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THE LIFTINGS OF PRODUCT FORM ON MEASURE SPACES

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ABSTRACT

In this paper we shall study about the following concept. Let $(X, \mathcal{B}_x, \mu_x)$ and $(Y, \mathcal{B}_y, \mu_y)$ be tow measure spaces with liftings ρ_x and ρ_y respectively. We denote by $(Z, \mathcal{B}_z, \mu_z)$ the product measure space of X and Y. Now we call a lifting ρ_z on Z of product form $\rho_x \times \rho_y$ if it satisfies the following conditions

$$\rho_z(A \times B) = \rho_x(A) \times \rho_y(B) \quad \text{for every } A \in \mathcal{B}_x \text{ and } B \in \mathcal{B}_y.$$

We shall show some properties the liftings of product form and some results connected with the strong lifting property.

§0. Introduction

The theory of lifting on measure spaces has been deveroped for the last fifteen years. In this paper we consider the liftings on product measure spaces. In §1 we write notations and preliminary results which are used in the following sections. In §2 we give the definitions and the characterizations of the lifting of product form. In §3 we show some necessarily and sufficient conditions of the existence of the lifting of product form.

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§1. Preliminaries

Recently A. and C. Ionescu Tulcea have shown many results about the lifting on measure spaces. We shall write several fundamental notations and theorems in their book [I.T] (Topics in the theory of lifting), which play many important role in this paper.

Let $(X, \mathfrak{B}, \mu)$ be measure space and we use the following notations. $\mathfrak{B}_0 = \{A \in \mathfrak{B} : \mu(A) < \infty\}$, $\mathfrak{B}^* = \{A \in \mathfrak{B} : \mu(A) > 0\}$, $\mathfrak{B}_0^* = \mathfrak{B}_0 \cap \mathfrak{B}^*$ and $\mathcal{N}_{\mathfrak{B}} = \{A \in \mathfrak{B} : \mu(A) = 0\}$. We denote by $M_R^\infty(X)$ the algebra of all bounde real-valued $\mathfrak{B}$-measurable functions on X. When for $f, g \in M_R^\infty(X)$ denote by $f \equiv g$ iff $\{x \in X : f(x) \neq g(x)\} \in \mathcal{N}_{\mathfrak{B}}$. Also for $A, B \in \mathfrak{B}$ denote by $A \equiv B$ iff $A \Delta B \in \mathcal{N}_{\mathfrak{B}}$.

Let ρ be a mapping of $M_R^\infty(X)$ into itself and consider the following conditions,

(I) $\rho(f) \equiv f$,

(II) $f \equiv g$ implies $\rho(f) = \rho(g)$,

(III) $\rho(1) = 1$,

(IV) $f \geqq 0$ implies $\rho(f) \geqq 0$,

(V) $\rho(\alpha\cdot f + \beta\cdot g) = \alpha\cdot\rho(f) + \beta\cdot\rho(g)$,

(VI) $\rho(f\cdot g) = \rho(f)\cdot\rho(g)$.

We call ρ is a (linear) lifting of $M_R^\infty(X)$ if it satisfies the conditions ((I) - (V)) (I) - (VI).

Let θ be a mapping of $\mathcal{B}$ into itself and consider the following conditions,

(I′) $\theta(A) \equiv A$,

(II′) $A \equiv B$ implies $\theta(A) = \theta(B)$,

(III′) $\theta(X) = X$ and $\theta(\phi) = \phi$,

(IV′) $\theta(A \cap B) = \theta(A) \cap \theta(B)$,

(V′) $\theta(A \cup B) = \theta(A) \cup \theta(B)$.

We call θ is a lower density of $\mathcal{B}$ if it satisfies the conditions (I′) - (IV′) and a lifting of $\mathcal{B}$ if it satisfies the conditions (I′) - (V′). There exists a one-to-one correspondence between the liftings of $M_R^\infty(X)$ and the liftings of $\mathcal{B}$ which are connected with the epuations.

