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A Note on Finite Element Approximation of Evolution Equations

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0. Introduction

In this note, we consider the finite element method for approximate solutions of mixed initial-boundary value problems of both parabolic and hyperbolic types, including the equation of elastodynamics. The main concern is the problem of stability in the sense of energy norm for hyperbolic problems, and in the sense of maximum norm for parabolic problems (i.e., the problem of *discrete maximum principle*).

We begin by defining some preparatory notions.

1. Stability Functions and Acuteness of Triangulation

1.1 Finite Element Spaces X_o^h, X^h, Y_o^h and Y^h

Assumption: Ω is an m-dimensional polyhedral domain with the boundary Γ.
Definition: Triangulation T^h of $\bar{\Omega}$.

T^h is a finite set of non-degenerate (closed) m-dimensional simplices Δ such that

(1) $\Omega = \bigcup_{\Delta \in T^h} \Delta$

(2) any face of $\Delta \in T^h$ is either a face of another m-simplex, or a portion of the boundary Γ.

Definition: barycentric fragments Δ'_λ, $\lambda=1,..,(m+1)!$ of $\Delta \in T^h$.

Δ'_λ, $\lambda=1,..,(m+1)!$, are m-simplices which satisfy

(1) $\bigcup_{\lambda=1}^{(m+1)!} \Delta'_\lambda = \Delta \quad (\Delta'_\lambda \in \Delta)$,

(2) a vertex of Δ'_λ is the barycenter of Δ, and another vertex of Δ'_λ is a vertex of Δ, and

(3) the intersection of Δ'_λ and a face of Δ (which is an (m-1)-simplex) is again a barycentric fragment of the face.

Definition: barycentric subdivision B_i^Δ (i=1,..,m+1) of a m-simplex $\Delta \in T^h$.

With each vertex P_i of Δ, B_i^Δ is defined to be

$$B_i^\Delta = \bigcup_{P_i \subset \Delta'_\lambda} \Delta'_\lambda.$$

Note: $\text{mes}(B_i^\Delta) = \text{mes}(\Delta)/(m+1)$, $i=1,..,m+1$.

Note: mes(B_i^{Δ}) = mes$(\Delta)/(m+1)$, $i=1,..,m+1$.

Definition: barycentric domain B_i associated with each vertex P_i of T^h.

$$B_i = \bigcup_{P_i \subset B_i^{\Delta}} B_i^{\Delta} .$$

Definition: $Y^h = Y^h(\Omega;T^h)$ and $Y_o^h = Y_o^h(\Omega;T^h)$ (piecewise linear finite element spaces).

$$Y^h = \{\hat{\phi};\ \hat{\phi}\in C^o(\bar{\Omega}) \text{ and } \hat{\phi}|_{\Delta} = \text{linear for each simplex } \Delta\in T^h\}$$

$$Y_o^h = \{\hat{\phi};\ \hat{\phi}\in Y^h \text{ and } \hat{\phi}|_{\Gamma} = 0\}$$

Definition: $X^h = X^h(\Omega;T^h)$ and $X_o^h = X_o^h(\Omega;T^h)$ (piecewise constant finite element spaces).

$$X^h = \{\bar{\phi};\ \bar{\phi}|_{B_i} = \text{constant for each barycentric domain } B_i \text{ of } T^h\}$$

$$X_o^h = \{\bar{\phi};\ \bar{\phi}\in X^h \text{ and } \bar{\phi}|_{\Gamma} = 0\}$$

Definition: We say that functions $\hat{\phi}\in Y^h$ and $\bar{\phi}\in X^h$ are *associative* if $\hat{\phi}(P_i) = \bar{\phi}(P_i)$ for all vertices P_i of T^h.

Definition: Let $\hat{\phi}_i\in Y^h$ and $\bar{\phi}_i\in X^h$ be such that

$$\hat{\phi}_i(P_j) = \delta_{ij}, \quad \text{and} \quad \bar{\phi}_i(P_j) = \delta_{ij} \qquad (\delta_{ij}\text{: Kronecker's delta}). \tag{1.1}$$

The sets $\{\hat{\phi}_i\}_{i=1}^N$ and $\{\bar{\phi}_i\}_{i=1}^N$ form the bases of Y^h and X^h, respectively. The space X^h is sometimes called "the lumped space", since it is used to define the so-called lumped mass type approximation.

1.2 Acuteness Σ of the Triangulation

Definition: κ_{min} (κ_{max}) is defined to be the minimum (maximum) perpendicular length of all the simplices Δ of T^h.

Definition: Acuteness σ_{Δ} of a simplex Δ of T^h is defined as:

$$\sigma_{\Delta} = \min_{i\neq j}\{-\cos(\nabla\hat{\lambda}_j,\nabla\hat{\lambda}_i)\} = \min_{i\neq j}\frac{-(\nabla\hat{\lambda}_j,\nabla\hat{\lambda}_i)_E}{|\nabla\hat{\lambda}_j|_E|\nabla\hat{\lambda}_i|_E} \tag{1.2}$$

where $\hat{\lambda}_i$ $(i=1,..,m+1)$ are the barycentric coordinates of a point $x\in\Delta$ with respect to the point P_i. The vector $\nabla\hat{\lambda}_j$, $j=1,..,m+1$, denotes the gradient of the function $\hat{\lambda}_j$, $j=1,..,m+1$.and $(\ ,\ )_E$ and $|\ |_E$ respectively denote the Euclidean inner product and Euclidean norm in R^m.

Note: $\hat{\lambda}_i = \hat{\phi}_i|_{\Delta}$ $(i=1,..,m+1)$.

Definition: Acuteness Σ of a triangulation T^h is defined to be the minimum of all the σ_Δ , i.e.,

(1.3) $$\Sigma = \min_{\Delta \in T^h} \sigma_\Delta .$$

Definition: We say that a triangulation is of acute type if $\Sigma \geq 0$, and of strictly acute type if $\Sigma > o$.

Example: m=1: By definition, $\Sigma = \sigma_\Delta = 1$.

m=2: Let η_i (i=1,2,3) be the three angles of a triangle Δ. Then,

(1.4) $$\sigma_\Delta = \min_i \{\cos(\eta_i)\}$$

m=3: Let $(i,j,k,\ell)=(1,2,3,4)$ be the vertices of a tetrahedron Δ, and η_{ij} be the angle made by the faces $P_i P_k P_\ell$ and $P_j P_k P_\ell$. Then,

(1.5) $$\sigma_\Delta = \min_{i,j} \{\cos(\eta_{ij})\}.$$

Hence, σ_Δ is of acute type if and only if $\eta_{ij} \leq \pi/2$.

1.3 Estimation of Stability Functions

Stability functions will play an essential role in establishing energy estimates of time-discrete finite element schemes.

Definition: Stability functions $\gamma_1 = \gamma_1(m;T^h)$ and $\gamma_2 = \gamma_2(m;T^h)$ are defined as:

(1.6) $$\gamma_1^2 = \kappa_{\min}^2 \cdot \sup_{\hat{w} \in Y^h} \sum_{\ell=1}^{m} \left\| \frac{\partial \hat{w}}{\partial x_\ell} \right\|^2 \Big/ \|\hat{w}\|^2$$

(1.7) $$\gamma_2^2 = \kappa_{\min}^2 \cdot \sup_{\hat{w} \sim \bar{w}} \sum_{\ell=1}^{m} \left\| \frac{\partial \hat{w}}{\partial x_\ell} \right\|^2 \Big/ \|\bar{w}\|^2 \qquad (\hat{w} \in Y^h,\ \bar{w} \in X^h).$$

An estimate of the stability functions γ_1 and γ_2 are given in [1], [2] for the cases m=1 and m=2. Here, we give an improved estimate of γ_1 and γ_2 with arbitrary space dimension m.

