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Nonconforming elements and patch test

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1. Introduction

The present report is a brief introduction to the following papers.

[1] Irons & Razzaque, Experimence with the patch test for convergence of finite elements, The Math. Found. of F.E.M. with Application to P.D.E.(Edited by Aziz).

[2] Strang, Variational crimes in the finite element method, ibid.

As well known, in seeking accurate approximate solutions of plate bending problems one meets a speacial difficulty due to second order derivatives appearing in the energy functional. To avoid this difficulty engineers employ so called " Non-conforming elements " frequently, in spite of the fact that such procedures can not be justified by the classic al variational principles, and it is also a well known fact that some nonconforming elements can give good approximate solutions. Therefore arose a new problem in the history of variational method that " find a simple criterion to ensure the convergence of nonconforming finite element solution".

Irons' idea and its mathematical formulation by Strang is the first approach to this problem. Briefly speaking their result is as follows. Nonconforming solution converges if the problems having exact solution of constant curvature can be solved exactly by the used nonconforming elements and by nonconforming way. Since it is not so difficult to check this assumption for each element it seems that they solved the above problem in very elegant way. But in the author's opinion the situations are not so changed yet, because their method can be applied only to the element of a speacial type.

But their approach is very elegant and unique. This is the reason why we report the followings.

2. Origins of the patch test

Irons says in [1] that in 1965 engineers believed that the inter-element continuity of the finite element was very important,and therefore a numerical experiment by Tocher and Kapur which demonstrated the convergence of so called ACM-so-

-lution could not be explained even by engineering intuition.

But " ... Some months later,research at Rolls-Royce on the Zienkiewicz nonconforming triangle ...... clarified the situation. .... It was observed (a) that every problem giving constant curvature over the whole domain was accurately solved by the conforming elements,whatever the mesh pattern,as was expected, (b) that the nonconforming element was also successful, but only for one particular mesh pattern. Thus the patch test was born. "

3. Patch test (formulation by Strang)

Assumption: We assume that the trialfunction used in the computation can represent the functions of constant curvature exactly on the whole domain,that is,the trial function must contain $P_1=(a_1+a_2x+a_3y;a_i$ is arbitraly) when second order equations are solved and $P_2=(a_1+a_2x+a_3y+.....+ a_6y^2$; a_i is arbitraly) for fourth order equations.

Definition: Let Ω_h be a decomposition of the domain Ω .

A patch is a subdomain(therefore connected) of Ω_h consisting of several element-subdomains of Ω_h.

Definition: The patch test for a basis $\{\varphi_i\}$ is passed if the following holds.

Give a boundary condition at the boundary of an arbitraly patch in $_h$ so that the exact solution has constant curvature. Then the approximate solution obtained by using $\{\varphi_i\}$ in nonconforming way coincides with this exact solution.

We shall demonstrate the above idea for the following boundary value problem.

(3.1) $$-\Delta u = f \quad \text{in} \quad \Omega_h , \qquad u\Big|_{\partial\Omega_h} = \text{given} .$$

We shall seek the approximate solution in the form

(3.2) $$u = \sum_{i=1}^{n_1+n_2} u_i \varphi_i \qquad (n_2 = \text{number of bry. nodes}).$$

Note that $u_{n_1+1}, \ldots\ldots, u_{n_2}$ are given values.

Some notations:

$$V_1 = \left\{ v_h = \sum_{i=1}^{n_1} u_i \varphi_i \right\} , \quad V_2 = \left\{ v_h = \sum_{i=1}^{n_1+n_2} u_i \varphi_i \right\}.$$

$$a(u,v) = (u_x, v_x) + (u_y, v_y), \qquad (u,v) = \int_{\Omega_h} uv\,dxdy$$

$$\tilde{a}(u,v) = \sum_e \left\{ (u_x, v_x)_e + (u_y, v_y)_e \right\}, \quad (u,v)_e = \int_e uv\,dxdy$$

Proposition 1. Patch test is passed if and only if

(3.3) $$a(p_1, v_h) = \tilde{a}(p_1, v_h)$$

for any $p_1 = a+bx+cy$ and any $v_h \in V_1$.

Remark. Definition of $a(u,v_h)$. Strictly speaking, the integral $a(u,v_h)$ is not well defined in the ordinal sence, since v_h is nonconforming (not continuous, in the present problem). But it becomes well defined if we understand as follows. There exists a sequence of smooth functions $v^{(n)}$ which support lie in Ω_h and converging to v_h in L_2 sence. Therefore,

$$a(u,v_h) = \lim a(u,v^{(n)}) = \lim (-\Delta u,v^{(n)}) = (-\Delta u,v_h)$$

The proof of the proposition is not difficult.

Remark. The eqality (3.3) is equivalent to the following equality.

$$\sum_e \int_{\partial e} \frac{dp_1}{dn} v_h \, ds = 0 \qquad (\forall p_1, \forall v_h$$

where n denotes the unit normal(outward)to the element boundary.

An example of basis passed the patch test.

Let e be the square $\{ |x|, |y| \leqq 1 \}$.

The following basis (called Wilson's element) satisfys (3.3). Note that the supports of the nonconforming parts lie in the element e.

