a unique solution $u \in C^\infty(\tilde{\Omega})$ for every $F \in C^\infty(\tilde{\Omega})$, $f \in C^\infty(\tilde{\omega}_1)$ and $g \in C^\infty(\tilde{\omega}_2)$, which satisfy the compatibility condition

$$R_{21}(f) = R_{21}(g) \quad \text{in } \omega_{21}. \qquad \bullet$$

In fact if we take a coordinate $\xi = x_3 + x_2$ and $\eta = x_3 - x_2$, then we have $P = D_1^2 - 4 D_\xi D_\eta$ and hence

$$A_1 = -4 D_\xi \quad \text{and} \quad B_1 = -4 D_\eta.$$

Therefore we can apply Theorem 4.5.1. and the uniqueness of solutions can be proved rather easily.

If A_1 or B_m is not normal in the previous theorem, then we have to solve some equations in ω_{21}. We stop to go further and state a result on a special case, the Goursat problem (see Tsutsumi [27] for a related result). The proof consists of repetitions of the argument in the proof of the previous theorem.

Theorem 4.5.3. Let α be a multi-index such that $\alpha_j \neq 0$, $j=1,2,\cdots,n$ and set $P(D)=D^\alpha+\sum_{|\beta|<|\alpha|} a_\beta(x)\cdot D^\beta$, where $a_\beta \in C^\infty(\Omega_0)$ Suppose that $\tilde{\Omega}$ is contained in the set $\Gamma^*=\{x\in\mathbb{R}^n;\ x_j \geqq 0,\ j=1,2,\cdots,n\}$ and set $\tilde{\omega}_j=\{x\in\tilde{\Omega};\ x_j=0\}$ and $\omega_{j0}=\{x\in\tilde{\Omega};\ x_j=0$ and $x_k>0,\ j\neq k\}$, $j=1,2,\cdots,n$. Let R_j be the trace to ω_{j0} and R_{jk} be the trace to $\omega_{jk}=\tilde{\omega}_j\cap\tilde{\omega}_k$. Assume that to every point $x\in\Omega$, the intersection of $x-\Gamma^*$ and $\tilde{\Omega}$ is compact. Then the equation

$$\begin{cases} P(D)u=F & \text{in } \Omega \\ R_1\circ D_1^{\lambda-1}(u)=f_\lambda^{(1)} & \text{in } \omega_{10},\ \lambda=1,2,\cdots,\alpha_1 \\ \vdots & \\ R_n\circ D_n^{\lambda-1}(u)=f_\lambda^{(n)} & \text{in } \omega_{n0},\ \lambda=1,2,\cdots,\alpha_n \end{cases}$$

has a unique solution $u\in C^\infty(\tilde{\Omega})$ for every $F\in C^\infty(\tilde{\Omega})$ and $f_\lambda^{(k)}\in C^\infty(\tilde{\omega}_k)$, $\lambda=1,2,\cdots,\alpha_k$ which satisfy the compatibility conditions

$$R_{jk}\circ D_j^{\lambda-1}(f_\mu^{(k)})=R_{jk}\circ D_k^{\mu-1}(f_\lambda^{(j)}),\quad j,k=1,2,\cdots,n$$
$$\lambda,\mu=1,2,3,\cdots.$$

Chapter V. Differential equations in $\mathcal{D}'(\tilde{\Omega}, \omega_1; E)$

§5.0. Introduction.

This chapter is devoted to studies on the existence and the prolongation of singular solutions of boundary problems for linear differential equations. When the boundary ω is empty, our results are mainly due to Hörmander (see [8, 13]). If there is non-empty boundary, we encounter with many technical difficulties, but the principle of the proof is almost the same. In section 5.1. some topological properties of the space $\mathcal{D}'(\tilde{\Omega}, \omega_1; E)$ will be studied. Essential difficulty of the proof arises from the well-known fact that a subspace of an (LF) space does not necessarily become an (LF) space again (Dieudonne-Schwartz [2] and Grothendieck [6]). Fortunately we can overcome this difficulty in our case (Lemma 5.1.2.).

In sections 5.2. and 5.3. we extend some results due to Hörmander to our boundary problems. In addition to an extended version of 'P-convexity condition with respect to singular support', we need another condition (5.B) on C^∞-extendability of solutions. This seems to be rather restrictive, but we have not studied it well enough. As in the previous chapter, the existence of solutions is closely related to the extension property of solutions. When differential operators have constant coefficients, we can obtain better results. These are presented in section 5.3. In this paper we have not studied the geometric meaning of P-convexity with respect to singular support. We refer the reader to the recent studies of Hörmander and others [13, 4]. We can use their results and obtain many geometric conditions of P-convexity rather easily.