Theorem 1. For any $\hat{w} \in Y^h$ and $\bar{w} \in X^h$ such that $\hat{w} \sim \bar{w}$, it holds that

(1.8) $$\sum_{\ell=1}^{m} \left\| \frac{\partial \hat{w}}{\partial x_\ell} \right\|^2 \leq \frac{A_m}{\kappa_{\min}^2} (m+1)(m+2) \|\hat{w}\|^2, \text{ and}$$

(1.9) $$\sum_{\ell=1}^{m} \left\| \frac{\partial \hat{w}}{\partial x_\ell} \right\|^2 \leq \frac{A_m}{\kappa_{\min}^2} (m+1) \|\bar{w}\|^2,$$

where the constant A_m is estimated as

$$(1.10)\qquad A_m = \begin{cases} 2 & (\Sigma \geq 0) \\ m+1 & (\Sigma < 0). \end{cases}$$

To prove Theorem 1, we first show the following

Lemma 1. For the basis functions $\hat{\phi}_i$ and $\bar{\phi}_i$ $(i=1,..,\bar{N})$, it holds that

$$(1.11)\qquad \sum_{\ell=1}^{m} \left\| \frac{\partial \hat{\phi}_i}{\partial x_\ell} \right\|^2 \leq \frac{1}{\kappa_{min}^2} \frac{(m+1)(m+2)}{2} \|\hat{\phi}_i\|^2 \quad (i=1,..,\bar{N}),$$

and

$$(1.12)\qquad \sum_{\ell=1}^{m} \left\| \frac{\partial \hat{\phi}_i}{\partial x_\ell} \right\|^2 \leq \frac{1}{\kappa_{min}^2} (m+1) \|\bar{\phi}_i\|^2 \qquad (i=1,..,\bar{N}).$$

Proof of Lemma 1. Let Δ be an m-simplex of T^h. Let P_i and κ_i be a vertex of Δ and the length of the perpendicular line from P_i, respectively. Then, (1.11) and (1.12) immediately follow from

$$\|\hat{\phi}_i\|_\Delta^2 = \frac{2\ \mathrm{mes}(\Delta)}{(m+1)(m+2)},$$

$$\|\bar{\phi}_i\|_\Delta^2 = \frac{\mathrm{mes}(\Delta)}{(m+1)}, \quad \text{and}$$

$$\sum_{\ell=1}^{m} \left\| \frac{\partial \hat{\phi}_i}{\partial x_\ell} \right\|_\Delta = \frac{\mathrm{mes}(\Delta)}{\kappa_i^2} \leq \frac{\mathrm{mes}(\Delta)}{\kappa_{min}^2}$$

where $\|\cdot\|_\Delta$ implies the L^2-integration over the simplex Δ.

Proof of Theorem 1. Suppose that $\hat{w}$ and $\bar{w}$ are expressed as

$$\hat{w} = \sum_{i=1}^{m+1} w_i \cdot \hat{\phi}_i \quad \text{and} \quad \bar{w} = \sum_{i=1}^{m+1} w_i \cdot \bar{\phi}_i \qquad \text{on the simplex } \Delta.$$

Then, we have that

$$\|\bar{w}\|_\Delta^2 = \frac{\mathrm{mes}(\Delta)}{m+1} \sum_{i=1}^{m+1} w_i^2, \quad \text{and}$$

$$\|\hat{w}\|_\Delta^2 = \frac{2\ \mathrm{mes}(\Delta)}{(m+1)(m+2)} \left(\sum_{i=1}^{m+1} w_i^2 + \sum_{i=1}^{m+1} \sum_{j=i+1}^{m+1} w_i w_j \right)$$

$$\geq \frac{2\ \mathrm{mes}(\Delta)}{(m+1)(m+2)} \left(\frac{1}{2} \sum_{i=1}^{m+1} w_i^2 \right).$$

On the other hand, we can show the inequality

$$(*)\qquad \sum_{\ell=1}^{m} \left\| \frac{\partial \hat{w}}{\partial x_\ell} \right\|_\Delta^2 \leq \frac{A_m\ \mathrm{mes}(\Delta)}{\kappa_{min}^2} \sum_{i=1}^{m+1} w_i^2,$$

where A_m is the constant given by (1.10). This is shown as follows:

$$\sum_{\ell=1}^{m} \left(\frac{\partial \hat{w}}{\partial x_\ell}\right)^2 = \sum_{i=1}^{m+1} w_i^2 |\nabla\hat{\phi}_i|_E^2 + \sum_{\substack{i,j=1 \\ i\neq j}}^{m+1} w_i w_j \, (\nabla\hat{\phi}_i, \nabla\hat{\phi}_j)_E .$$

The first term of the right hand is bounded by $\sum_{i=1}^{m+1} w_i^2/\kappa_{min}^2$, and the second term is written as $\{W\}^T[K]\{W\}$, where $\{W\}^T = (w_1, \ldots, w_{m+1})$, and $K_{ii}=0$, $K_{ij} = (\nabla\hat{\phi}_i, \nabla\hat{\phi}_j)_E$ $(i\neq j)$. We estimate the spectral radius ρ of the matrix $[K]$. We make use of the fact that $\sum_{j=1}^{m+1} \nabla\hat{\phi}_j = 0$ (since $\sum_{j=1}^{m+1} \hat{\phi}_j \equiv 1$ on Δ). Now, the largest Gerschgorin radius r of $[K]$, i.e., $r = \max_i \sum_{j\neq i}^{m+1} |K_{ij}|$ is estimated as

$$r \begin{cases} = \max_i \sum_{j\neq i}^{m+1} -(\nabla\phi_i, \nabla\phi_j)_E = \max_i |\nabla\phi_i|_E^2 \le 1/\kappa_{min}^2 & (\Sigma \ge 0) \\ \le \max_i \sum_{j\neq i}^{m+1} |\nabla\phi_i|_E |\nabla\phi_j|_E = \max \sum_{j\neq i}^{m+1} \dfrac{1}{\kappa_i \kappa_j} \le m/\kappa_{min}^2 & (\Sigma < 0). \end{cases}$$

Hence, the inequality (*) follows, completing the proof of Theorem 1.

2. Approximation of Second Order Hyperbolic Equations and of Equation of Elastodynamics

In this section, we discuss briefly the finite element approximation of hyperbolic equations of second order and the equation of elastodynamics. The main concern is the question of stability in the sense of energy for the scheme under consideration. We note that the convergence of the finite element approximate solution follows almost automatically from the stability under appropriate assumptions on the exact solution. See, [1] and [2].

2.1 Model Problems and Accociated Energy Forms

The domain Ω is assumed to be an m-dimensional polyhedron. We take the following as the model problems:

Problem 1. *Wave Equation* $(m \ge 1)$

$$\begin{array}{lrcll} (2.1) & \dfrac{\partial^2 u}{\partial t^2} & = & \Delta u + f & \text{in } \Omega \times (0,T] \\ (2.2) & u & = & 0 & \text{on } \Gamma_D \times (0,T] \\ (2.3) & \dfrac{\partial u}{\partial n} & = & 0 & \text{on } \Gamma_N \ (0,T] \end{array}$$

where $\Gamma = \bar{\Gamma}_D \cup \Gamma_N$, $\Gamma_D \cap \Gamma_N = \phi$ and n is the outward normal direction.

Problem 2. Equation of Elastodynamics (m = 2 or 3)

(2.4) $$\rho \frac{\partial^2 u_i}{\partial t^2} = \sum_{j=1}^{m} \frac{\partial}{\partial x_j} \tau_{ij} + f_i, \quad i=1,..,m,$$

(2.5) $$\tau_{ij}[u] = \sum_{k,\ell=1}^{m} C_{ijk\ell}\, \varepsilon_{k\ell}[u], \quad i,j=1,..,m,$$

(2.6) $$\varepsilon_{ij}[u] = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \varepsilon_{ji}[u], \quad i,j=1,..,m,$$

with the symmetry assumption on the generalized Hooke coefficients:

(2.7) $$C_{ijk\ell} = C_{jik\ell} = C_{k\ell ij}$$

also, for any symmetric tensor ε_{ij},

(2.8) $$\sum_{ijk\ell}^{m} C_{ijk\ell}\varepsilon_{ij}\, \varepsilon_{k\ell} \geq \mu_o \sum_{ij}^{m} \varepsilon_{ij}^2 .$$

Note. (the isotropic case)

(2.9) $$C_{ijk\ell} = \lambda\delta_{ij}\delta_{k\ell} + \mu(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk})$$

where λ and μ are the Lamé coefficients.

with the boundary condition:

(2.10) $$u_i = 0, \quad i=1,..,m \quad \text{on } \Gamma_D \times (0,T]$$

(2.11) $$\sum_{j=1}^{m} \tau_{ij} n_j = 0, \quad i=1,..,m \quad \text{on } \Gamma_N \times (0,T].$$

Associated Bilinear and Quadratic Forms

We introduce a bilinear and a quadratic form corresponding to each problem. *Problem 1.*

(2.12) $$W[u,u\#] = \sum_{i=1}^{m} \left\langle \frac{\partial u}{\partial x_i}, \frac{\partial u\#}{\partial x_i} \right\rangle, \qquad u,\ u\# \in H^1(\Omega)$$

(2.13) $$W[\,u\,] = W[u,u] = \sum_{i=1}^{m} \left\| \frac{\partial u}{\partial x_i} \right\|^2,$$

Problem 2.

