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THE RANDOM ERGODIC THEOREM AND TEMPORALLY WEAKELY QUASI-HOMOGENEOUS PROCESSES

By Takeshi Yoshimoto

The random ergodic theorem concerning measure preserving transformations has been extensively studied. Here we consider the corresponding random average behaviour of non-singular transformations and derive some analogue results. Moreover, representation of temporally weakly quasi-homogeneous processes by means of non-singular semiflows will be discussed.

1. Discrete random ergodic theorems. Let $(X, \mathcal{B}, m)$ be a σ-finite measure space, Φ the set of bimeasurable positively non-singular transformations φ of X onto itself and $\mathcal{A}$ the σ-algebra of subsets of Φ. Given a sequence $\{\mu_n\}$ of probability measures defined on $\mathcal{A}$, we consider the one-sided product space $(\Phi_d^*, \mathcal{A}_d^*, \mu_d^*) = \bigotimes_{n=1}^{\infty} (\Phi_n, \mathcal{A}_n, \mu_n)$ $((\Phi_n, \mathcal{A}_n) = (\Phi, \mathcal{A})$ $n=1,2,\ldots)$ on which the one-sided shift transformation σ is defined in the usual way. From now on we assume that $\{(x, \varphi^*): \varphi_{n_k} \cdots \varphi_{n_1} x \in E\}$ is measurable in $X \otimes \Phi_d^*$ for any $E \in \mathcal{B}$, where φ_k denotes the k-th coordinate of φ^* and $(n_1, \ldots, n_k)$ is an arbitrary sequence of positive integers with $n_1 < \cdots < n_k$.

It follows then by the assumption that $f(\varphi_k \cdots \varphi_1 x)$ is measurable in $X \otimes \Phi_d^*$ whenever $f(x)$ is measurable in X. Since for any $\varphi^* \in \Phi_d^*$ and $k=1,2,\ldots$, the successive transformations $\varphi_k \cdots \varphi_1$ are positively non-singular, there exists a family $\{ \beta_{(k, \varphi^*)}(x) \}$ of measurable non-negative functions such that

$$m(\varphi_k \cdots \varphi_1 E) = \int_E \beta_{(k, \varphi^*)}(x)\, dm$$

for any $E \in \mathcal{B}$ and $k=1,2,\ldots$. There is no loss of generality in supposing that every $\beta_{(k, \varphi^*)}(x)$ is positive everywhere on X.

By considering approximating sums to the integrals in question, one can easily check that

$$\int_X f(x)\, dm = \int_X f(\varphi_k \cdots \varphi_1 x) \beta_{(k, \varphi^*)}(x)\, dm$$

for all integrable functions $f(x)$ on X and $k=1,2,\ldots$. It follows easily that

$$\beta_{(i+j,\ \varphi^*)}(x) = \beta_{(i,\ \sigma^j\varphi^*)}(\varphi_j\cdots\varphi_1 x)\beta_{(j,\ \varphi^*)}(x)$$

almost everywhere on X, for $i,j=1,2,\ldots$, and it can be assumed with no loss of generality that the above equality holds everywhere on X. We know that there exists a sequence $\{\beta_k^*(x,\ \varphi^*)\}$ of $\mathcal{B}\otimes\mathcal{A}_d^*$-measurable versions of $\beta_{(k,\ \varphi^*)}(x)$, $k=1,2,\ldots$, such that except for a set of μ_d^*-measure zero, $\beta_k^*(x,\ \varphi^*) = \beta_{(k,\ \varphi^*)}(x)$ almost everywhere on X, for $k=1,2,\ldots$. Let us consider the skew product transformation S on $X\otimes\Phi_d^*$ given by $S(x,\ \varphi^*) = (\varphi_1 x, \sigma\varphi^*)$. Note here that S is a bi-measurable positively non-singular transformation.