§5.1. Basic properties of the space $\mathcal{D}'(\tilde{\Omega}, \omega_1; E)$.

Let M be a σ-compact C^∞ manifold, and E a complex C^∞ vector bundle over M. Let Ω be an open subset of M, ω an open subset of the boundary of Ω, and ω_1 a subset of ω. Other notations will be the same as in sections 2.1. and 2.2. Take three open sets Ω_0, Ω_1, Ω_2 which satisfy the conditions of Proposition 2.2.1.

Theorem 5.1.1. The space $C_0^\infty(\tilde{\Omega}, \omega_1; E)$ is a strict inductive limit of Frechet-Schwartz spaces. Therefore it is separable complete bornological Montel. Its dual space is isomorphic to $\mathcal{D}'(\tilde{\Omega}, \omega_2; E)$:

$$C_0^\infty(\tilde{\Omega}, \omega_1; E)' \cong \mathcal{D}'(\tilde{\Omega}, \omega_2; E). \tag{5.1.1}$$

Proof. Choose a family of open sets U_j, $j = 1,2,\cdots$ in Ω_0 such that U_j's cover Ω_0 and the closure of each U_j is compact and is contained in U_{j+1}. From the definition $C_0^\infty(\Omega_0; E)$ is the strict inductive limit of $\overset{\circ}{C}{}^\infty(\overline{U_j}; E)$, $j = 1,2,\cdots$:

$$C_0^\infty(\Omega_0; E) = \varinjlim \overset{\circ}{C}{}^\infty(\overline{U_j}; E). \tag{5.1.2}$$

Since the union of all $\overline{U_j \cap \Omega_2}$, $j = 1,2,\cdots$ is equal to $\tilde{\Omega}_2$, the space $\overset{\circ}{C}{}_0^\infty(\tilde{\Omega}_2; E)$ is the union of all $\overset{\circ}{C}{}^\infty(\overline{U_j \cap \Omega_2}; E)$, $j = 1,2,\cdots$ as a set. By $\mathcal{E}_j$ we denote the set of all C^∞ sections φ in $C_0^\infty(\Omega_1; E)$

such that there exists $\psi \in \overset{\circ}{C}^{\infty}(\overline{U_j \cap \Omega_2};E)$ and its restriction to Ω_1 is equal to φ. Since the restriction mapping of $\overset{\circ}{C}{}_0^{\infty}(\tilde{\Omega}_2;E)$ onto $C_0^{\infty}(\tilde{\Omega}, \omega_1;E)$ induces a surjection ρ_j of $\overset{\circ}{C}^{\infty}(\overline{U_j \cap \Omega_2};E)$ onto $\mathcal{E}_j$, we give the topology to $\mathcal{E}_j$ which is induced from $\overset{\circ}{C}^{\infty}(\overline{U_j \cap \Omega_2};E)$ by ρ_j. Then $C_0^{\infty}(\tilde{\Omega}, \omega_1;E)$ is the union of all $\mathcal{E}_j$, $j = 1,2,\cdots$ as a set.

<u>Lemma</u> 5.1.2. The space $\overset{\circ}{C}{}_0^{\infty}(\tilde{\Omega}_2;E)$ is the strict inductive limit of $\overset{\circ}{C}^{\infty}(\overline{U_j \cap \Omega_2};E)$, $j = 1,2,\cdots$:

$$\overset{\circ}{C}{}_0^{\infty}(\tilde{\Omega}_2;E) \cong \varinjlim \overset{\circ}{C}^{\infty}(\overline{U_j \cap \Omega_2};E). \qquad (5.1.3)$$

<u>Proof</u>. The natural injection of $\overset{\circ}{C}^{\infty}(\overline{U_j \cap \Omega_2};E)$ into $\overset{\circ}{C}{}_0^{\infty}(\tilde{\Omega}_2;E)$ is continuous, and hence the natural bijection of $\varinjlim \overset{\circ}{C}^{\infty}(\overline{U_j \cap \Omega_2};E)$ onto $\overset{\circ}{C}{}_0^{\infty}(\tilde{\Omega}_2;E)$ is continuous. Then we only have to prove that it is an open map.