(2.14) $$W[u,u\#] = \sum_{i,k=1}^{m} \langle \tau_{ik}[u],\ \varepsilon_{ik}[u\#] \rangle, \qquad u,\ u\# \in (H^1(\Omega))^m$$

(2.15) $$W[\,u\,] = W[u,u] = \sum_{i,k=1}^{m} \langle \tau_{ik}[u],\ \varepsilon_{ik}[u] \rangle, \quad u \in (H^1(\Omega))^m.$$

Remark: Korn's Inequality

Under appropriate assumptions on the boundary condition, the so-called Korn's inequality holds:

$$(2.16) \qquad \sum_{i,k=1}^{m} \left\| \frac{\partial u_i}{\partial x_k} \right\|^2 \leq C \cdot W[\, u \,]$$

where C is a constant depending only on the region Ω. Hence, W[u] can be an equivalent norm in the space $(W_2^{(1)})^m$. See K.O. Friedrichs[5] for the case of isotropic elasticity with the first boundary condition, I.Hlavacek-J. Necas [6] or G. Duvaut-J.L. Lions [7] for general cases.

2.2 Stability Function for Energy Quadratic Form

Definition: Let

$$(2.17) \qquad \nu_o = \frac{1}{2} \max_{i,j} \sum_{k,\ell=1}^{m} C_{ijk\ell}.$$

Remark: For the isotropic case (2.9), ν_o turns out to be

$$(2.18) \qquad \nu_o = \lambda + \frac{m}{2}\mu,$$

where λ and μ are the Lamé coefficients.

Theorem 2. For any associative functions $\hat{w} \in (Y^h)^m$ and $\bar{w} \in (X^h)^m$, it holds that

$$(2.19)_a \qquad W[\, \hat{w} \,] \leq \frac{\nu_o}{\rho} \frac{2(\gamma_C^m)^2}{\kappa_{min}^2} \frac{\rho}{2} \sum_{i=1}^{m} \|\hat{w}_i\|^2,$$

and

$$(2.20)_a \qquad W[\, \hat{w} \,] \leq \frac{\nu_o}{\rho} \frac{2(\gamma_L^m)^2}{\kappa_{min}^2} \frac{\rho}{2} \sum_{i=1}^{m} \|\bar{w}_i\|^2,$$

where the constants γ_C^m and γ_L^m are given by

$$(2.19)_b \qquad \gamma_C^m = \{A_m\ (m+1)(m+2)\}^{1/2}$$

$$(2.20)_b \qquad \gamma_L^m = \{A_m\ (m+1)\}^{1/2}$$

in which A_m is the quantity defined by (1.10).

2.3 Finite Element Scheme. Definition†

Let Y_*^h (resp. X_*^h) be the set of functions $\hat{\phi} \in Y^h$ (resp. $\bar{\phi} \in X^h$) which satisfy the geometric boundary condition, i.e., $\hat{\phi} = 0$ (resp. $\bar{\phi}=0$) on Γ_D.

Definition: Finite Element Continuous-Time Scheme of Consistent Mass Type

Seek a function $\hat{v} = \hat{v}(\cdot,t) \in (Y_*^h)^m$ for each t, $0 < t \le T$, such that

$$(2.21) \qquad \langle \rho \frac{\partial^2}{\partial t^2} \hat{v}_i, \hat{\phi} \rangle + \sum_{j=1}^{m} \langle \hat{\tau}_{ij}, \frac{\partial}{\partial x_j} \hat{\phi} \rangle = \langle f_i, \hat{\phi} \rangle, \quad 0 < t \le T, \quad i=1,..,m,$$

for any test function $\hat{\phi} \in Y_*^h$, where the finite element stress tensor $\hat{\tau}_{ij}$ is defined to be

$$(2.22) \qquad \hat{\tau}_{ij} = \tau_{ij}[\hat{v}] = \sum_{k,\ell=1}^{m} C_{ijk\ell} \hat{\varepsilon}_{k\ell}, \ i,j=1,..,m,$$

$$(2.23) \qquad \hat{\varepsilon}_{ij} = \varepsilon_{ij}[\hat{v}] = \frac{1}{2} \left(\frac{\partial \hat{v}_i}{\partial x_j} + \frac{\partial \hat{v}_j}{\partial x_i} \right), \ i,j=1,..,m.$$

Definition: Finite Element Continuous-Time Scheme of Lumped Mass Type

Seek associative functions $\hat{v} = \hat{v}(\cdot,t) \in (Y_*^h)^m$ and $\bar{v} = \bar{v}(\cdot,t) \in (X_*^h)^m$ for each t, $0 < t \le T$, such that

$$(2.24) \qquad \langle \rho \frac{\partial^2}{\partial t^2} \bar{v}_i, \bar{\phi} \rangle + \sum_{j=1}^{m} \langle \hat{\tau}_{ij}, \frac{\partial}{\partial x_j} \hat{\phi} \rangle = \langle f_i, \bar{\phi} \rangle, \quad 0 < t \le T, \ i=1,..,m,$$

for any associative test functions $\hat{\phi} \in Y_*^h$ and $\bar{\phi} \in X_*^h$.

It is easily seen that the "continuous-time schemes" (2.21) and (2.24) are reduced to systems of ordinary differential equations in terms of the nodal displacement vector $\{V_i\} = (v_{i,1}, \dots, v_{i,J})^T$, i=1,..,m, where J is the number of nodes which are in Ω or on Γ_N. For detail, please refer [2]. For practical numerical computations, we need discretize the schemes with respect to time. Let $t_n = n\Delta t$, where $\Delta t = T/p$ and p is an integer. Let $\hat{v}^n$ and $\bar{v}^n$ denote $\hat{v}(\cdot,t_n)$ and $\bar{v}(\cdot,t_n)$, respectively.

† Here, we give the definition of the finite element schemes only for the Problem 2. Definition for the Problem 1 is similarly given.

Definition: Finite Element β *Scheme of Consistent Mass Type* (β ≥ 0)

$$(2.25)\quad \langle \rho D_t D_{\bar t} \hat{v}_i^n, \hat{\phi}\rangle + \sum_{j=1}^{m} \langle \hat{\tau}_{ij}^n, \frac{\partial}{\partial x_j}\hat{\phi}\rangle + \beta\cdot\Delta t^2 \sum_{j=1}^{m} \langle D_t D_{\bar t} \hat{\tau}_{ij}^n, \frac{\partial}{\partial x_j}\hat{\phi}\rangle = \langle f_i^n, \hat{\phi}\rangle ,$$

$$n = 1,2,..,p-1,\ i=1,..,m,$$

for any test function $\hat{\phi}\in Y_*^h$.

Definition: Finite Element β *Scheme of Lumped Mass Type* (β ≥ 0)

$$(2.26)\quad \langle \rho D_t D_{\bar t} \bar{v}_i^n, \bar{\phi}\rangle + \sum_{j=1}^{m} \langle \hat{\tau}_{ij}^n, \frac{\partial}{\partial x_j}\hat{\phi}\rangle + \beta\cdot\Delta t^2 \sum_{j=1}^{m} \langle D_t D_{\bar t} \hat{\tau}_{ij}^n, \frac{\partial}{\partial x_j}\hat{\phi}\rangle = \langle f_i^n, \bar{\phi}\rangle ,$$

$$n = 1,2,..,p-1,\ i=1,..,m,$$

for any associative test functions $\hat{\phi}\in Y_*^h$ and $\bar{\phi}\in X_*^h$.

Remark: A special case β=0 is the so-called central difference scheme. It is noted that the central difference scheme with lumped mass type approximation is of explicit in the sense that it can be solved step-by-step explicitly, while all the other cases are of implicit type.

2.4 Energy Stability of the Finite Element Schemes

Obviously, for the "continuous-time" schemes of both consistent mass and lumped mass types the *a priori* energy estimate of the form

$$(2.27)\qquad (\tilde{K} + \hat{W})(T) \le C\,\{(\tilde{K} + \hat{W})(0) + \int_0^T \|f\|^2 dt\}$$

holds, where $\hat{W}(t) = W[\hat{v}(\cdot,t)]$ (see Eq.(2.13) or (2.15)), and $\tilde{K}(t)$ denotes either

$$\tilde{K}(t) = K[\hat{v}(\cdot,t)] = \frac{\rho}{2}\sum_{i=1}^{m}\left\|\frac{\partial \hat{v}_i}{\partial t}\right\|^2 \quad \text{for the consistent mass case,}$$

or

$$\tilde{K}(t) = K[\bar{v}(\cdot,t)] = \frac{\rho}{2}\sum_{i=1}^{m}\left\|\frac{\partial \bar{v}_i}{\partial t}\right\|^2 \quad \text{for the lumped mass case.}$$

With regards to the time-discrete scheme, that is, the β scheme (2.25) or (2.26), the discrete analogue of the energy inequality does not hold, unless some restriction on the ratio $(\Delta t/\kappa_{min})$ is fulfilled. In fact, we can show the following

Theorem 3. (Energy Stability of the β Scheme of Consistent Mass Type)

The β scheme of consistent mass type (2.25) is unconditionally stable if $\beta > 1/4$, or stable under the condition

$$(2.28) \qquad \frac{\Delta t}{\kappa_{min}} \le \frac{1}{\sqrt{\nu_o/\rho}} \frac{\sqrt{2}(1-\zeta)}{\sqrt{1-4(1-\zeta)\beta}} \frac{1}{\gamma_C^m} \quad (\forall \zeta > 0)$$

if $\beta \le 1/4$, where ζ is any positive constant, γ_C^m is the quantity given by $(2.19)_b$, in the sense that, in both cases, the following estimate holds:

$$(2.29) \qquad \frac{\rho}{2} \sum_{i=1}^{m} \|D_{\bar{t}} \hat{v}_i^r\|^2 + W[\hat{v}^r] \le C\{\frac{\rho}{2} \sum_{i=1}^{m} \|D_t \hat{v}_i^o\|^2 + W[\hat{v}^o] + \sum_{n=1}^{r-1} \Delta t \sum_{i=1}^{m} \|f_i^n\|^2\}$$

$r = 2,3,..,p$; where $p \cdot \Delta t = T$ and C is a constant independent of Δt and the triangulation.