THEOREM 1. Let $h(x)$ be any non-negative measurable function defined on X satisfying that after dropping a set of μ_d^*-measure zero,

$$\sum_{k=1}^{\infty} h(\varphi_k\cdots\varphi_1 x)\beta_{(k,\ \varphi^*)}(x) = \infty$$

almost everywhere on X. Then for all $f(x)\in L_1(X)$, there exists a μ_d^*-null set D^* such that for any $\varphi^*\in\Phi_d^* - D^*$, the limit

$$\lim_{n\to\infty}\frac{\sum_{k=1}^{n} f(\varphi_k\cdots\varphi_1 x)\beta_{(k,\ \varphi^*)}(x)}{\sum_{k=1}^{n} h(\varphi_k\cdots\varphi_1 x)\beta_{(k,\ \varphi^*)}(x)}$$

exists and is finite almost everywhere on X.

THEOREM 2. Let the measure m be finite and suppose there exists a positive constant K such that $m(\varphi_1^{-1}\cdots\varphi_k^{-1}E)\leqq K\cdot m(E)$ $(k=1,2,\ldots)$ for all $\varphi^*\in\Phi_d^*$ and every $E\in\mathcal{B}$. Then for every $f(x)\in L_1(X)$, there exist a μ_d^*-null set D^* and a function $G_{\varphi^*}(x)\in L_1(X)$ such that for any $\varphi^*\in\Phi_d^* - D^*$,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f(\varphi_k\cdots\varphi_1 x) = G_{\varphi^*}(x)$$

almost everywhere on X and $G_{\varphi^*}(x)$ is also the limit in the mean of order 1.

REMARK. If we take $\varphi_k = \varphi_1$ for $k=2,3,\ldots$, Theorems 1 and 2 are reduced to Dowker's theorem and Dunford-Miller's theorem respectively.

It is worth while to notice that if for any $\varphi^* \in \Phi_d^*$, we write $\psi_{(k,\varphi^*)} = \varphi_k \cdots \varphi_1$, then $\{\psi_{(k,\varphi^*)}: k=1,2,\dots\}$ turns out to be a quasi semigroup associated with σ, i.e. $\psi_{(i+j,\varphi^*)} = \psi_{(j,\sigma^i\varphi^*)}\psi_{(i,\varphi^*)}$. In fact, this property plays an essential role in discrete random ergodic theorems but can not be expected in a continuous parameter case, unless $\varphi^* = (\varphi_t: t \geqq 0)$ satisfies the semigroup property (i.e. semi-flow). So the method used above is of no service to the continuous parameter one.

2. Continuous random ergodic theorems (1). Let $\{\varphi_t: -\infty < t < \infty\}$ be a measurable positively non-singular flow on X and $\{\psi_{(t,x)}: x \in X, -\infty < t < \infty\}$ a quasi semigroup of bimeasurable positively non-singular transformations of another measure space (Y, Σ, μ) associated with $\{\varphi_t\}$. Define a transformation group $\{Z_t: -\infty<t<\infty\}$ on $X \otimes Y$ by $Z_t(x, y) = (\varphi_t x, \psi_{(t,x)}y)$.

PROPOSITION 1. $\{Z_t: -\infty<t<\infty\}$ is the measurable positively non-singular skew product flow of $\{\varphi_t\}$ with $\{\psi_{(t,x)}\}$.

Let $\beta_t^*(x,y)$ be the Radon-Nikodym's derivative associated with $\{Z_t\}$ and $\beta^*(t,x,y)$ denote the (t,x,y)-measurable version of $\beta_t^*(x,y)$.

THEOREM 3. Let m be finite. For every $f(y) \in L_1(Y)$ and an arbitrary positive measurable function $h(y)$ with

$$\int_0^\infty h(\psi_{(t,x)}y)\beta^*(t,x,y)\,dt = \infty$$

$m\otimes\mu$-almost everywhere on $X \otimes Y$, there exists an m-null set N such that for any $x \in X - N$,

$$\lim_{\alpha \to \infty} \frac{\int_0^\alpha f(\psi_{(t,x)}y)\beta^*(t,x,y)\,dt}{\int_0^\alpha h(\psi_{(t,x)}y)\beta^*(t,x,y)\,dt}$$

exists and is finite almost everywhere on Y.

THEOREM 4. Let φ_t and $\psi_{(t,x)}$ be one-to-one and $m\otimes\mu$ finite.