Let p be a continuous seminorm on $\varinjlim \overset{\circ}{C}^{\infty}(\overline{U_j \cap \Omega_2};E)$. Then the restriction of p to each $\overset{\circ}{C}^{\infty}(\overline{U_j \cap \Omega_2};E)$ is continuous. Hence there exists a family of continuous seminorms p_j on $\overset{\circ}{C}^{\infty}(\overline{U_j};$ $;E)$ such that p_j is equal to p in $\overset{\circ}{C}^{\infty}(\overline{U_j \cap \Omega_2};E)$, $j = 1,2,\cdots$. Now take C^{∞} functions χ_j in $\overset{\circ}{C}^{\infty}(\overline{U_j} \setminus U_{j-2})$ such that $\sum_{j=1}^{\infty} \chi_j = 1$ in Ω_0. Then define a seminorm on $C_0^{\infty}(\Omega_0;E)$ by

$$q(\varphi) = \sum_{j=1}^{\infty} p_j(\chi_j \cdot \varphi), \quad \varphi \in C_0^{\infty}(\Omega_0;E).$$

The restriction of q to $\overset{\circ}{C}{}_0^{\infty}(\tilde{\Omega}_2;E)$ is larger than p. Moreover q is continuous. In fact for every $\varphi \in \overset{\circ}{C}^{\infty}(\overline{U_j};E)$ we have

$$q(\varphi)=\sum_{k=1}^{j+1} p_k(\chi_k\cdot\varphi). \tag{5.1.4}$$

Since p_k's are continuous, there exist functions ψ_k in $C_0^\infty(\Omega_0)$, real numbers s_k, $k=1,2,\cdots,j-1$ and a constant $C>0$ such that

$$\sum_{k=1}^{j+1} p_k(\varphi)\leqq C\sum_{k=1}^{j+1}\|\psi_k\cdot\varphi\|_{(s_k)},\quad \varphi\in\mathcal{E}^\infty(\overline{U_j};E).$$

Then it follows from (5.1.4) that

$$q(\varphi)\leqq C\sum_{k=1}^{j+1}\|(\psi_k\cdot\chi_k)\cdot\varphi\|_{(s_k)},\quad \varphi\in\mathcal{E}^\infty(\overline{U_j};E).$$

Hence q is continuous on each $\overset{\circ}{C}{}^\infty(\overline{U_j};E)$, and then from (5.1.2) it is continuous on $C_0^\infty(\Omega_0;E)$. We have thus proved that p is smaller than the restriction of a continuous seminorm on $C_0^\infty(\Omega_0;E)$. Therefore p is a continuous seminorm on $\overset{\circ}{C}{}_0^\infty(\tilde{\Omega}_2;E)$, and the proof of Lemma 5.1.2. is complete.

<u>Lemma</u> 5.1.3. The space $C_0^\infty(\tilde{\Omega},\omega_1;E)$ is the strict inductive limit of $\mathcal{E}_j$, $j=1,2,\cdots$:

$$C_0^\infty(\tilde{\Omega},\omega_1;E)\cong\varinjlim\mathcal{E}_j. \tag{5.1.5}$$

<u>Proof</u>. To each $j=1,2,\cdots$ we have the following commutative diagram:

$$\begin{array}{ccc}
\mathcal{E}^\infty(\overline{U_j\cap\Omega_2};E) & \hookrightarrow & \varinjlim\overset{\circ}{\mathcal{E}}{}^\infty(\overline{U_k\cap\Omega_2};E)\\
\downarrow\rho_j & & \downarrow\rho\\
\mathcal{E}_j & \hookrightarrow & \varinjlim\mathcal{E}_j
\end{array}$$

Then it is easy to prove that ρ is an epimorphism of $\mathcal{E}_0^\infty(\tilde{\Omega}_2;E)$

onto $\varinjlim \mathcal{E}_j$. Hence $C_0^\infty(\tilde{\Omega}, \omega_1;E)$ is isomorphic to $\varinjlim \mathcal{E}_j$ as locally convex spaces.

Proof of Theorem 5.1.1. (continued) Since every $\mathcal{E}_j$ is Frechet-Schwartz, $C_0^\infty(\tilde{\Omega}, \omega_1;E)$ is separable complete bornological Montel. Moreover its dual space is isomorphic to the projective limit of $\mathcal{E}_j'$, $j = 1,2,\cdots$:

$$C_0^\infty(\tilde{\Omega}, \omega_1;E)' \cong \varprojlim \mathcal{E}_j'. \qquad (5.1.5)$$