Theorem 4. (Energy Stability of the β Scheme of Lumped Mass Type)

The β scheme of lumped mass type (2.26) is unconditionally stable if $\beta > 1/4$, or stable under the condition

$$(2.29) \qquad \frac{\Delta t}{\kappa_{min}} \le \frac{1}{\sqrt{\nu_o/\rho}} \frac{\sqrt{2}(1-\zeta)}{\sqrt{1-4(1-\zeta)\beta}} \frac{1}{\gamma_L^m} \quad (\forall \zeta > 0)$$

if $\beta \le 1/4$, where ζ is any positive constant, γ_L^m is the constant given by $(2.20)_b$, in the sense that, in both cases, the following estimate holds:

$$(2.31) \qquad \frac{\rho}{2} \sum_{i=1}^{m} \|D_{\bar{t}} \bar{v}_i^r\|^2 + W[\hat{v}^r] \le C\{\frac{\rho}{2} \sum_{i=1}^{m} \|D_t \bar{v}_i^o\|^2 + W[\hat{v}^o] + \sum_{n=1}^{r-1} \Delta t \sum_{i=1}^{m} \|f_i^n\|^2\},$$

$r = 2,3,..,p$; where $p \cdot \Delta t = T$ and C is a constant independent of Δt and the triangulation.

Remark: For the proof, see [1], [2] or [4], where the stability conditions and the convergence in energy norm are discussed for two-dimensional elastodynamics and wave equation. These Theorems give an improved estimate of those results, including three-dimensional cases also. It is also noted that the convergence of those schemes to the solution of the original equation can also be shown under the assumption that the stability conditions are satisfied.

The key to those discussions is the estimate of the stability function γ_C^m or γ_L^m given by Theorem 2.

2.5 A Remark on Linear Visco-Elastodynamics

An interesting generalization of the results in the previous section is the problem of linear visco-elastodynamics.

The linear visco-elasticity with materials of long memory is characterized by

$$\tau_{ij}(t) = \sum_{k,\ell=1}^{m} C_{ijk\ell}\, \varepsilon_{k\ell}(t) + \int_{o}^{t} B_{ijk\ell}(t-s)\, \varepsilon_{k\ell}(s)\, ds, \tag{2.32}$$

where $C_{ijk\ell}$ are the generalized Hooke coefficients which satisfy the conditions (2.7) and (2.8), and $B_{ijk\ell} = B_{ijk\ell}(x,t)$ are bounded functions of x and t, such that

$$(2.33)_a \qquad B_{ijk\ell} = B_{jik\ell},$$

and that

$$(2.33)_b \qquad B_{ijk\ell},\ \partial B_{ijk\ell}/\partial t,\ \partial^2 B_{ijk\ell}/\partial t^2 \in L^{\infty}(\Omega\times(o,T))$$

The equation of motion of linear visco-elastodynamics is again given by Eq.(2.4), i.e.,

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \sum_{j=1}^{m} \frac{\partial}{\partial x_j} \tau_{ij}[u] + f_i, \quad i = 1,..,m, \tag{2.34}$$

with the stress-strain relations (2.6).

For the sake of illustration, let us consider the phenomena of wave propagation on a one-dimensional material of long memory. Eq.(2.32) is reduced to

$$\tau[u] = C\cdot\varepsilon[u] + \int_o^t B(t-s)\cdot\varepsilon[u](s)ds. \tag{2.35}$$

Since $\varepsilon[u] = \partial u/\partial x$, Eq.(2.34) is written as

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(C \frac{\partial u}{\partial x}\right) + \int_o^t \frac{\partial}{\partial x}\left(B(t-s) \frac{\partial u}{\partial x}(s)\right) ds + f, \tag{2.36}$$

which is a differential-integral equation.

Existence and uniqueness of the solution of Eqs.(2.32)-(2.34) are discussed in Duvaut-Lions [7].

Let us construct a finite element approximate scheme for Eqs.(2.32)-(2.34):

$$(2.37)\quad \begin{aligned}\hat{\tau}_{ij}^{n} &= \sum_{k,\ell}^{m} C_{ijk\ell}\,\hat{\varepsilon}_{k\ell}^{n} + \sum_{s=0}^{n}\Delta t \sum_{k,\ell}^{m} B_{ijk\ell}^{n-s}\,\hat{\varepsilon}_{k\ell}^{s}\\ &\equiv (\hat{\tau}_{o})_{ij}^{n} \qquad + (\hat{\tau}_{1})_{ij}^{n}.\end{aligned}$$

Corresponding finite element β scheme may be obtained by substituting this expression into Eq.(2.25) or Eq.(2.26).

Remark: In practical computations, the expression (2.37) may cause difficulty in storage of those element-wise strain data $\hat{\varepsilon}_{k\ell}^{s}$, s=0,1,..,n. In many cases, however, the functions $B_{ijk\ell}(.,t)$ take an exponential form in t, and which makes it possible to compute $(\hat{\tau}_1)_{ij}^{n}$ recursively from $(\hat{\tau}_1)_{ij}^{n-1}$ and $\hat{\varepsilon}_{k\ell}^{n}$.

Our main result is that the introduction of long memory term does not destroy the energy stability. In fact, we have the following

Theorem 5. (Energy Stability of the Finite Element β Scheme for Linear Visco-Elastodynamics)

The finite element β scheme (2.25) (resp.(2.26)), with the memory term (2.37) is stable in energy, i.e., in the sense of Eq.(2.29) (resp.Eq.(2.31)) under the same condition on the ratio $\Delta t/\kappa_{min}$, i.e., Eq.(2.28) (resp. Eq. (2.29)).

Since all the novelty comes from the long memory term $\hat{\tau}_1^n$, we show only a brief discussion about the treatment of this term: We let

$$\Pi[B^n;u,v] = \sum_{ijk\ell} \langle B_{ijk\ell}^{n}\,\varepsilon_{k\ell}(u),\ \varepsilon_{ij}(v)\rangle ,$$

and

$$\sup(B) = \max_{ijk\ell}\ \sup_{x,t}\left|B_{ijk\ell}(x,t)\right|,\quad \sup(B') = \max_{ijk\ell}\ \sup_{x,t}\left|\frac{\partial}{\partial t} B_{ijk\ell}(x,t)\right|.$$

What is necessary is to show the inequality

$$(2.38)\quad \begin{aligned}&\sum_{q=0}^{p-1}\Delta t \sum_{s=0}^{q} \Delta t\cdot\Pi[B^{q-s};\hat{v}^{s},\tfrac{1}{2}(D_t+D_{\bar t})\hat{v}^{q}]\\ &\quad \le \delta\cdot C_1\{\ \frac{\rho}{2}\sum_{i=1}^{m}\|D_{\bar t}\hat{v}^{p}\|^{2} + W[\hat{v}^{p}]\ \} + C_2(1+1/\delta)\sum_{s=0}^{p}\Delta t\cdot W[\hat{v}^{s}],\\ &(\forall\delta > 0).\end{aligned}$$