(L) $\rho(1_A) = 1_{\theta(A)}$ for every $A \in \mathcal{B}$,

where 1_A denotes the characteristic function of the set A ([I.T], Ch.3). So we use the same symbol ρ for a lifting of $M_R^\infty(X)$ and a lifting of $\mathcal{B}$ which are connected with the equations (L) and say it a lifting on X. When X is a topological space with a topology $\mathcal{T}$, we denote by $C_R^\infty(X,\mathcal{T})$ the algebra of all bounded real-valued $\mathcal{T}$-continuous functions on X. A (linear)lifting ρ of $M_R^\infty(X)$ is called strong on $(X,\mathcal{T})$ if $C_R^\infty(X,\mathcal{T}) \subset M_R^\infty(X)$ and it satisfies the following conditions,

(VII) $\rho(f) = f$ for every $f \in C_R^\infty(X,\mathcal{T})$.

We shall introduce some results given in [I.T],[2],[4] and [6] which are used throughout this paper. From now on we shall assume that every measure space $(X,\mathcal{B},\mu)$ is complete, semi-finite localizable measure space and $\mathcal{B}$ is-identical with the σ-algebra of all μ-measurable subsets of X. We say that $(X,\mathcal{B},\mu)$ has the direct sum property if there exists a disjoint family $\{X_\iota\}_{\iota\in I}$ in $\mathcal{B}_0^*$ such that $X=\bigcup_{\iota\in I}X_\iota$ and for every $A\in\mathcal{B}_0^*$ there exists $\iota\in I$ such that $A\cap X_\iota\in\mathcal{B}_0^*$. It was proved that $(X,\mathcal{B},\mu)$ has the direct sum property iff there exists a lifting on X ([I.T] and [6]). A derivation basis on X is a family $\mathcal{F}=\{\mathcal{F}(x)\}_{x\in X}$, where for $x\in X$ $\mathcal{F}(x)$ is a filter basis on $\mathcal{B}_0^*$. For $A\in\mathcal{B}$ and $x\in X$ define

$$\text{(D)} \qquad D(A,\mathcal{F})(x)=\lim_{\mathcal{F}(x)}\mu(A\cap B)/\mu(B)$$

if the limit of the right side exists. A derevation basis $\mathcal{F}$ is said to be weak if for every $A\in\mathcal{B}$ the set $N_{\mathcal{F}}^A$ belongs to $\mathcal{N}_{\mathcal{B}}$, where $N_{\mathcal{F}}^A$ is the set of all $x\in X$ such that $D(A,\mathcal{F})(x)$ isn't defined or $D(A,\mathcal{F})(x)\neq 1_A(x)$. Let $\mathcal{F}$ be a weak derivation basis on X. For $A\in\mathcal{B}$, put $\theta_{\mathcal{F}}(A)=\{x\in X : D(A,\mathcal{F})(x)=1\}$, then $\theta_{\mathcal{F}}$ is a lower density of $\mathcal{B}$ ([4] and [5]). Let $L_R^\infty(X)=M_R^\infty(X)/\equiv$, $\mathbb{B}(X)=\mathcal{B}/\equiv$ and denote by $f\to\dot f$ (respectively $A\to\dot A$) the canonical mapping of $M_R^\infty(X)$ onto $L_R^\infty(X)$ (respectively $\mathcal{B}$ onto $\mathbb{B}(X)$). Since X is localizable, $\mathbb{B}(X)$ is a complete Boolean algebra and by $\dot A\to\dot 1_A$ it is isomorphic to the Boolean algebra of all idemportents in $L_R^\infty(X)$. Denote by $\check X$ the Stone representation space of $\mathbb{B}(X)$ i.e. X is the set of all maximal ideals of $\mathbb{B}(\check X)$ with a totally and extremally disconnected compact hausdorff topology and $\mathbb{B}(X)$ is isomorphic to the Boolean algebra of all clopen subsets of $\check X$. Since $C_R(\check X)$ and $L_R^\infty(X)$ are generated by the Boolean algebra of all their idemportents, $C_R(\check X)$ is isomorphic to $L_R^\infty(X)$ and hence $\check X$ is also the Gelfand representation space of $L_R^\infty(X)$. It was proved that a lifting on X induces a mapping from X into $\check X$ ([I.T] and [2]). Let ρ be a lifting on X. For $x\in X$, put $\mathcal{J}_\rho(x)=\{\dot A\in\mathbb{B}(X) : x\in\rho(A)^c\}$. It is easy to show that $\mathcal{J}_\rho(x)\in\check X$. We note that for $x\in X$ and $\dot f\in L_R^\infty(X)$ put $\chi_x(\dot f)=\rho(f)(x)$ then χ_x is a character of $L_R^\infty(X)$ and the corresponding maximal ideal $\{\dot f\in L_R^\infty(X) : \chi_x(\dot f)=0\}$ is generated by $\{\dot 1_A :$