But from Theorem 2.3.2. the space $\mathcal{E}_j'$ is isomorphic to $\mathcal{E}'(\overline{U_j \cap \Omega}, \omega_2 \cap U_j;E)$. Then their projective limit $\varprojlim \mathcal{E}_j'$ can be identified with $\mathcal{D}'(\tilde{\Omega}, \omega_2;E)$ as sets. Since $\mathcal{E}_j'$ is topologically a subspace of $\overset{\circ}{C}{}^\infty(\overline{U_j \cap \Omega_2};E)' \cong \mathcal{E}'(\overline{U_j \cap \Omega_2};E)$, their projective limit is topologically a subspace of $\varprojlim \mathcal{E}'(\overline{U_j \cap \Omega_2};E) \cong \mathcal{D}'(\tilde{\Omega}_2;E)$. Hence it follows that

$$\varprojlim \mathcal{E}_j' \cong \mathcal{D}'(\tilde{\Omega}, \omega_2;E). \qquad (5.1.6)$$

From (5.1.5) and (5.1.6) we obtain (5.1.1). Then the remaining part of the theorem is obvious, and the proof of Theorem 5.1.1. is complete.

§5.2. Existence theorems in $\mathcal{D}'(\tilde{\Omega}, \omega_1;E)$.

Let E and F be complex C^∞ vector bundles over M, and P a linear differential operator (with C^∞ coefficients) of $\mathcal{D}'(M;E)$ into $\mathcal{D}'(M;F)$. Take a positive integer l. We consider the following two properties:

(5.A) To every compact subset K of $\tilde{\Omega}$ there exists a compact subset K' of $\tilde{\Omega}$ such that we can take $K'=\phi$ if $K=\phi$, and the following statement holds. If $\psi\in C_0^1(\tilde{\Omega}, \omega_2;F)$ and ${}^tP(\psi)|_{\Omega_2\setminus K}\in C^\infty(\overline{\Omega_2\setminus K};E)$, then the singular support of ψ is contained in $\tilde{\Omega}\cap K'$, i.e. ψ is C^∞ in $\Omega_2\setminus K'$.

(5.B) To every $x\in\omega_1$ there exists an open neighbourhood V of x in Ω_0 such that $\psi\in C^1(\overline{\Omega\cap V}, \omega_2\cap V;F)$, ${}^tP(\psi)\in C^\infty(\overline{\Omega\cap V};E)$, and $\psi\in C^\infty(\Omega_2\cap V;F)$ implies $\psi\in C^\infty(\overline{\Omega\cap V};F)$.

The next theorem is a generalization of a result due to Hörmander [13]. The principle of the proof is the same as his, although it becomes more complicated.

<u>Theorem</u> 5.2.1. If the conditions (5.A) and (5.B) are valid, then for every $f\in\mathcal{D}'(\tilde{\Omega}, \omega_1;F)$ there exists $u\in\mathcal{D}'(\tilde{\Omega}, \omega_1;E)$ such that $f-P(u)\in C^\infty(\tilde{\Omega}, \omega_1;F)$.

Therefore if in addition we can find a solution $u\in C^\infty(\tilde{\Omega}, \omega_1;E)$ of the equation

$$P(u)=f \tag{5.2.1}$$

for any $f\in C^\infty(\tilde{\Omega}, \omega_1;F)$, then the equation (5.2.1) is solvable in $\mathcal{D}'(\tilde{\Omega}, \omega_1;E)$ for any $f\in\mathcal{D}'(\tilde{\Omega}, \omega_1;F)$.

<u>Lemma</u> 5.2.2. If (5.A) and (5.B) hold, then the following condition holds:

(5.C) To every compact subset K of $\tilde{\Omega}$ there exists a compact subset K' of $\tilde{\Omega}$, which can be taken to be empty if K is empty, such that $\psi \in C_0^1(\tilde{\Omega}, \omega_2; F)$ and ${}^tP(\psi)|_{\Omega_2 \setminus K} \in C^\infty(\overline{\Omega_2 \setminus K}; E)$ implies $\psi|_{\Omega_2 \setminus K'} \in C^\infty(\overline{\Omega_2 \setminus K'}; F)$.

Moreover it is obvious that (5.C) implies (5.A) and (5.B).

Proof. Let K be a compact subset of $\tilde{\Omega}$. Then there exists another compact subset K' of $\tilde{\Omega}$ which satisfies the condition (5.A). We can assume that $K \subset K'$. Take a compact subset K'' of Ω_0 such that K' is contained in the interior of K''. From the condition (5.B) there exists a locally finite open covering $\{V_\alpha;\ \alpha \in A\}$ of $\omega_1 \setminus K$ such that $V_\alpha \subset \Omega_0 \setminus K'$ and the condition (5.B) is valid if V is replaced by V_α.