Suppose there is a positive constant K such that $\mu(\Psi^{-1}_{(t,\varphi_t^{-1}x)}B) \leq K\cdot\mu(B)$ for all $B \in \Sigma$ and $(t,x) \in (-\infty, \infty)\otimes X$. Then for every $f(y) \in L_1(Y)$, there exists a function $F_x(y) \in L_1(Y)$ such that except for an m-null set,

$$\lim_{\alpha \to \infty} \frac{1}{\alpha}\int_0^\alpha f(\Psi_{(t,x)}y)\,dt = F_x(y)$$

almost everywhere on Y and

$$\lim_{\alpha \to \infty} \left\| F_x(y) - \frac{1}{\alpha}\int_0^\alpha f(\Psi_{(t,x)}y)\,dt \right\|_{L_1(Y)} = 0.$$

3. Continuous random ergodic theorems (2). Now consider the product space $(\Phi_c^*, \mathcal{A}_c^*, \mu_c^*) = \bigotimes_{t\geq 0}(\Phi_t, \mathcal{A}_t, \mu_t)$, $(\Phi_t, \mathcal{A}_t) = (\Phi, \mathcal{A})$, $t \geq 0$, and suppose that

$$\{(t, \varphi^*, x):\quad \varphi_t x \in E\} \in \mathcal{L}_+ \otimes \mathcal{A}_c^* \otimes \mathcal{B}$$

for any $E \in \mathcal{B}$, where φ_t denotes the t-th coordinate of $\varphi^* \in \Phi_c^*$ and $\mathcal{L}_+$ denotes the Lebesgue class of $[0, \infty)$. Let $B_{(t,\varphi^*)}(x)$ be the Radon-Nikodym's derivative associated with φ_t.

PROPOSITION 2. 1°. If f(x) is x-measurable, then $f(\varphi_t x)$ is (t, φ^*, x)-measurable.

2°. There exists an $\mathcal{L}_+\otimes\mathcal{A}_c^*\otimes\mathcal{B}$-measurable version $B(t,\varphi^*, x)$ of $B_{(t,\varphi^*)}(x)$ such that except for a set of points (t, φ^*) of product measure zero, $B(t, \varphi^*, x) = B_{(t,\varphi^*)}(x)$ almost everywhere on X.

3°. If $|f(x)|^p$ $(p \geq 1)$ is integrable in X, then $|f(\varphi_t x)|^p B(t,\varphi^*,x)$ is integrable in $I\otimes\mathcal{A}_c^*\otimes X$, where I is an arbitrary finite interval in $[0, \infty)$.

We shall restrict ourselves to real valued functions. In what follows p will denote an arbitrary positive integer and E_k $(k=1,2,\ldots)$ will denote a one-dimensional Borel set. We consider the product set $\Theta = \bigotimes_{t\geq 0} R(t)$, $R(t) = (-\infty, \infty)$, whose elements are denoted by $\theta = (\xi_t(\theta):\ t \geq 0)$ and let $\{Z_t:\ t \geq 0\}$ be the semigroup of shift transformations on Θ. Put $f(t,\varphi^*)(x) = f(\varphi_t x)B(t,\varphi^*,x)$ for $f(x) \in L_p(X)$ and define

$$\Gamma_f(u_1,\dots,u_p) = \{(\varphi^*,x):\ f(u_1,\varphi^*)(x)\in E_1,\dots,f(u_p,\varphi^*)(x)\in E_p\},$$

$$\Gamma_{(\varphi^*;f)}(u_1,\dots,u_p) = [\Gamma_f(u_1,\dots,u_p)](\varphi^*) \quad (\varphi^*\text{-section}),$$

$$\Lambda_\xi(u_1,\dots,u_p) = \{\theta:\ \xi_{u_1}(\theta)\in E_1,\dots,\ \xi_{u_p}(\theta)\in E_p\}.$$

Consider a mapping Π_f from $\Phi_c^*\otimes X$ to Θ :

$$\Pi_f(\varphi^*, x) = \theta, \qquad \xi_t(\theta) = f(t,\varphi^*)(x).$$

Then $\Pi_f^{-1}\Lambda_\xi(u_1,\dots,u_p) = \Gamma_f(u_1,\dots,u_p)$. By $\mathcal{L}_f$ denote the σ-algebra generated by all finite unions of sets $\Lambda_\xi(u_1,\dots,u_p)$ and put $\lambda_f(\Lambda) = \mu_c^*\otimes m(\Pi_f^{-1}\Lambda)$ for $\Lambda\in\mathcal{L}_f$. Then $\{Z_t:\ t\geq 0\}$ is $\mathcal{L}_f$-measurable.