$\dot{A}\in\mathcal{J}_\rho(x)\}$. Now we define a mapping $\mathcal{J}_\rho(\cdot)$ from X into $\check{X}$. Denote by $\mathcal{T}_\rho$ the weak topology induced by $\mathcal{J}_\rho(\cdot)$. Since $\check{X}$ has a base $\{E(\dot{A}) : A\in\mathcal{B}\}$, where $E(\dot{A}) = \{\mathcal{J}\in\check{X} : \dot{A}\in\mathcal{J}\}$ and $\mathcal{J}_\rho^{-1}(E(\dot{A}^c)) = \rho(A)$ $\mathcal{T}_\rho$ has a base $\{\rho(A) : A\in\mathcal{B}\}$. It was shown that $\mathcal{T}_\rho\subset\mathcal{B}$, $C_R^\infty(X,\mathcal{T}_\rho) = \{\rho(f) : f\in M_R^\infty(X)\}$ and $C_R^\infty(X,\mathcal{T}_\rho)$ is isomorphic to $L_R^\infty(X)$ by $f \to \dot{f}$ ([I.T],Ch.5). Next we consider a mapping from X into $\check{X}$ and construct a lifting from it under suitable conditions.

LEMMA 1. Let $\mathcal{J}(\cdot)$ be a mapping from X into $\check{X}$. For $A\in\mathcal{B}$, put $\rho_\mathcal{J}(A) = \{x\in X : \dot{A}^c\in\mathcal{J}(x)\}$. Then $\rho_\mathcal{J}$ is a mapping from $\mathcal{B}$ into $\mathcal{P}(X)$, where $\mathcal{P}(X)$ is a family of all subsets of X, and $\rho_\mathcal{J}$ satisfies the conditions (II′) - (V′) of a lifting.

Proof. It is clear that $\rho_\mathcal{J}$ satifies (II′) - (IV′). By (IV′) we have $\rho_\mathcal{J}(A)\cup\rho_\mathcal{J}(B)\subset\rho_\mathcal{J}(A\cup B)$. If $x\in\rho_\mathcal{J}(A\cup B)$ then $(A\dot{\cup}B)^c = \dot{A}^c\cap\dot{B}^c\in\mathcal{J}(x)$. Since $\mathcal{J}(x)$ is maximal, $\dot{A}^c\in\mathcal{J}(x)$ or $\dot{B}^c\in\mathcal{J}(x)$ and we have then $x\in\rho_\mathcal{J}(A)$ or $x\in\rho_\mathcal{J}(B)$. So $\rho_\mathcal{J}$ satisfies (V′).

q.e.d.

The above lemma implies the following. If $\rho_\mathcal{J}$ satisfies the condition (I′) then $\rho_\mathcal{J}$ is a lifting on X. We shall give a condition that $\rho_\mathcal{J}$ satisfies the condition(I′).

PROPOSITION 1. Let θ be a lower density of $\mathcal{B}$. For $x\in X$, put $\mathcal{J}_\theta(x) = \{\dot{A}\in\mathbb{B}(X) : x\in\theta(A^c)\}$ then $\mathcal{J}_\theta(x)$ is a proper ideal of $\mathbb{B}(X)$. Let $N_0\in\mathcal{N}_\mathcal{B}$. If $\mathcal{J}(x)\supset\mathcal{J}_\theta(x)$ for every $x\in X - N_0$ then $\theta(A) - N_0\subset\rho_\mathcal{J}(A)$ for every $A\in\mathcal{B}$. In this case $\rho_\mathcal{J}$ satisfies the condition (I′).