Now let $\psi \in C_0^1(\tilde{\Omega}, \omega_2; F)$ and ${}^tP(\psi)|_{\Omega_2 \setminus K} \in C^\infty(\overline{\Omega_2 \setminus K}; E)$. From (5.A) it follows that sing supp $\psi \subset \tilde{\Omega} \cap K'$. Then we have $\psi|_{\Omega_2 \setminus K'} \in C^\infty(\Omega_2 \setminus K'; E)$. Taking the restriction of ψ to $\Omega_2 \cap V_\alpha$, we obtain $\psi|_{\Omega_2 \cap V_\alpha} \in C^1(\overline{\Omega \cap V_\alpha}, \omega_2 \cap V_\alpha; F)$, ${}^tP(\psi|_{\Omega_2 \cap V_\alpha}) =$ $= {}^tP(\psi)|_{\Omega_2 \cap V_\alpha} \in C^\infty(\overline{\Omega \cap V_\alpha}; E)$, and $\psi|_{\Omega_2 \cap V_\alpha} \in C^\infty(\Omega_2 \cap V_\alpha; F)$. Hence (5.B) implies $\psi|_{\Omega_2 \cap V_\alpha} \in C^\infty(\overline{\Omega \cap V_\alpha}, \omega_2 \cap V_\alpha; F)$. Therefore it follows that $\psi|_{\Omega_2 \setminus K''} \in C^\infty(\overline{\Omega_2 \setminus K''}; F)$, or $\psi \in C^\infty(\tilde{\Omega}, \omega_2; F)$ if $K = \phi$, and the proof of Lemma 5.2.2. is finished.

The proof of Theorem 5.2.1. is complete if we can prove the following lemma. The remaining details may be left to the reader, because they are just a repetition of the argument due to Hörmander [13] Theorem 1.2.4.

Lemma 5.2.3. Let $0 < C < C'$, and p be a continuous seminorm on $C_0^\infty(\tilde{\Omega}, \omega_2; E)$. Let r be a continuous seminorm on $C_0^\infty(\tilde{\Omega}, \omega_2; F)$ such that

$$\inf\{\|\psi'\|_{(1+n)};\ \psi'|_{\Omega_2} = \psi \text{ and } \psi' \in C_0^\infty(\Omega_0; F)\} \leqq r(\psi) \tag{5.2.2}$$

for all $\psi \in C_0^\infty(\tilde{\Omega}, \omega_2; F)$. Suppose that compact sets K and K' in $\tilde{\Omega}$ satisfy the condition (5.C). Moreover take two compact subsets K'' and K''' of Ω_0 such that K' is contained in the interior of K'' and $K'' \subset K'''$. Let μ_j, $j = 1, 2, \cdots$ be elements of $C_0^\infty(\tilde{\Omega}, \omega_1; F)$ with locally finite supports. If $\psi \in C_0^\infty(\tilde{\Omega}, \omega_2; F)$ and $\operatorname{supp}\psi \subset \tilde{\Omega} \wedge K''$ implies

$$r(\psi) \leqq C \cdot (p({}^tP(\psi)) + \sum_{j=1}^{\infty} |\langle\psi, \mu_j\rangle|), \tag{5.2.3}$$

Then there exist a continuous seminorm q on $C^\infty(\overline{\Omega_2 \setminus K}; E)$ and a finite number of elements $\nu_1, \cdots, \nu_s$ of $C_0^\infty(\tilde{\Omega}, \omega_1; F)$ with support in $\tilde{\Omega} \wedge K'$, such that $\psi \in C_0^\infty(\tilde{\Omega}, \omega_2; F)$ and $\operatorname{supp}\psi \subset \tilde{\Omega} \wedge K'''$ implies

$$r(\psi) \leqq C' \cdot (p({}^tP(\psi)) + \sum_{j=1}^{\infty} |\langle\psi, \mu_j\rangle| + $$

$$+ q \circ \gamma({}^tP(\psi)) + \sum_{j=1}^{s} |\langle\psi, \nu_j\rangle|), \tag{5.2.4}$$

where γ is the restriction mapping to $\Omega_2 \setminus K$.