When this inequality is established, then the first term is absorbed by the energy at the left hand of Eq.(2.28) for small $\delta > 0$, and the second term can be, thanks to the discrete Gronwall lemma, eliminated, which leads to the desired energy stability. Now;

$$\sum_{q=0}^{p-1}\Delta t\sum_{s=0}^{q}\Delta t\cdot\Pi[B^{q-s};\hat{v}^s,\tfrac{1}{2}(D_t+D_{\bar{t}})\hat{v}^q]$$

$$=\sum_{s=0}^{p-1}\Delta t\sum_{q=s}^{p-1}\Delta t\cdot\Pi[B^{q-s};\hat{v}^s,\tfrac{1}{2}(D_t+D_{\bar{t}})\hat{v}^q]$$

$$=-\sum_{s=0}^{p-1}\Delta t\sum_{q=s}^{p-1}\Delta t\cdot\Pi[\tfrac{1}{2}(D_t+D_{\bar{t}})_{(q)}B^{q-s};\hat{v}^s,\hat{v}^q]$$

$$+\frac{1}{2}\sum_{s=0}^{p-1}\Delta t\ \{\Pi[B^{p-1-s};\hat{v}^s,\hat{v}^p]+\Pi[B^{p-2-s};\hat{v}^s,\hat{v}^p]-\Pi[B^{p-1-s};\hat{v}^s,\Delta tD_{\bar{t}}\hat{v}^p]\}$$

$$-\frac{1}{2}\sum_{s=0}^{p-1}\Delta t\ \{\Pi[B^0;\hat{v}^s,\hat{v}^s]+\Pi[B^{-1};\hat{v}^s,\hat{v}^{s+1}]\}$$

$$=\ (I)+(II)+(III)+(IV)+(V)+(VI).$$

With the help of Korn's inequality (2.16) and Theorem 2, we see that

$$|\ I\ |\ \le\ \sup(B')\cdot T\sum_{s=0}^{p-1}\Delta t\sum_{i,j}\|\varepsilon_{ij}(\hat{v}^s)\|^2\ \le\ C\sum_{s=0}^{p-1}\Delta t\cdot W[\hat{v}^s],$$

$$|II|+|III|\le\ \sup(B)\sum_{i,j}\|\varepsilon_{ij}(\hat{v}^p)\|\sum_{s=0}^{p-1}\Delta t\sum_{i,j}\|\varepsilon_{ij}(\hat{v}^s)\|$$

$$\le\ \delta\cdot C\cdot W[\hat{v}^p]+\frac{C'}{\delta}\sum_{s=0}^{p-1}\Delta t\cdot W[\hat{v}^s]\ ,$$

$$|V|+|VI|\ \le\ \sup(B)\sum_{s=0}^{p}\Delta t\sum_{i,j}\|\varepsilon_{ij}(\hat{v}^s)\|^2\ \le\ C\sum_{s=0}^{p}\Delta t\cdot W[\hat{v}^s],$$

and

$$|IV|\ \le\ \sup(B)\sum_{s=0}^{p-1}\Delta t\sum_{i,j}\|\varepsilon_{ij}(\hat{v}^s)\|\cdot\|\varepsilon_{ij}(\Delta t\cdot D_{\bar{t}}\hat{v}^p)\|$$

$$\le\ C\ \{\frac{1}{\delta}\sum_{s=0}^{p-1}\Delta t\cdot W[\hat{v}^s]+\delta\cdot T\cdot W[\Delta t\cdot D_{\bar{t}}\hat{v}^p]\}$$

$$\le\ C\ \{\frac{1}{\delta}\sum_{s=0}^{p-1}\Delta t\cdot W[\hat{v}^s]+\delta\cdot T\frac{\nu_o}{\rho}\ \frac{\Delta t^2}{\kappa_{min}^2}(\gamma_C^m)^2\ \rho\sum_{i=1}^{m}\|D_{\bar{t}}\hat{v}_i^p\|^2\}.$$

Combining the above estimates, we obtain the inequality (2.38).

3. Parabolic Problems. Discrete Maximum Principle and L^2-sense Stability

It is an interesting question whether or not a parabolic scheme constructed in a Finite Element-Galerkin manner still retains the maximum principle property. In this section, we investigate this discrete maximum principle property, as well as stability in the mean square sense.

3.1 Finite Element Scheme

We consider the Dirichlet problem of the simplest parabolic equation, that is, the heat equation:

$$(3.1)\quad \frac{\partial u}{\partial t} = \alpha_o \sum_{\ell=1}^{m} \frac{\partial^2 u}{\partial x_\ell^2} + f \quad \text{in } \Omega x(0,T], \quad (\alpha_o = \text{constant} > 0),$$

$$(3.2)\quad u = g \quad \text{on } \Gamma x(0,T],$$

subject to the initial condition

$$(3.3)\quad u\big|_{t=0} = u_o \quad \text{in } \Omega.$$

Here, appropriate smoothness of the data f, g and u_o is assumed. The weak form corresponding to the differential problem (3.1)-(3.2) is formulated as:

Seek a function $u \in H^1(\Omega)$ such that $u-g \in H_o^1(\Omega)$, and that

$$(3.4)\quad \langle\frac{\partial u}{\partial t}, \phi\rangle + \sum_{\ell=1}^{m} \langle\alpha_o \frac{\partial u}{\partial x_\ell}, \frac{\partial \phi}{\partial x_\ell}\rangle = \langle f, \phi\rangle,\ 0 < t \leq T, \text{ for any } \phi \in H_o^1(\Omega).$$

The finite element scheme of consistent mass type is defined to be:

Seek a function $\hat{v} = \hat{v}(\cdot,t) \in Y^h$ such that

$$(3.5)_a\quad \hat{v} - \hat{g} \in Y_o^h \text{ and}$$

$$(3.5)_b\quad \langle\frac{\partial v}{\partial t}, \hat{\phi}\rangle + \sum_{\ell=1}^{m} \langle\alpha_o \frac{\partial \hat{v}}{\partial x_\ell}, \frac{\partial \hat{\phi}}{\partial x_\ell}\rangle = \langle f, \hat{\phi}\rangle,\ 0 < t \leq T, \text{ for any } \hat{\phi} \in Y_o^h.$$

Similarly, *the finite element scheme of lumped mass type* is defined with the aid of the lumped spaces X^h and X_o^h: Seek a function $\hat{v} \in Y^h$ and its associative function $\bar{v} \in X^h$ such that

$$(3.6)_a\quad \hat{v} - \hat{g} \in Y_o^h \text{ and}$$

$$(3.6)_b\quad \langle\frac{\partial \bar{v}}{\partial t}, \bar{\phi}\rangle + \sum_{\ell=1}^{m} \langle\alpha_o \frac{\partial \hat{v}}{\partial x_\ell}, \frac{\partial \hat{\phi}}{\partial x_\ell}\rangle = \langle f, \bar{\phi}\rangle,\ 0 < t \leq T,$$

for any $\hat{\phi} \in Y_o^h$ and its associative function $\bar{\phi} \in X_o^h$.

Here, in Eqs.(3.5) and (3.6), $\hat{g}$ is defined to be a function of Y^h which coincides with g at each node points.†

As in the hyperbolic case, the scheme (3.5) or (3.6) is reduced to a system of ordinary differential equations

$$(3.7)_a \qquad [M^o;M^\partial]\,\frac{d}{dt}\begin{Bmatrix}V\\G\end{Bmatrix} + [K^o;K^\partial]\begin{Bmatrix}V\\G\end{Bmatrix} = \{F\}, 0 < t \le T,$$

where $[M^o;M^\partial]$ and $[K^o;K^\partial]$ are N by $\bar{N}$ matrices (N: number of nodes in Ω, $\bar{N}$: number of nodes in $\bar{\Omega}$) with

$$(3.7)_b \qquad M_{ij} = \begin{cases} \langle\hat{\phi}_j, \hat{\phi}_i\rangle & \text{(consistent mass)} \\ \langle\bar{\phi}_j, \bar{\phi}_i\rangle & \text{(lumped mass)} \end{cases} \qquad \begin{pmatrix} 1 \le i \le N \\ 1 \le j \le \bar{N} \end{pmatrix},$$

$$(3.7)_c \qquad K_{ij} = \sum_{\ell=1}^{m} \langle\alpha_o \frac{\partial\hat{\phi}_j}{\partial x_\ell}, \frac{\partial\hat{\phi}_i}{\partial x_\ell}\rangle \qquad \begin{pmatrix} 1 \le i \le N \\ 1 \le j \le \bar{N} \end{pmatrix}.$$

Corresponding to the continuous-time scheme (3.7), we make use of a family of finite difference approximation in time with a parameter Θ, $1 \le \Theta \le 1$: for $n = 0,1,2,..,p-1$; $p\cdot\Delta t = T$,

$$(3.8) \qquad [M^o;M^\partial]\,D_t\begin{Bmatrix}V^n\\G^n\end{Bmatrix} + \Theta\cdot[K^o;K^\partial]\begin{Bmatrix}V^{n+1}\\G^{n+1}\end{Bmatrix} + (1-\Theta)\cdot[K^o;K^\partial]\begin{Bmatrix}V^n\\G^n\end{Bmatrix} = \{F^n\}$$

where D_t denotes the forward difference operator in time.

Note: $\Theta = 0$: the forward difference scheme

$\Theta = 1/2$: the Crank-Nicolson scheme

$\Theta = 1$: the backward difference scheme

$\Theta = 2/3$: proposed by Zienkiewicz in [8].