Proof. It is clear that $\mathcal{J}_\theta(x)$ is a proper ideal of $\mathbb{B}(X)$. Now we assume $\mathcal{J}(\cdot)$ satisfies the conditions. If x $\theta(A) - N_0$ then $\dot{A}^c\in\mathcal{J}_\theta(x)\subset\mathcal{J}(x)$ and hence $x\in\rho_\mathcal{J}(A)$. So $\theta(A) - N_0\subset\rho_\mathcal{J}(A)$ for every $A\in\mathcal{B}$. Since $\rho_\mathcal{J}(A) = \rho_\mathcal{J}(A^c)^c\subset\theta(A^c)^c\cup N_0$ and $A\equiv\theta(A) - N_0\equiv\theta(A^c)^c\cup N_0$ we have $\rho_\mathcal{J}(A)\equiv A$.

q.e.d.

By Zorn's lemma every proper ideal of $\mathbb{B}(X)$ is contained in a maximal ideal, the existence of a lower density of $\mathcal{B}$ implies the existence of a mapping from X into $\check{X}$ which satisfies the conditions of above proposition, i.e. the existence of a lower density of $\mathcal{B}$ implies the existence of a lifting.

REMARK. We note that $\{1_A^{\cdot} : \dot{A} \in \mathcal{Q}_\theta(x)\}$ generates a closed proper ideal of $L_R^\infty(X)$ and it coincides with J_x in [I.T],Ch.5, Proposition 2. Moreover if $\mathcal{J}(\cdot)$ and $N_0 \in \mathcal{N}_\mathcal{B}$ is same in above proposition then for $x \in X - N_0$, $\{1_A^{\cdot} : \dot{A} \in \mathcal{J}(x)\}$ generates a maximal ideal of $L_R^\infty(X)$ and the character χ_x of $L_R^\infty(X)$ corresponding to it vanishes on J_x. Consequently without the topology $\mathcal{T}_\theta$ ([I.T],Ch.5), we can construct same lifting in [I.T],Ch.5, Proposition 2.

§2. The lifting of product form

Let $(X, \mathcal{B}_x, \mu_x)$ and $(Y, \mathcal{B}_y, \mu_y)$ be measure spaces with liftings ρ_x and ρ_y respectively. Let $(Z, \mathcal{B}_z, \mu_z)$ be a measure space such that $Z = X \times Y$ and $\mathcal{B}_z$ is the σ-algebra of all $\mu_x \times \mu_y$-measurable subsets of Z, where $\mu_x \times \mu_y$ is the product measure of μ_x and μ_y in the sence of [1],Ch.6 and μ_z is the contracted measure of $\widehat{\mu_x \times \mu_y}$ which is the extension of $\mu_x \times \mu_y$ on $\mathcal{B}_z$, i.e.

$$\mu_z(A) = \sup\{\widehat{\mu_x \times \mu_y}(B) : B \in \mathcal{B}_z,\ B \subset A \text{ and } \widehat{\mu_x \times \mu_y}(B) < \infty\}.$$

In the following we shall write X,Y and Z instead of $(X, \mathcal{B}_x, \mu_x)$, and $(Y, \mathcal{B}_y, \mu_y)$ and $(Z, \mathcal{B}_z, \mu_z)$ respectively. We note if X and Y are σ-finite then Z is the usual completion of the product measure space $(X \times Y, \mathcal{B}_x \times \mathcal{B}_y, \mu_x \times \mu_y)$.

LEMMA 2. Z is strictly localizable i.e. a semi-finite and complete measure space with the direct sum property.

