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Equivariant point theorems and their applications

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I will report the results obtained in [1].

Let N be a closed manifold with a free involution, and M be a closed manifold with an involution which is free or trivial. We shall give theorems on the existence of equivariant point of a continuous map $f : N \longrightarrow M$. The theorem for free case is similar to the classical Lefschetz fixed point theorem. The theorems are applied to the problem of which finite groups can act freely on a sphere, and are also used to show a generalization of the Borsuk-Ulam theorem and a formula relating the semicharacteristic class to the Stiefel-Whitney classes.

Throughout this report, the homology and cohomology with coefficients in Z_2 are to be understood. For brevity, manifolds and actions on them are assumed to be differentiable.

1. The equivariant Lefschetz class

Let N and M be closed manifolds with involution T, and assume that the involution on N is free. Regard the product $M^2 = M \times M$ as a manifold with involution by defining $T(x_1, x_2) = (x_2, x_1)$, and define an equivariant embedding $d' : M \rightarrow M^2$ by $d'(x) = (x, Tx)$. Consider the orbit manifolds $N \underset{T}{\times} M$ and $N \underset{T}{\times} M^2$ under the diagonal action, and define

$$\Delta \in H^m(N \underset{T}{\times} M^2)$$

to be the Poincaré dual of the image of the fundamental class $[N \underset{T}{\times} M] \in H_{n+m}(N \underset{T}{\times} M)$ under the homomorphism $(1 \underset{T}{\times} d')_* :$

$H_{n+m}(N \underset{T}{\times} M) \longrightarrow H_{n+m}(N \underset{T}{\times} M^2)$, where $n = \dim N$, $m = \dim M$.

Let $f : N \longrightarrow M$ be a continuous map. Define $\hat{f} : N \longrightarrow N \times M^2$ by $\hat{f}(y) = (y, f(y), fT(y))$. Since $\hat{f}$ is equivariant, it induces a map $\hat{f}_T : N_T \longrightarrow N \underset{T}{\times} M^2$ between the orbit manifolds. We call the element

$$\hat{f}_T^*(\Delta) \in H^m(N_T)$$

the <u>equivariant Lefschetz class</u> of f. If $n = m$, the number

$$\hat{I}(f) = \langle \hat{f}_T^*(\Delta), [N_T] \rangle \in Z_2$$

is called the <u>equivariant point index</u> of f.

Denote by $A(f)$ the set of equivariant points of f:

$$A(f) = \{y \in N \; ; \; fT(y) = Tf(y)\}.$$

We have

<u>Theorem 1</u>. If $\hat{f}_T^*(\Delta) \neq 0$ then $\dim A(f) \geqq n - m$. In particular, if $\hat{I}(f) \neq 0$ then f has an equivariant point.

2. The equivariant point theorem —— free case

In this section we give the equivariant point theorem in the case the involution T on M is free.

If M is a closed manifold with a free involution T, it can be proved that a non-degenerate symplectic pairing $\circ : H^*(M) \otimes H^*(M) \longrightarrow Z_2$ is given by

$$\alpha \circ \beta = \langle \alpha \vee T^*\beta, [M] \rangle.$$

Therefore the vector space $H^*(M)$ has a symplectic basis, i.e. a basis $\{\mu_1, \cdots, \mu_r, \mu_1', \cdots, \mu_r'\}$ such that

$$\mu_i \circ \mu_j = 0, \quad \mu_i' \circ \mu_j' = 0, \quad \mu_i \circ \mu_j' = \delta_{ij}.$$