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†

$$\hat{g} = \sum_{i=N+1}^{\bar{N}} g(P_i)\cdot\hat{\phi}_i.$$

3.2 Study of Discrete Maximum Principle

In this section, we study the discrete maximum principle problem of the finite element Θ scheme (3.8). The key in this argument is the notion of "triangulation of acute type", introduced by Ciarlet and Raviart [9]. The first step is the following

Theorem 6. Assume that the triangulation T^h is of acute type. Assume also that the time increment Δt is taken so as to satisfy

(3.8) $$M_{ii} - (1-\Theta)\Delta t \cdot K_{ii} \geq 0 \quad \text{for all } i,\ 1 \leq i \leq N,$$

and

(3.9) $$M_{ij} + \Theta\Delta t \cdot K_{ij} \leq 0 \quad \text{for all } i \text{ and } j \text{ such that } i \neq j,\ 1 \leq i \leq N,\ 1 \leq j \leq \bar{N}.$$

Then, the *discrete maximum principle*

(3.10) $$\min\{\,0,\ g^{n+1}_{\min},\ v^n_{\min}\} + \Delta t \cdot f^n_{\min} \leq v^{n+1}_i \leq \max\{\,0,\ g^{n+1}_{\max},\ v^n_{\max}\} + \Delta t \cdot f^n_{\max} \qquad (1 \leq i \leq N),$$

holds for $n = 0,1,2,..,p-1$, where

$$v^n_{\max} = \max\{0, \max_{1\leq j\leq \bar{N}} v^n_j\},\quad v^n_{\min} = \min\{0, \min_{1\leq j\leq \bar{N}} v^n_j\},$$

(3.11) $$f^n_{\max} = \max\{0, \sup_{x\in\Omega} f^n(x)\},\quad f^n_{\min} = \min\{0, \inf_{x\in\Omega} f^n(x)\},$$

and $$g^n_{\max} = \max\{0, \max_{1\leq j\leq N^\partial} g^n_j\},\quad g^n_{\min} = \min\{0, \min_{1\leq j\leq N^\partial} g^n_j\}.$$

For the proof, see [10].

The conditions (3.8)-(3.9) guarantee the maximum principle (3.10) of the finite element scheme (3.8). Those conditions, however, give no information whether or not the maximum principle holds for a given pair of Δt, Θ and a triangulation T^h, *until* the matrices [M] and [K] are actually computed. Also, Theorem 6 gives no guiding principle how to modify the triangulation T^h along with Δt and Θ, *when they violate the conditions* (3.8)-(3.9). Thus, we need some criteria to check the conditions (3.8)-(3.9) *a priori*.

Lemma 2. Assume that the triangulation T^h is of acute type. Then, a sufficient condition for (3.8) is given by

(3.12) $$\alpha_o(1-\Theta)\frac{\Delta t}{\kappa^2_{\min}} \leq \frac{2}{(m+1)(m+2)} \qquad \text{for the consistent mass case,}$$

or by

$$(3.13) \qquad \alpha_o\ (1-\Theta)\ \frac{\Delta t}{\kappa^2_{min}} \leq \frac{1}{(m+1)} \qquad \text{for the lumped mass case.}$$

Proof is easily obtained from the estimates in Lemma 1. With regard to the second condition (3.9), we remark that it does not impose any restriction on the time increment Δt, *if* [M] *is the lumped mass matrix,* and *if the triangulation is of acute type.* While for the consistent mass type approximation, the situation becomes to be rather restrictive. In fact, the following lemma implies that the time increment Δt cannot be taken too small (and, at the same time, it cannot be too large from the condition (3.12)).

Lemma 3. Assume that the triangulation is of *strictly* acute type. Suppose that [M] is the mass matrix of consistent type given by Eq.$(3.7)_b$. Then, if Δt is chosen as

$$(3.14) \qquad \alpha_o\Theta\ \frac{\Delta t}{\kappa^2_{max}} \geq \frac{1}{\Sigma\cdot(m+1)(m+2)},$$

then the condition (3.9) is satisfied.

Proof is given from Eq(1.2) and $<\hat{\phi}_i,\hat{\phi}_j>_\Delta = mes(\Delta)/(m+1)(m+2)$, $|\nabla\cdot\hat{\phi}_i|_E = 1/\kappa_i$. For detail, see [10]. Now, combining Theorem 6 with Lemmas 2 and 3, we finally obtain the following

Theorem 7. (Maximum Principle for Lumped Mass Type Scheme)

Assume that the triangulation T^h is of acute type. Then, the solution of the finite element scheme of lumped mass type (3.7) satisfies the discrete maximum principle (3.10) if

$$(3.15) \qquad \dot{\alpha_o}\ (1-\Theta)\ \frac{\Delta t}{\kappa^2_{min}} \leq \frac{1}{(m+1)}.$$

Theorem 8. (Maximum Principle for Consistent Mass Type Scheme)

Assume that the triangulation T^h is of strictly acute type, that is, $\Sigma > 0$. Then, the solution of the finite element scheme of consistent mass type (3.7) satisfies the discrete maximum principle (3.10) if

$$(3.16) \qquad \alpha_o\ (1-\Theta)\ \frac{\Delta t}{\kappa^2_{min}} \leq \frac{2}{(m+1)(m+2)}$$

and

$$(3.17) \qquad \alpha_o\ \Theta\ \frac{\Delta t}{\kappa^2_{max}} \geq \frac{1}{\Sigma\cdot(m+1)(m+2)}.$$

Remark: From Theorem 8, we see that the value of Θ cannot be taken arbitrary. For example, let us consider a triangulation with *regular* simplices. Let $\kappa_{min} = \kappa_{max} = \kappa$. It is easily seen that $\Sigma = 1/m$, and that the conditions (3.16) and (3.17) are reduced to

$$\frac{m}{\Theta\cdot(m+1)(m+2)} \le \lambda \le \frac{2}{(1-\Theta)(m+1)(m+2)} \tag{3.18}$$

where we put $\lambda \equiv \alpha_o \Delta t/\kappa^2$. In order to let the two inequalities hold simultaneously, Θ must be greater than or equal to $m/(m+2)$, i.e.,

$$\begin{aligned} &\Theta \ge 1/3 \quad \text{for one-dimensional case,} \\ &\Theta \ge 1/2 \quad \text{for two-dimensional case,} \\ \text{and } &\Theta \ge 3/5 \quad \text{for three-dimensional case.} \end{aligned} \tag{3.19}$$

3.3 Stability in the Mean Square Sense

In this section, we assume for simplicity $g\equiv 0$. We say that the finite element scheme (3.7) is stable in the mean square sense if the solution $\hat{v}^n$ satisfies the inequality

$$\|\hat{v}^r\|^2 + \zeta \sum_{n=0}^{r-1} \left\| \frac{\partial \hat{v}^n_\Theta}{\partial x_\ell} \right\|^2 \le \|\hat{v}^o\|^2 + C \sum_{n=1}^{r-1} \Delta t \, \|f^n\|^2 \quad (\zeta > 0), \tag{3.20}$$

$$r = 1,2,..,p;\ p\Delta t = T,$$

where $\hat{v}^n_\Theta = \Theta\cdot\hat{v}^{n+1} + (1-\Theta)\cdot\hat{v}^n$.

Such *a priori* estimates have been used by several authors to study stability and convergence of discrete approximations for parabolic equations. Douglas and Dupont [11]have investigated a class of step-by-step Galerkin schemes with the case $\Theta \ge 1/2$, and obtained unconditional stability and error estimates in the mean square sense.

This problem, i.e., obtaining *a priori* estimates of (3.20) type is essentially that of *a priori* estimation of the spectral radius of $[M^o]^{-1}[K^o]$, and Theorem 1 has already provide an effective tool for this purpose.

Theorem 9. (Stability in the Mean Square Sense for Lumped Mass Type Scheme)

The finite element scheme of lumped mass type (3.7) is stable in the sense of (3.20), if

$$\max\{\, 0,\ \alpha_o(1-2\Theta)\frac{\Delta t}{\kappa^2_{min}} \,\} \le \frac{2}{A_m(m+1)}, \tag{3.21}$$

where

$$A_m = \begin{cases} 2 & (\Sigma \ge 0) \\ m+1 & (\Sigma < 0). \end{cases} \tag{3.22}$$

Theorem 10. (Stability in the Mean Square Sense for Consistent Mass Type Sc.)

The finite element scheme of consistent mass type (3.7) is stable in the sense of (3.20), if

$$\text{(3.23)} \qquad \max\{\, 0,\ \alpha_0(1-2\Theta)\,\frac{\Delta t}{\kappa^2_{min}}\,\} \ \le\ \frac{2}{A_m(m+1)(m+2)}\ ,$$

where A_m is the constant defined by Eq. (3.22).