We now set up the following statements: There are two positive constants K_1, K_2 such that

(I) $\displaystyle\limsup_{\alpha\to\infty}\frac{1}{\alpha}\int_0^\alpha \lambda_f(Z_t^{-1}\Lambda)\,dt \leq K_2\cdot\lambda_f(\Lambda) \quad (\Lambda\in\mathcal{L}_f),$

(II) $\displaystyle\liminf_{\alpha\to\infty}\frac{1}{\alpha}\int_0^\alpha \lambda_f(Z_t^{-1}\Lambda)\,dt \geq K_1\cdot\lambda_f(\Lambda) \quad (\Lambda\in\mathcal{L}_f),$

(III) $\displaystyle\liminf_{\alpha\to\infty}\frac{1}{\alpha}\int_0^\alpha \lambda_f(Z_t^{-1}\Lambda)\,dt > 0 \quad \text{if } \lambda_f(\Lambda) > 0.$

THEOREM 4. Under the conditions I and III, there exists a function $f_{\varphi^*}(x)\in L_p(X)$ such that except for a μ_c^*-null set,

$$\lim_{\alpha\to\infty}\frac{1}{\alpha}\int_0^\alpha f(\varphi_t x)B(t,\varphi^*,x)\,dt = f_{\varphi^*}(x)$$

almost everywhere on X. Furthermore, under the conditions I and II, if $1 < p < \infty$, then

$$\lim_{\alpha\to\infty}\left\| f_{\varphi^*}(x) - \frac{1}{\alpha}\int_0^\alpha f(\varphi_t x)B(t,\varphi^*,x)\,dt\right\|_{L_p(X)} = 0,$$

and if m is finite, then the limit also exists in the norm of order 1.

Detailed information concerning applications of random ergodic theorems may be found in Beck and Schwartz's paper and Revesz's paper.

4. Representation of temporally weakly quasi-homogeneous processes by means of non-singular semiflows. Let $M(R_+)$ be the set of real valued Lebesgue measurable functions defined on R_+ and $\Omega = \bigotimes_{n=1}^{\infty} M_n(R_+)$, $M_n(R_+) = M(R_+)$, where $R_+ = [0, \infty)$, $\mathcal{B}_+$ = Borel class of R_+.
An element $\omega \in \Omega$ is an equivalence class of measurable functions and denoted by $\omega = (\omega_1(t), \omega_2(t), \dots)$, $\omega_n(t) \in M_n(R_+)$ $(n=1,2,\dots)$.

We assume that $m(X) = 1$ and there is given a sequence $F(t,x) = \{F_n(t,x): t \geq 0\}$ of real valued measurable component processes $F_1(t,x)$, $F_2(t,x), \dots$ on $(X, \mathcal{B}, m)$. Let us define

$$\Gamma_{E_1,\dots,E_p}^{(n_1,\dots,n_p)}(u_1,\dots,u_p) = \left\{x: F_{n_1}(u_1,x) \in E_1, \dots, F_{n_p}(u_p,x) \in E_p\right\}$$

$$(*) \quad \Lambda_{E_1,\dots,E_p}^{(n_1,\dots,n_p)}(u_1,\dots,u_p) = \left\{\omega: \omega_{n_1}(u_1) \in E_1, \dots, \omega_{n_p}(u_p) \in E_p\right\}$$

and a mapping Π from X to Ω by $\Pi(x) = \omega$, where $\omega_n(u) = F_n(u,x)$, $n=1,2,\dots$. Then

$$\Pi^{-1}\Lambda_{E_1,\dots,E_p}^{(n_1,\dots,n_p)}(u_1,\dots,u_p) = \Gamma_{E_1,\dots,E_p}^{(n_1,\dots,n_p)}(u_1,\dots,u_p).$$