For the equivariant Lefschetz class Δ, we have

$$\Delta = \sum_{i=1}^{r} \phi^*(1 \times \mu_i \times \mu_i'),$$

where $\phi^* : H^*(N \times M^2) \longrightarrow H^*(N \underset{T}{\times} M^2)$ is the transfer homomorphism. It is also seen that for a continuous map $f : N \longrightarrow M$, the number

$$\hat{\chi}(f) = \sum_{i=1}^{r} \langle f^*(\mu_i) \cup T^*f^*(\mu_i'), [N] \rangle \in Z_2$$

is independent of the choice of symplectic basis for $H^*(M)$, and if $\{\nu_1, \cdots, \nu_s, \nu_1', \cdots, \nu_s'\}$ is a symplectic basis for $H^*(N)$ and

$$f^*(\mu_i) = \sum_i a_{ij}\nu_i + \sum_i c_{ij}\nu_i'$$

$$f^*(\mu_i') = \sum_i b_{ij}\nu_i + \sum_i d_{ij}\nu_i'$$

then

$$\hat{\chi}(f) = \mathrm{trace}\ ({}^tAD + {}^tBC).$$

We call $\hat{\chi}(f)$ the <u>equivariant Lefschetz number</u> of f.

We have the following theorem analogous to the classical Lefschetz fixed point theorem.

<u>Theorem 2</u>. Let N and M be the same dimensional closed manifolds with free involution, and $f : N \longrightarrow M$ be a continuous map. Then the equivariant point index $\hat{I}(f)$ is equal to the equivariant Lefschetz number $\hat{\chi}(f)$.

By Theorems 1 and 2, we have

<u>Theorem 3</u>. Let N and M be the same dimensional closed manifolds with free involution, and let $f : N \longrightarrow M$ be a continuous map such that $\hat{\chi}(f) \not\equiv 0$. Then f has an equivariant point.

For a closed manifold M such that the dimension of the vector space $H_*(M)$ is even, the number

$$\hat{\chi}(M) = \frac{1}{2} \dim H_*(M) \mod 2$$

is called <u>the semicharacteristic</u> of M.

From Theorem 3 we obtain the following generalizations of Theorem 1 in Milnor [2].

<u>Corollary 1</u>. Let M be a closed manifold with free involution, such that $\hat{\chi}(M) \not\equiv 0 \mod 2$. Let T, T' be free involutions on M such that $T_* = T'_* = id : H_*(M) \longrightarrow H_*(M)$. Then, any continuous map $f : M \longrightarrow M$ of odd degree has an equivariant point. In particular, T and T' have an coincidence.

<u>Corollary 2</u>. Let M be a closed manifold with a free involution T, and assume $\hat{\chi}(M) \not\equiv 0 \mod 2$. Then, any continuous map $f : M \longrightarrow M$ such that $f_* = id : H_*(M) \longrightarrow H_*(M)$ has an equivariant point.

<u>3. Applications of Theorem 3</u>

From Corollaries 1 and 2 we get immediately

<u>Theorem 4</u>. Let M be a closed manifold such that $\dim H_*(M) \equiv 2 \mod 4$, and G be a group acting freely on M. Then we have

i) G can have at most one element T of order 2 such that $T_* = id : H_*(M) \longrightarrow H_*(M)$.

ii) If $T \in G$ is an element of order 2 such that $T_* = id : H_*(M) \longrightarrow H_*(M)$, T lies in the center of G.

iii) If $T \in G$ is an element of order 2, T lies in the centralizer of $G_0 = \{S \in G \; ; \; S_* = id : H_*(M) \longrightarrow H_*(M)\}$.

Let $D(2\ell)$ denote the dihedral group with presentation $(X, Y \; ; \; X^2 = (XY)^2 = Y^\ell = 1)$.

Theorem 4 implies

Theorem 5. Let M be a closed manifold on which $D(2\ell)$ acts freely. Assume that $\hat{\chi}(M) \not\equiv 0$ and ℓ is an odd > 1. Then the representation of $D(2\ell)$ on $H_*(M)$ given by sending $S \in D(2\ell)$ to $S_* : H_*(M) \longrightarrow H_*(M)$ is faithful.

Let $Q(8n, k, \ell)$ denote the group with presentation $(X, Y, A \; ; \; X^2 = (XY)^2 = Y^{2n}, A^{k\ell} = 1, XAX^{-1} = A^r, YAY^{-1} = A^{-1})$, where $8n, k, \ell$ are pairwise relatively prime positive integers, $r \equiv -1 (\mathrm{mod}\ k)$ and $r \equiv +1 (\mathrm{mod}\ \ell)$. Milnor asks in [2] if $Q(8n, k, \ell)$ can act freely on a 3-sphere. Recently R.Lee [3] has obtained the following partial answer.