Proof. Since the locally convex space

$$\{\psi \in C_0^\infty(\tilde{\Omega}, \omega_1; F);\ \operatorname{supp}\psi \subset \tilde{\Omega}\setminus K'\} \tag{5.2.5}$$

is separable from Theorem 5.1.1., there exists a dense sequence $\lambda_1, \lambda_2, \cdots$ in the space (5.2.5). Let q_k, $k = 1, 2, \cdots$ be a countable basis of Spec $C^\infty(\overline{\Omega_2 \setminus K}; E)$ such that $2q_k \leqq q_{k+1}$, $k = 1, 2, \cdots$. If (5.2.4) does not hold, then there exists a sequence of C^∞ sections $\psi_1, \psi_2, \cdots$ in $C_0^\infty(\tilde{\Omega}, \omega_2; F)$ such that

$$\operatorname{supp}\psi_k \subset \tilde{\Omega} \wedge K''', \quad r(\psi_k) = C', \tag{5.2.6}$$

and

$$p({}^tP(\psi_k)) + \sum_{j=1}^{\infty} |\langle \psi_k, \mu_j\rangle| + q_k \circ \gamma({}^tP(\psi_k)) + $$
$$+ k\sum_{j=1}^{k} |\langle \psi_k, \lambda_j\rangle| \leqq 1. \tag{5.2.7}$$

Therefore we have

$$\gamma({}^tP(\psi_k)) \longrightarrow 0, \ k \longrightarrow \infty \text{ in } C^\infty(\overline{\Omega_2 \setminus K}; E) \tag{5.2.8}$$

and

$$|\langle \psi_k, \lambda_j\rangle| \longrightarrow 0, \ k \longrightarrow \infty \text{ for } j = 1, 2, \cdots. \tag{5.2.9}$$

Now by Φ we denote the completion of the space

$$\{\psi \in C_0^\infty(\tilde{\Omega}, \omega_2; F);\ \operatorname{supp}\psi \subset \tilde{\Omega} \wedge K'''\}$$

with respect to the seminorms r and $q \circ \gamma \circ {}^tP$, $q \in$ Spec $C^\infty(\overline{\Omega_2 \setminus K}; F)$. Then Φ is a Frechet space. From (5.2.2) we have

$$\Phi \subset \{\psi \in C_0^\ell(\tilde{\Omega}, \omega_2; F);\ \operatorname{supp}\psi \subset \tilde{\Omega} \wedge K'''\}.$$

From the definition of the topology of Φ it follows that $\gamma({}^tP(\psi)) \in C^\infty(\overline{\Omega_2 \setminus K}; E)$ for all $\psi \in \Phi$. Then (5.C) implies

$\psi|_{\Omega_2\setminus K'} \in C^\infty(\overline{\Omega_2\setminus K'};F)$ for all $\psi \in \Psi$. By γ' we denote the restriction mapping of Ψ into $C^\infty(\overline{\Omega_2\setminus K'};F)$. Then γ' is continuous from the closed graph theorem.

From (5.2.6) and (5.2.8) the set $\{\psi_k;\ k=1,2,\cdots\}$ is bounded in Ψ, and then $\{\gamma'(\psi_k);\ k=1,2,\cdots\}$ is bounded in $C^\infty(\overline{\Omega_2\setminus K'};F)$. Since $C^\infty(\overline{\Omega_2\setminus K'};F)$ is a Montel space from Theorem 2.3.2., there exists a subsequence of $\gamma'(\psi_k)$, $k=1,2,\cdots$ which converges to some ψ_0 in $C^\infty(\overline{\Omega_2\setminus K'};F)$. We may denote this subsequence by the same symbol $\gamma'(\psi_k)$, $k=1,2,\cdots$. From (5.2.9) we obtain

$$|\langle\gamma'(\psi_k),\ \lambda_j\rangle| = |\langle\psi_k,\ \lambda_j\rangle| \longrightarrow |\langle\psi_0,\ \lambda_j\rangle| = 0,$$

as $k\longrightarrow\infty$, for all $j=1,2,\cdots$. Hence $\gamma'(\psi_k)$ converges to zero weakly in $C^\infty(\overline{\Omega_2\setminus K'};F)$, which is a Montel space. Therefore we have

$$\gamma'(\psi_k)\longrightarrow 0,\ k\longrightarrow\infty \text{ in } C^\infty(\overline{\Omega_2\setminus K'};F). \tag{5.2.10}$$

Next take a function χ in $C_0^\infty(\Omega_0)$ with support in K'' such that $\chi=1$ in a neighbourhood of K'. We write $\psi_k'=\chi\cdot\psi_k$ and $\psi_k''=(1-\chi)\cdot\psi_k$. Then ψ_k'' tends to zero in $C_0^\infty(\tilde{\Omega},\ \omega_2;F)$ from (5.2.10). From (5.2.6) and (5.2.7) we have for sufficiently large k

$$c\cdot(p({}^tP(\psi_k')) + \sum_{j=1}^{\infty}|\langle\psi_k',\ \mu_j\rangle| \leqq r(\psi_k'),$$

which contradict with (5.2.3). This finishes the proof.