Proof. The semi-finiteness and completeness follows from the construction. Since there exist liftings on X and Y, X and Y have the direct sum Property. Let $\{X_\iota\}_{\iota \in I}$ (respectively $\{Y_\kappa\}_{\kappa \in K}$)

be a disjoint family in $\mathcal{B}_{x0}^*$ (respectively $\mathcal{B}_{y0}^*$) by which the direct sum property is satisfied. It is clear that $\{X_\iota \times Y_\kappa\}_{(\iota,\kappa)\in I\times K}$ is a disjoint family in $\mathcal{B}_{z0}^*$ and $Z = \bigcup_{(\iota,\kappa)\in I\times K} X_\iota \times Y_\kappa$. For $A \in \mathcal{B}_{z0}^*$ there exists $A' \in \mathcal{B}_x \times \mathcal{B}_y$ and $N \in \mathcal{N}_{\mathcal{B}_z}$ such that $A \subset A' \cup N$, $A' \cap N = \phi$ and $\mu_z(A) = \mu_x \times \mu_y(A')$. If $\mu_z(A \cap (X_\iota \times Y_\kappa)) = 0$ for every $(\iota,\kappa) \in I \times K$ then $\mu_x \times \mu_y(A' \cap (X_\iota \times Y_\kappa)) = 0$ for every $(\iota,\kappa) \in I \times K$. From the definition of $\mu_x \times \mu_y$ we have then $\mu_x \times \mu_y(A') = 0$. This contradicts to $\mu_x \times \mu_y(A') = \mu_z(A) > 0$.

q.e.d.

This lemma implies that there always exist liftings on Z. So we consider the following definition.

DEFINITION. A lifting ρ_z on Z is said to be of product form $\rho_x \times \rho_y$ if it satisfies the following conditions,

(P) $\rho_z(A \times B) = \rho_x(A) \times \rho_y(B)$ for every $A \in \mathcal{B}_x$ and $B \in \mathcal{B}_y$.

We shall show some characterizations of the lifting of product form $\rho_x \times \rho_y$.

THEOREM . Let ρ_x, ρ_y and ρ_z be liftings on X, Y and Z respectively. Then for these ρ_x, ρ_y and ρ_z the following conditions are equivalent to each others.

i) ρ_z is of product form $\rho_x \times \rho_y$,

ii) $\rho_z(A \times Y) \subset \rho_x(A) \times Y$ and $\rho_z(X \times B) \subset X \times \rho_y(B)$ for $A \in \mathcal{B}_x$ and $B \in \mathcal{B}_y$,

iii) $\rho_x(A) \times Y \subset \rho_z(A \times Y)$ and $X \times \rho_y(B) \subset \rho_z(X \times B)$ for $A \in \mathcal{B}_x$ and $B \in \mathcal{B}_y$,

iv) $\mathcal{T}_{\rho_x} \times \mathcal{T}_{\rho_y} \subset \mathcal{T}_{\rho_z}$ where $\mathcal{T}_{\rho_x} \times \mathcal{T}_{\rho_y}$ means the product topology on Z of $\mathcal{T}_{\rho_x}$ and $\mathcal{T}_{\rho_y}$,

v) ρ_z is strong on $(Z, \mathcal{T}_{\rho_x} \times \mathcal{T}_{\rho_y})$,

vi) $\rho_z(f \times 1) = \rho_x(f) \times 1$ and $\rho_z(1 \times g) = 1 \times \rho_y(g)$ for $f \in M_R^\infty(X)$ and $g \in M_R^\infty(Y)$.

Proof. i) → ii) and i) → iii) are obvious. ii) → i) Since $A \times B = (A \times Y) \cap (X \times B)$ we have $\rho_z(A \times B) \subset \rho_x(A) \times \rho_y(B)$. Since $\rho_z(C^c) =$