3.4 Some Numerical Illustrations

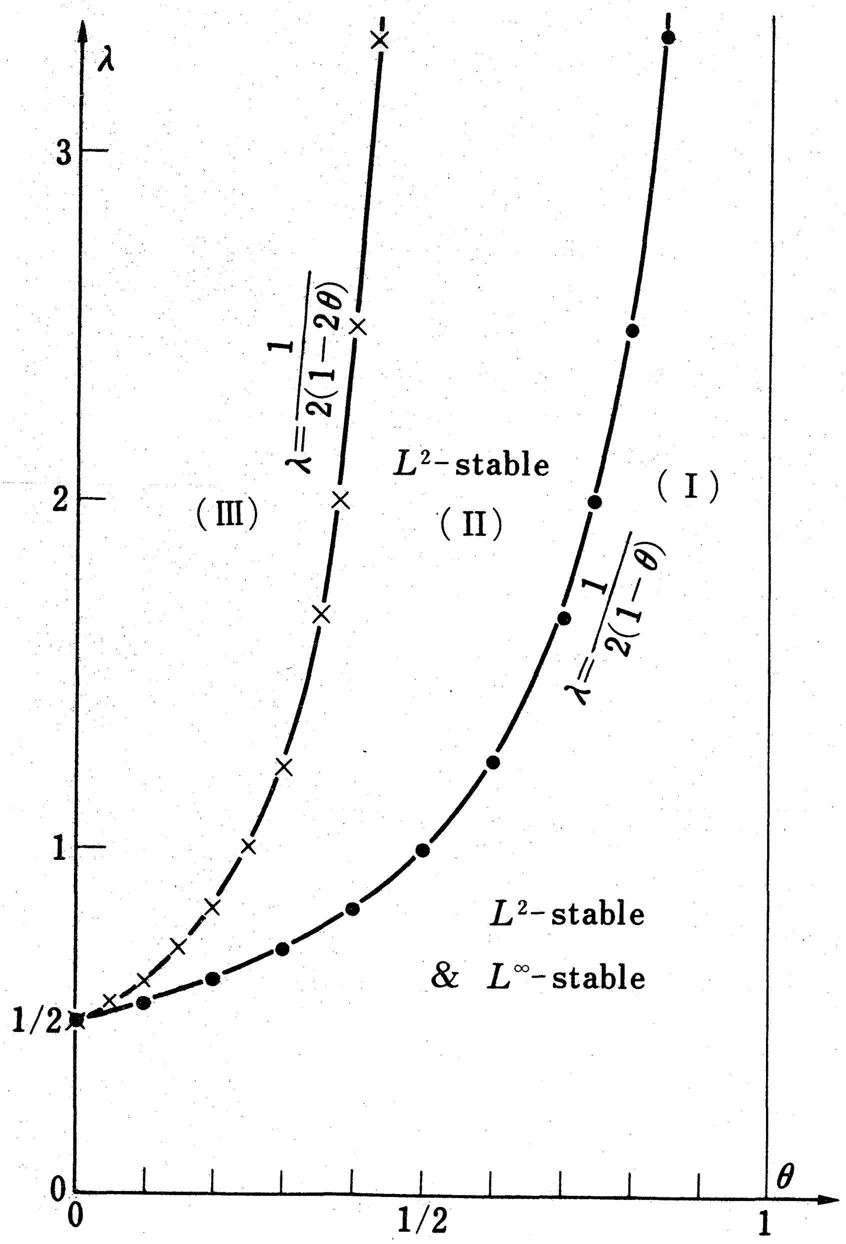

Fig.1 Stability Region for Lumped Mass Type Scheme

In the following, we give some numerical illustrations on L^2-sense stability and the maximum principle sense stability (L^∞-stability). All the examples are one-dimensional; the acuteness assumption is automatically satisfied.

Fig.1 shows the stability region for the lumped mass type scheme, where λ is $\alpha_0\Delta t/\kappa^2$. (Mesh spacing is assumed to be uniform.) In this case, the situation is not so much different from that of finite difference cases.

Fig.2 gives the corresponding stability region for consistent mass type scheme.

It is seen from Fig.2 that the consistent mass type scheme is more restrictive from the standpoint of L^∞-stability. For a fixed Θ, too small λ as well as too large λ may yield L^∞-unstable solution, and which is clearly shown in Fig.5.

Fig.3 and Fig.4 show an example of L^2/L^∞-stable computation (Fig.3), and an example of L^2-stable, but not L^∞-stable computation (Fig.4).