Corollary 5.2.4. If (5.A) and (5.B) hold, then for every $f \in \mathcal{D}'(\tilde{\Omega}_1;F)$ and $u \in \mathcal{D}'(\overline{\Omega_1 \setminus \Omega};E)$, which satisfy $P(u) = f$ in $\Omega_1 \setminus \tilde{\Omega}$, there exists $\tilde{u} \in \mathcal{D}'(\tilde{\Omega}_1;E)$ such that u is equal to the restriction of $\tilde{u}$ to $\Omega_1 \setminus \tilde{\Omega}$ and

$$P(\tilde{u}) - f \in C^\infty(\tilde{\Omega}, \omega_1;F).$$

Proof. Take f and u which satisfy the above hypotheses. There exists $u_1 \in \mathcal{D}'(\tilde{\Omega}_1;E)$, whose restriction to $\Omega_1 \setminus \tilde{\Omega}$ is equal to u. Write $g = f - P(u_1)$. Then g belongs to $\mathcal{D}'(\tilde{\Omega}, \omega_1;F)$. From Theorem 5.2.1. there exists $v \in \mathcal{D}'(\tilde{\Omega}, \omega_1;E)$ such that $P(v) - g \in C^\infty(\tilde{\Omega}, \omega_1;F)$. Write $\tilde{u} = u_1 + v$. Then $\tilde{u}$ satisfies the required condition, and the proof is complete.

Now consider the conditions (5.A) and (5.B).

From the results of the previous chapter, (5.B) is satisfied if ${}^tP$ is hyperbolic with respect to $\Omega_1 \setminus \Omega$ for some Ω_1. But this is too restrictive.

The condition (5.A) is a generalization of Hörmander's 'P-convexity with respect to singular support'. Then we can make the following definition:

Definition 5.2.1. The pair $(\tilde{\Omega}, \omega_1)$ is called P-convex with respect to singular support if and only if to every compact set $K \subset \tilde{\Omega}$ there exists another compact set $K' \subset \tilde{\Omega}$, which can be taken to be void if K is void, such that $\psi \in \mathcal{E}'(\tilde{\Omega}, \omega_2; F)$ and ${}^tP(\psi)|_{\Omega_2 \setminus K} \in C^\infty(\overline{\Omega_2 \setminus K}; E)$ implies sing supp $\psi \subset \tilde{\Omega}_\wedge K'$.

If ω is empty, this definition agrees with that of Hörmander. Then we can survey the geometric meaning of this P-convexity condition as Hörmander did in the case of no boundary (cf. [4, 13]).

The next theorem is obvious:

<u>Theorem</u> 5.2.5. If P is hypoelliptic, then any pair $(\tilde{\Omega}, \omega_1)$ is P-convex with respect to singular support.

§5.3. Differential equations with constant coefficients

Let $M = \mathbb{R}^n$, $E = \mathbb{R}^n \times \mathbb{C}^L$, $F = \mathbb{R}^n \times \mathbb{C}^N$, $L \geqq N$, and let $P = P(D)$ be a linear differential operator of $\mathcal{D}'(\Omega_0)^L$ into $\mathcal{D}'(\Omega_0)^N$ with constant coefficients. Our consideration is limited to the case of determined or under-determined systems of differential equations, because in the case of over-determined systems there arise essentially different problems (see Ehrenpreis [5]). Under the above restrictions we can prove the following theorem:

<u>Theorem</u> 5.3.1. If the condition (5.B) holds, then the following three statements are equivalent:

(1) The pair $(\tilde{\Omega}, \omega_1)$ is P-convex with respect to singular support.

(2) For every $f \in \mathcal{D}'(\tilde{\Omega}, \omega_1)^N$ there exists $u \in \mathcal{D}'(\tilde{\Omega}, \omega_1)^L$ such that $P(u) - f \in C^\infty(\tilde{\Omega}, \omega_1)^N$.

(3) If $f \in \mathcal{D}'(\tilde{\Omega}_1)^N$, $u \in \mathcal{D}'(\overline{\Omega_1 \setminus \Omega})^L$, and $P(u) = f$ in $\Omega_1 \setminus \tilde{\Omega}$, then there exists $\tilde{u} \in \mathcal{D}'(\tilde{\Omega}_1)^L$ such that $u = \tilde{u}$ in $\Omega_1 \setminus \tilde{\Omega}$ and $f - P(\tilde{u}) \in C^\infty(\tilde{\Omega}, \omega_1)^N$.