$\rho_z(C)^c$ and $(A\times B)^c = (A^c\times Y)\cap(X\times B^c)$ we have $\rho_z(A\times B)^c \subset (\rho_x(A)\times\rho_y(B))^c$. So $\rho_z(A\times B) = \rho_x(A)\times\rho_y(B)$. iii) $\to$ i) follows similarly from the case ii) $\to$ i). Thus it is sufficient to prove i) $\to$ iv) $\to$ v) $\to$ vi) $\to$ ii). i) $\to$ iv) The topologies $\mathcal{T}_{\rho_x}$ and $\mathcal{T}_{\rho_y}$ have bases $\{\rho_x(A) : A\in\mathcal{B}_x\}$ and $\{\rho_y(B) : B\in\mathcal{B}_y\}$ respectively. So we can take as a base of $\mathcal{T}_{\rho_x}\times\mathcal{T}_{\rho_y}$ the family $\{\rho_x(A)\times\rho_y(B) : A\in\mathcal{B}_x$ and $B\in\mathcal{B}_y\}$. From i) $\rho_x(A)\times\rho_y(B) = \rho_z(A\times B)$ and $\rho_z(A\times B)$ is contained in $\mathcal{T}_{\rho_z}$, we have then $\mathcal{T}_{\rho_x}\times\mathcal{T}_{\rho_y}\subset\mathcal{T}_{\rho_z}$. iv) $\to$ v) $\mathcal{T}_{\rho_x}\times\mathcal{T}_{\rho_y}\subset\mathcal{T}_{\rho_z}$ implies $C_R^\infty(Z,\mathcal{T}_{\rho_x}\times\mathcal{T}_{\rho_y})\subset C_R^\infty(Z,\mathcal{T}_{\rho_z})$. Since $C_R^\infty(Z,\mathcal{T}_{\rho_z}) = \{\rho_z(f) : f\in M_R^\infty(Z)\}$ and $\rho_z(\rho_z(f)) = \rho_z(f)$, ρ_z is strong on $(Z,\mathcal{T}_{\rho_x}\times\mathcal{T}_{\rho_y})$. v) $\to$ vi) Since $\rho_x(f)\times 1$ is $\mathcal{T}_{\rho_x}\times\mathcal{T}_{\rho_y}$-continuous, we have $\rho_z(f\times 1) = \rho_z(\rho_x(f)\times 1) = \rho_x(f)\times 1$. Similarly $\rho_z(1\times g) = 1\times\rho_y(g)$. vi) $\to$ ii) $\rho_z(1_A\times 1) = \rho_x(1_A)\times 1$ implies $\rho_z(A\times Y) = \rho_x(A)\times Y$, and $\rho_z(1\times 1_B) = 1\times\rho_y(1_B)$ implies $\rho_z(X\times B) = X\times\rho_y(B)$.

Q.E.D.

This theorem is similar to [I.T],Ch.5,Th.3 but to gether with its proof this is more simplified. We note that $C_R^\infty(Z,\mathcal{T}_{\rho_x}\times\mathcal{T}_{\rho_y})$ is isomorphic to $L_R^\infty(X)\otimes_\lambda L_R^\infty(Y)$, where λ is the least cross norm, and also isomorphic to a closed subalgebra of $L_R^\infty(Z)$ by $f\to\dot{f}$.

EXAMPLES. 1) Either X or Y is purely atomic, i.e. every measurable set of positive measure contains an atom, then there exists a lifting of product form $\rho_x\times\rho_y$ for every pair of liftings ρ_x on X and ρ_y on Y.

2) Let $X = Y$ be the ordinary Lebesgue measure space of R. Then Z becomes the ordinary Lebesgue measure space of R^2. For $x\in R$, let $\mathcal{F}_1(x)$ be the filter basis consisting of the sections of the family $\{I :$ bounded open interval in R and $x\in I\}$. For $(x,y)\in R^2$, let $\mathcal{F}_2(x,y)$ be the filter basis consisting of the sections of the family $\{I\times J :$ I and J are bounded open intervals in R such that $x\in I$ and $y\in J\}$. Then $\mathcal{F}_1 = \{\mathcal{F}_1(x)\}_{x\in R}$ (respectively $\mathcal{F}_2 = \{\mathcal{F}_2(z)\}$