Theorem 6. If n is even and $\ell > 1$, the group $Q(8n, k, \ell)$ can not act freely on any mod 2 homology sphere whose dimension is 3 mod 8.

By applying Theorem 4, we have another proof of this result.

R.Lee states in [3] that the group $O(48 \; ; \; k, \ell)$ with $\ell \not\equiv 0$ (mod 2), $\ell \not\equiv 0$ (mod 3), can not act freely on any mod 2 homology sphere whose dimension is 3 mod 8. His proof is incorrect if $\ell = 1$, and application of Theorem 4 gives another proof of the result for $\ell \neq 1$.

By applying Theorem 2 we can also prove the following theorem on the semicharacteristic.

Theorem 7. Let N be a closed manifold with a free involution T. Denote by $\tau(N_T)$ the tangent bundle of N_T, and ρ the line bundle associated to the $O(1)$-bundle $\pi : N$

$\longrightarrow N_T$. Then we have

$$\hat{\chi}(N) = \langle \chi(\rho \otimes \tau(N_T)),\ [N_T] \rangle,$$

where χ stands for the Euler class mod 2.

From this we get the following formula of Uchida.

Corollary. For a closed manifold with free involution, we have

$$\hat{\chi}(N) = \langle \sum_{k=0}^{n} c^{n-k} w_k,\ [N_T] \rangle,$$

where c is the 1-st Stiefel-Whitney class of the bundle $\pi : N \longrightarrow N_T$, and w_k is the k-th Stiefel-Whitney class of N_T.

4. The equivariant point theorem — trivial case

In this section we give the equivariant point theorem in the case the involution T on M is trivial.

For a closed manifold N with a free involution, we consider the operation

$$Q : H^r(N) \longrightarrow H^{2r}(N_T)$$

defined by Bredon [4]. This is defined to be the composition

$$H^r(N) \xrightarrow{P} H^{2r}(E \underset{T}{\times} N^2) \xrightarrow{(1 \underset{T}{\times} d')^*} H^{2r}(E \underset{T}{\times} N) \overset{q_T^*}{\cong} H^{2r}(N_T),$$

where E is the universal Z_2- bundle, P is the external Steerod square and $q_T : E \underset{T}{\times} N \longrightarrow N_T$ is the projection.

We have

Theorem 8. Let N be a closed manifold with free involution, and let $f : N \longrightarrow M$ be a continuous map to a closed manifold M. Let $\{\alpha_1, \alpha_2, \cdots, \alpha_s\}$ be a basis for the vector space $H^*(M)$, and define $\eta_{jk} \in Z_2$ $(j, k = 1,2,\cdots, s)$ by

$$X = (\xi_{jk}), \qquad \xi_{jk} = \langle \alpha_j \cup \alpha_k, [M] \rangle,$$
$$Y = (\eta_{jk}), \qquad Y = X^{-1}.$$

Then the equivariant Lefschetz class $f_T^*(\Delta)$ is equal to

$$\sum_{i=0}^{[m/2]} c^{m-2i} Q(f^* v_i) + \sum_{j<k} (\eta_{jk} + \eta_{jj}\eta_{kk}) \phi^*(f^*\alpha_j \cup T^* f^* \alpha_k),$$

where v_i is the i-th Wu class of M, and ϕ^* is the transfer homomorphism.

By Theorems 1 and 8, we have the following generalization of the Borsuk-Ulam theorem [5].

<u>Theorem 9</u>. Let N be a closed manifold with a free involution T, and let $f : N \longrightarrow M$ be a continuous map to a manifold M. Assume that $c^m \neq 0$, and $f_* : \tilde{H}_*(N) \longrightarrow \tilde{H}_*(M)$ is trivial. Then the dimension of $A(f) = \{y \in N ; f(Ty) = f(y)\}$ is at least $n - m$.

Theorem 8 can be applied also to prove Theorem 5.

References

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