Proof. From Theorem 5.2.1. and Corollary 5.2.4. it remains to prove that (2) implies (1) and that (3) implies (2). The proof of (2)$\Longrightarrow$(1) is just a repetition of the argument in Hörmander [8], Theorem 3.6.3., therefore we leave the details for the reader. The proof of (3)$\Longrightarrow$(2) is very easy. In fact let $f \in \mathcal{D}'(\tilde{\Omega}, \omega_1)^N$. Define $u \in \mathcal{D}'(\overline{\Omega_1 \setminus \Omega})^L$ by $u(x) = 0$, $x \in \Omega_1 \setminus \tilde{\Omega}$. Then $P(u) = f = 0$ in $\Omega_1 \setminus \tilde{\Omega}$. Hence (3) implies the existence of $\tilde{u} \in \mathcal{D}'(\tilde{\Omega}, \omega_1)^L$ such that $f - P(\tilde{u}) \in C^\infty(\tilde{\Omega}, \omega_1)^N$. This completes the proof.

Under more restrictions we can prove the following two theorems:

Theorem 5.3.2. Suppose that $\tilde{\Omega}$ is contained in the closed half space $H \subset \mathbb{R}^n$ and ω_1 is an open subset of the boundary of H. Moreover assume that P(D) is determined, that is, $L = N$ and $\det P(\xi) \not\equiv 0$, and the condition (5.B) holds. Then the following three statements are equivalent:

(1) P(D) is evolutional with respect to H and the pair $(\tilde{\Omega}, \omega_1)$ is strongly P(D)-convex, i.e. P(D)-convex with respect to support and singular support.

(2) For every $f \in \mathcal{D}'(\tilde{\Omega}, \omega_1)^N$ there exists a solution $u \in \mathcal{D}'(\tilde{\Omega}, \omega_1)^N$ of the equation

$$P(D)u = f. \tag{5.3.1}$$

(3) If $f \in \mathcal{D}'(\tilde{\Omega}_1)^N$, $u \in \mathcal{D}'(\overline{\Omega_1 \setminus \Omega})^N$, and (5.3.1) holds in $\Omega_1 \setminus \tilde{\Omega}$, then there exists $\tilde{u} \in \mathcal{D}'(\tilde{\Omega}_1)^N$ such that the restriction of $\tilde{u}$ to $\Omega_1 \setminus \tilde{\Omega}$ is equal to u and $P(D)\tilde{u} = f$ in Ω_1.

Proof. (1) implies (2) from Theorems 4.4.1 and 5.2.1. It is obvious that (2) and (3) are equivalent. Now assume that (2) is true. We can suppose that $0 \in \omega_1$. Then there exists a distribution Φ in $\mathcal{D}'(\tilde{\Omega}, \omega_1)$ such that $\det P(D)\Phi = \delta$ in Ω_1. There exist a compact neighbourhood K of 0 in Ω_1, positive constants C and C' and an integer m such that $\varphi \in C_0^\infty(\Omega_1)$ and supp $\varphi \subset K$ implies

$$\begin{aligned} |\varphi(0)| &= |\langle \varphi, \delta \rangle| = |\langle \varphi, \det P(D)\Phi \rangle| = |\langle \det P(-D)\varphi, \Phi \rangle| \\ &\leq C \cdot \inf \Big\{ \sum_{|\alpha| \leq m} \sup_{x \in K} |D^\alpha \psi(x)| \;;\; \psi \in C_0^\infty(\Omega_0) \text{ and } \det P(-D)\varphi = \psi \text{ in} \\ &\leq C' \cdot \sum_{|\alpha| \leq 2m} \sup_{x \in H} |D^\alpha \det P(-D)\varphi(x)| . \end{aligned}$$

Hence P(D) is evolutional (see the proof of Theorem 4.4.2), and this completes the proof.

Theorem 5.3.3. Let the hypotheses of Theorem 5.3.2. be fulfilled except (5.B). In addition we assume that P(D) is hypoelliptic. Then the following condition (1') is equivalent with (2) and (3) of the previous theorem.

(1') The pair $(\tilde{\Omega}, \omega_1)$ is P(D)-convex with respect to support and P(D) is parabolic in the sense that there exists a real number τ_0 such that

$$\det P(\xi - i \cdot \tau \cdot \nu) \neq 0 \text{ if } \xi \in \mathbb{R}^n \text{ and } \tau > \tau_0,$$

where ν is the inner normal to H.

Proof. ($1'$) implies (2) from Theorem 5.2.5., 5.3.2., and a well-known result (e.g. [8] Theorem 5.8.2.). Then we have to prove that (2) implies the parabolicity of P(D). We can prove this fact directly using the argument of Theorem 5.8.1. in [8]. Details may be omitted.

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