We denote by $\mathfrak{A}$ the σ-algebra generated by all finite unions of sets of the form (*) and put $\mu = m \circ \Pi^{-1}$. Let $\{Z_t: t \geq 0\}$ be a shift transformation semigroup on Ω given by

$$\omega = (\omega_1(u), \omega_2(u), \dots) \longrightarrow Z_t\omega = (\omega_1(u+t), \omega_2(u+t), \dots),$$

which is $\mathcal{B}_+ \otimes \mathfrak{A}$-measurable. We shall say that $F(t,x)$ is a temporally weakly quasi-homogeneous process (abbreviated TWQHP) provided that there exists a positive constant K such that for any $(n_1,\dots,n_p)$ with $n_1 < \dots < n_p$ and any $E_1,\dots,E_p$ (Borel sets),

$$(**) \quad \limsup_{\alpha \to \infty} \frac{1}{\alpha}\int_0^{\alpha} m(\Gamma_{E_1,\dots,E_p}^{(n_1,\dots,n_p)}(u_1+t,\dots,u_p+t))\,dt \leq K \cdot m(\Gamma_{E_1,\dots,E_p}^{(n_1,\dots,n_p)}(u_1,\dots,u_p)).$$

Then it follows from (**) that for any $\Lambda \in \mathfrak{A}$,

$$\limsup_{\alpha \to \infty} \frac{1}{\alpha}\int_0^{\alpha} \mu(Z_t^{-1}\Lambda)\,dt \leq K \cdot \mu(\Lambda).$$

Moreover, we see that there exists a probability measure ν on $\mathfrak{A}$ such that

(1) $\nu(\Lambda) \leqq \tilde{K} \cdot \mu(\Lambda)$ $(\Lambda \in \mathfrak{A}$, $\tilde{K}$ constant),

(2) $\nu(Z_t^{-1}\Lambda) = \nu(\Lambda)$ $(\Lambda \in \mathfrak{A})$,

(3) ν is equivalent to μ,

(4) μ is non-singular with respect to $\{Z_t: t \geqq 0\}$.

Let $\mathfrak{A}_\mu$ and $\mathfrak{A}_\nu$ be the completions of $\mathfrak{A}$ by μ and ν respectively. Then $\mathfrak{A}_\mu = \mathfrak{A}_\nu$ which will be denoted by $\mathfrak{F}$. For an $\mathfrak{F}$-measurable function $f(\omega)$, there are two $\mathfrak{A}$-measurable functions $f_1(\omega)$, $f_2(\omega)$ such that

$$f_1(\omega) \leqq f(\omega) \leqq f_2(\omega), \quad \int_\Omega \{f_2(\omega) - f_1(\omega)\} d\mu = 0.$$

Then

$$f_1(Z_t\omega) \leqq f(Z_t\omega) \leqq f_2(Z_t\omega),$$

$$\int_0^u \int_\Omega \{f_2(Z_t\omega) - f_1(Z_t\omega)\} d\nu \ dt$$

$$\leqq \tilde{K} \cdot u \cdot \int_\Omega \{f_2(\omega) - f_1(\omega)\} d\mu = 0 \qquad (u > 0),$$

which implies that $\{Z_t: t \geqq 0\}$ is $\overline{\mathfrak{B}_+ \otimes \mathfrak{A}}$-measurable.

We say that $\{Z_t: t \geqq 0\}$ is the ($\overline{\mathfrak{B}_+ \otimes \mathfrak{A}}$-measurable) non-singular semiflow on (Ω, μ) determined by the TWQHP $F(t,x)$.

THEOREM 5. Suppose there is given a measurable TWQHP $F(t,x)$ and consider the non-singular semiflow $\{Z_t: t \geqq 0\}$ on (Ω, μ) determined by the TWQHP. Then there exists a measurable TWQHP $G(t, \omega)$ on (Ω, μ) such that

$$G(t, \omega) = G(0, Z_t\omega), \qquad (t, \omega) \in R_+ \otimes \Omega,$$

$$G(t, \omega) \sim F(t, x) \quad \text{in probability law.}$$

Finally we note that it has been proved by G. Maruyama that if $F(t,x)$ is a real measurable stationary process, then $\{Z_t: t \geqq 0\}$ can be chosen as a measurable measure preserving topological semiflow.

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Science University of Tokyo