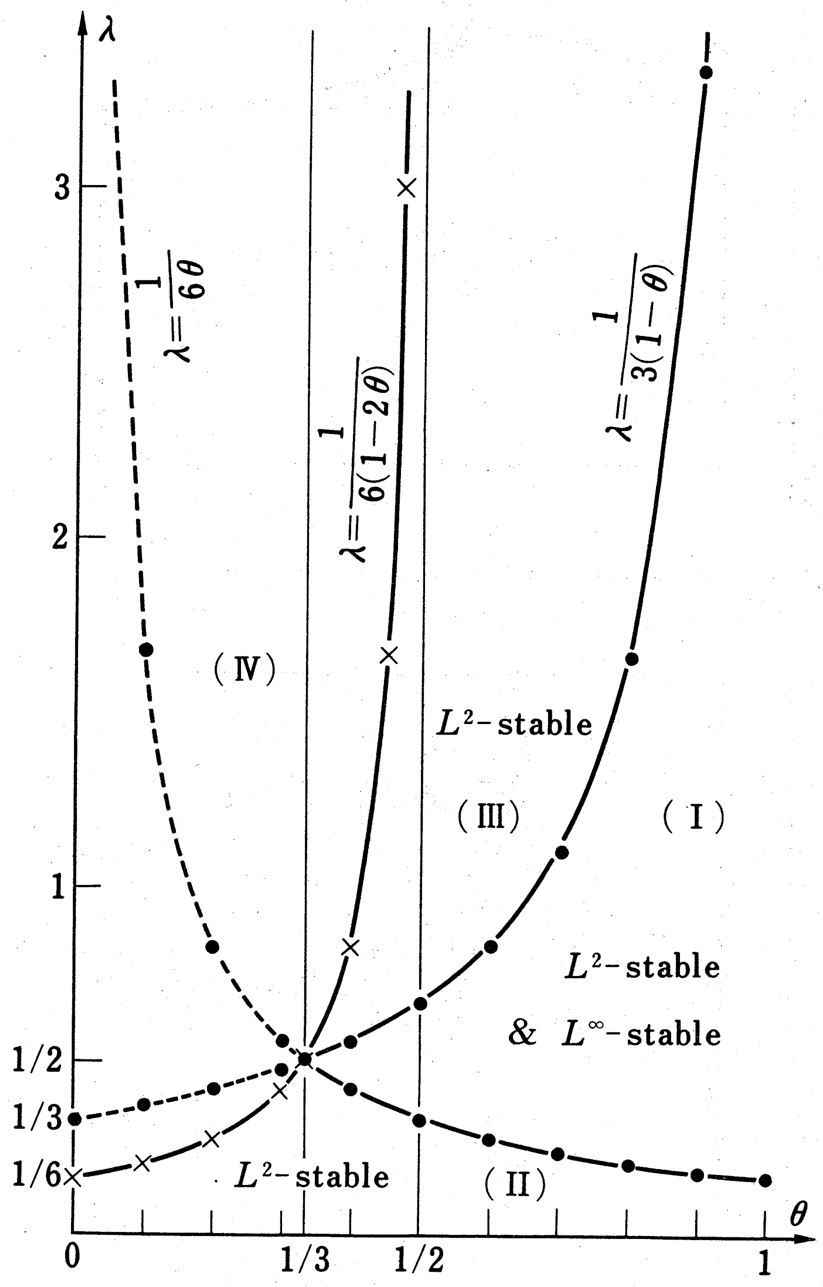

Fig.2 Stability Region for Consistent Mass Type Scheme

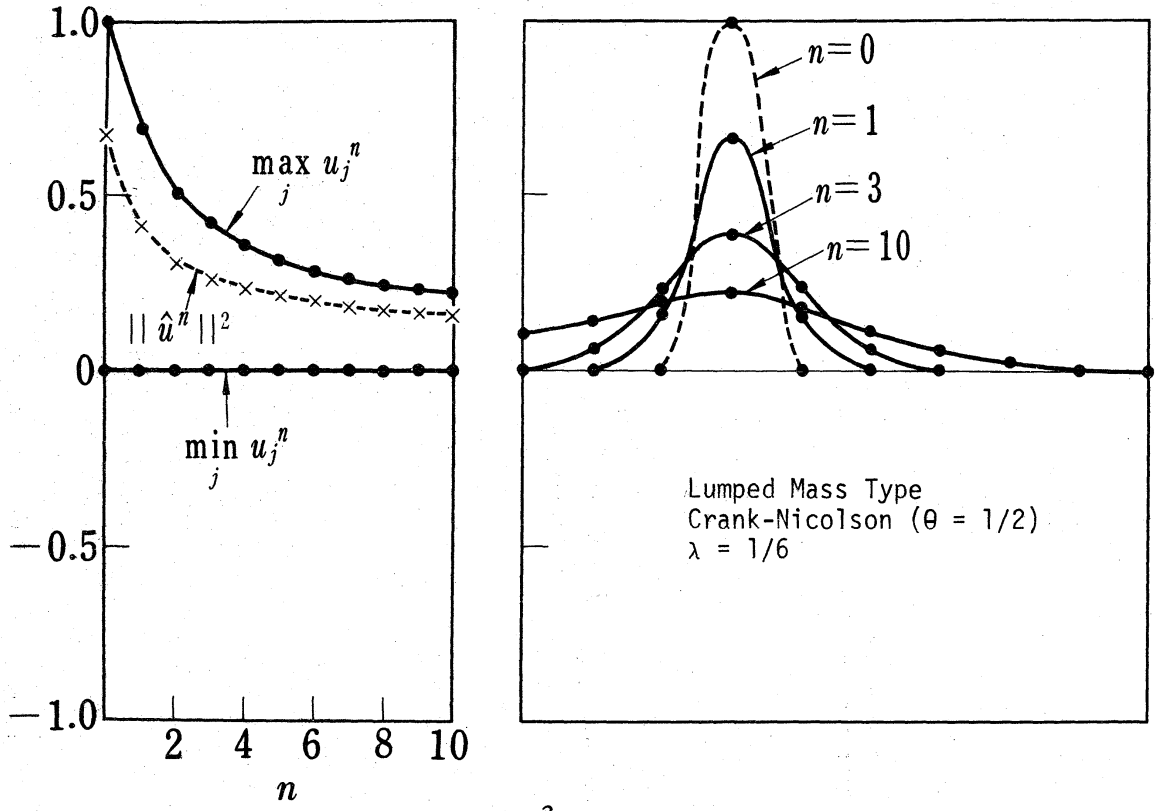

Fig. 3 L^2-Stable and L^∞-Stable Computation

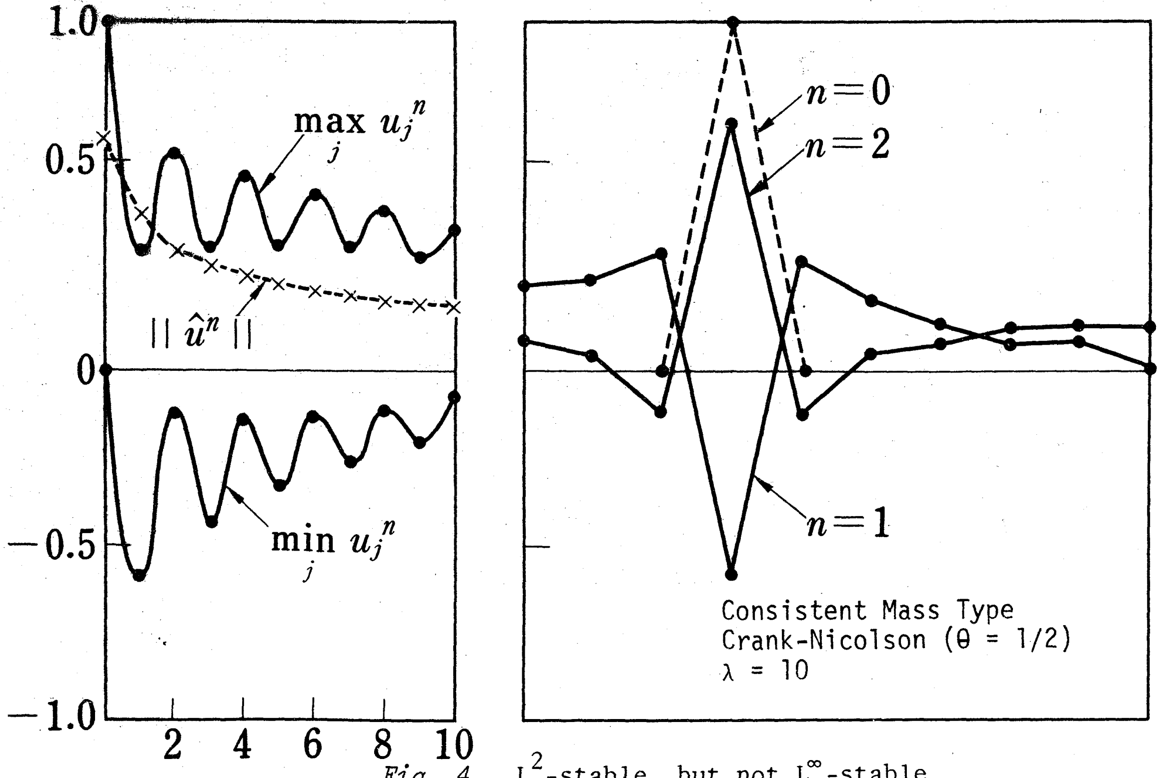

Fig. 4 L^2-stable, but not L^∞-stable

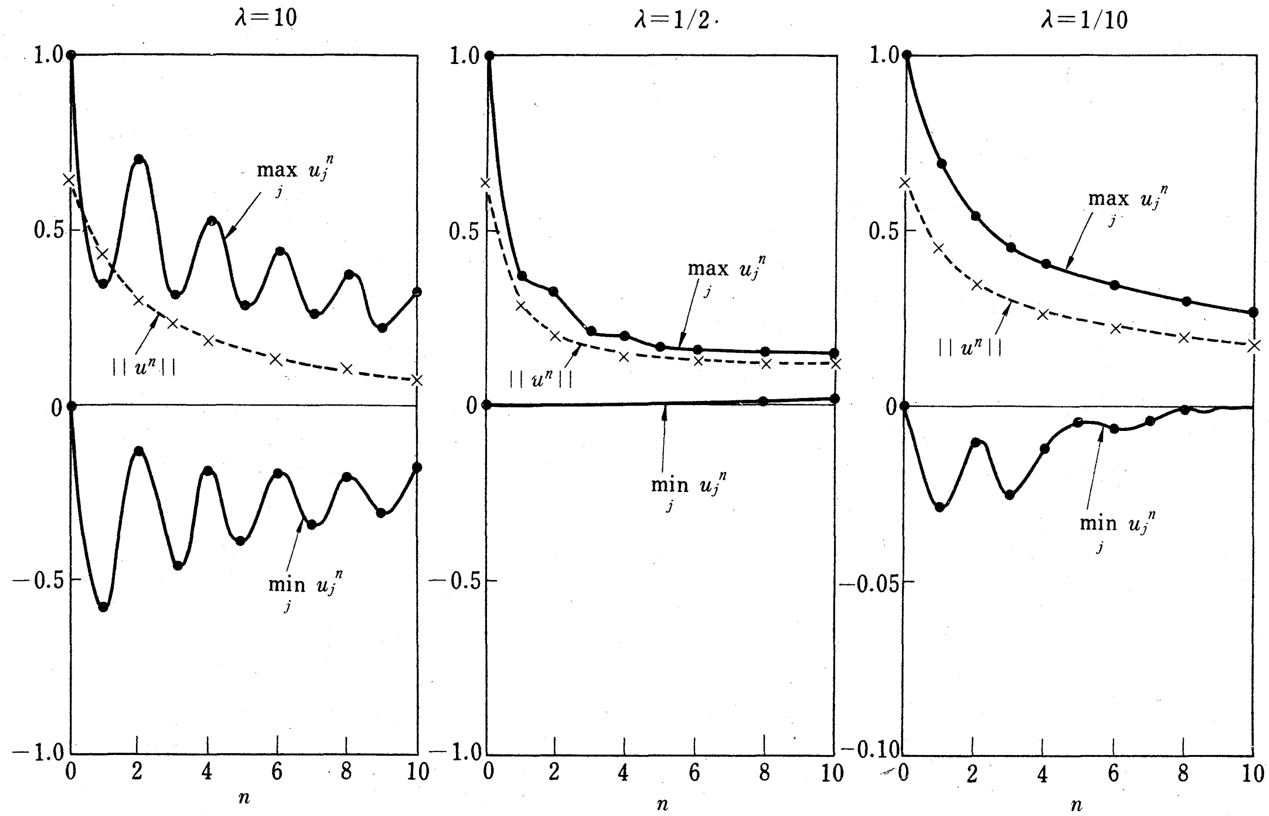

Fig.5 Too small λ causes L^∞-instability, as well as too large λ!

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