$\varphi_{1,e} \sim \varphi_{4,e}$: basis corresponding to the shape function $a_0+a_1x+a_2y+a_3xy$, $\varphi_{5,e} = 1-x^2$, $\varphi_{6,e} = 1-y^2$.

4. Patch test and convergence of nonconforming solutions.

Let u and $\hat{u}$ be the exact and approximate solution of the our model problem respectively. Then hold

$$(4.1)\qquad \sum_e (u, \varphi)_{1,e} = (f, \varphi) + \sum_e \int_{\partial e} \frac{du}{dn} \varphi \, ds ,$$

$$(4.2)\qquad \sum_e (\hat{u}, \varphi)_{1,e} = (f, \varphi)$$

for any function φ in V_2. Substituting (4.2) from (4.1)

$$\sum_e (u-\hat{u}, \varphi)_{1,e} = \sum_e \int_{\partial e} \frac{du}{dn} \varphi \, ds,$$

therefore we have

$$(4.3)\qquad \| u-\hat{u} \| \equiv \sqrt{\sum_e (u-\hat{u}, u-\hat{u})_{1,e}} \geqq \sup_{\varphi \in V_1} \frac{\sum \int_{\partial e} \frac{du}{dn} \varphi \, ds}{\| \varphi \|} \equiv \Delta_0 .$$

On the other hand, for any $\hat{w}$ in V_2 it holds that

$$\| \hat{w} - \hat{u} \|^2$$

$$= \sum_e (\hat{w}-u, \hat{w}-\hat{u})_{1,e} + \sum_e (u-\hat{u}, \hat{w}-\hat{u})_{1,e} .$$

But the second term of the right side is bounded by

$$\Delta_0 \cdot \| \hat{w} - \hat{u} \| .$$

Therefore we have

$$\|\hat{w} - \hat{u}\| \leqq \|\hat{w} - u\| + \Delta_0 ,$$

and thus

$$\|u - \hat{u}\| \leqq \Delta_0 + 2 \inf_{\hat{w}} \|\hat{w} - u\| .$$

Proposition 2. Let Δ_0 be defined by (4.3). Then holds the following inequality for the approximate solution $\hat{u}$.

$$\Delta_0 \leqq \|u - \hat{u}\| \leqq \Delta_0 + 2 \inf_{\hat{w}} \|\hat{w} - u\| . \tag{4.4}$$

Now, we shall show that for the Willson's element Δ_0 tends to zero as the element size tend to zero. We denote the conforming part by ω_e and non-conforming parts by φ_e and ψ_e. Therefore $\hat{v}$ is expressed as follows.

$$\hat{v} = \sum_e a_e \omega_e + b_e \varphi_e + c_e \psi_e .$$

The contribution from the element e to the integral of Δ_0 can be calculated as follows.

$$\int_{\partial e} \frac{du}{dn} \hat{v}\, ds = \int_{\partial e} \frac{du}{dn}(b_e \varphi_e + c_e \psi_e) ds$$

$$= \int_{\partial e} \frac{d}{dn}(u - p_1)(b_e \varphi_e + c_e \psi_e) ds$$

$$= (\Delta(u-p_1), b_e\varphi_e + c_e\psi_e)_e + (\partial[u-p_1], \partial[b_e\varphi_e + c_e\psi_e])_e^{(*)}$$

$$\leq C\|u\|_{2,e} \cdot \|b_e\varphi_e + c_e\psi_e\|_e + \|u-p_1\|_{1,e}^{(*)} \cdot \|b_e\varphi_e + c_e\psi_e\|_{1,e}.$$

On the other hand it can be easily verified that

$$\|b_e\varphi_e + c_e\psi_e\|_e \leq Ch\|b_e\varphi_e + c_e\psi_e\|_{1,e}$$

$$\inf_{p_1}\|u - p_1\|_{1,e} \leq Ch\|u\|_{2,e} .$$

Substituting these estimates we have

$$\int_{\partial e} \frac{du}{dn}\hat{v}ds \leq Ch\|u\|_{2,e} \cdot \|b_e\varphi_e + c_e\psi_e\|_{1,e} .$$

Moreover it is proved by a elementally calculation that

$$\|b_e\varphi_e + c_e\psi_e\|_{1,e} \leq 2\|a_e\omega_e + b_e\varphi_e + c_e\psi_e\|_{1,e}$$

Substituting this into the above estimate and summing over all e and deviding both side by $\|\hat{v}\|_1$ we have

$$\Delta_0 \leq \text{const.}\ h\|u\|_2 ,$$

which is the desired estimate.

(*) $\|u\|_{1,e} = (\partial u, \partial u)_e = \sum_i (u_{,i}, u_{,i})_e$

5. Some comments

For the convergence proof given above it is essential that the support of the non-conforming part of the basis is in the element and thus we can take p_1 separately for the individual element. But,the usual non-conforming elements which are used in actual computatin have no such property. For such non-conforming elements the above idea to prove the convergence is not valid.

Although Strang's formulation sketched above is incomlete,but this does not mean that the patch test proposed by Irons is not worth considering,because it is known empirically that the elements passed the patch test can give good approximate solutions.