$z\in R^2$) is a weak derivation basis in R (respectively R^2)([3]). Put $\mathcal{J}_{\mathcal{F}_1}(x)=\{\dot{A}\in\mathbb{B}(R) : D(A,\mathcal{F}_1)(x)=0\}$ and $\mathcal{J}_{\mathcal{F}_2}(z)=\{\dot{A}\in\mathbb{B}(R^2) : D(A,\mathcal{F}_2)(z)=0\}$. It is easy to show that $\mathcal{J}_{\mathcal{F}_1}(x)=\mathcal{J}_{\theta_{\mathcal{F}_1}}(x)$ and $\mathcal{J}_{\mathcal{F}_2}(z)=\mathcal{J}_{\theta_{\mathcal{F}_2}}(z)$. So $\mathcal{J}_{\mathcal{F}_1}(x)$ and $\mathcal{J}_{\mathcal{F}_2}(z)$ are proper ideals of $\mathbb{B}(R)$ and $\mathbb{B}(R^2)$ respectively. Let $\mathcal{J}_1(\cdot)$ be a mapping from R into $\check{R}$ such that $\mathcal{J}_1(x)\supset\mathcal{J}_{\mathcal{F}_1}(x)$ for every $x\in R$ and ρ_1 be the lifting on R defined by $\rho_1(A)=\{x\in R : \dot{A}^c\in\mathcal{J}_1(x)\}$. We shall show the existence of a lifting of product form $\rho_1\times\rho_1$.

LEMMA 3. For $(x,y)\in R^2$ denote by $\mathcal{J}_2((x,y))$ the ideal of $\mathbb{B}(R^2)$ generated by $\mathcal{J}_1(x)\times\dot{R}$, $\dot{R}\times\mathcal{J}_1(y)$ and $\mathcal{J}_{\mathcal{F}_2}((x,y))$. Then $\mathcal{J}_2((x,y))$ is proper.

Proof. If $\mathcal{J}_2((x,y))$ isn't proper then there exist $\dot{A}\in\mathcal{J}_1(x)$, $\dot{B}\in\mathcal{J}_1(y)$ and $\dot{C}\in\mathcal{J}_{\mathcal{F}_2}((x,y))$ such that $\dot{R}^2=(\dot{A}\times\dot{R})\cup(\dot{R}\times\dot{B})\cup\dot{C}$. Consequently $\dot{A}^c\times\dot{B}^c\subset\dot{C}$ and this implies $\dot{A}^c\times\dot{B}^c\in\mathcal{J}_{\mathcal{F}_2}((x,y))$. From the definition we have $D(A^c\times B^c,\mathcal{F}_2)((x,y))=\lim_{\mathcal{F}_2((x,y))}\mu_2((A^c\times B^c)\cap(I\times J))/\mu_2(I\times J)$ $=\lim_{\mathcal{F}_2((x,y))}\mu_1(A^c\cap I)\cdot\mu_1(B^c\cap J)/\mu_1(I)\cdot\mu_1(J)=0$. So $D(A^c,\mathcal{F}_1)(x)=0$ or $D(B^c,\mathcal{F}_1)(y)=0$. If $D(A^c,\mathcal{F}_1)(x)=0$ then $\dot{A}^c\in\mathcal{J}_1(x)$. This contradicts to $\dot{A}\in\mathcal{J}_1(x)$ since $\mathcal{J}_1(x)$ is proper. Similarly $D(B^c,\mathcal{F}_1)(y)=0$ implies a contradiction.

q.e.d.

Let $\mathcal{J}(\cdot)$ be a mapping from R^2 into $\check{R}^2$ such that $\mathcal{J}(z)\supset\mathcal{J}_2(z)$ for every $z\in R^2$. Put $\rho(A)=\{z\in R^2 : \dot{A}^c\in\mathcal{J}(z)\}$ then ρ is a lifting on R^2. For $A\in\mathcal{B}_X$ and $x\in\rho_1(A)$, $\dot{A}^c\times R\in\mathcal{J}_1(x)\times\dot{R}\subset\mathcal{J}((x,y))$ for every $y\in R^2$, so $(x,y)\in\rho(A\times R)$ for every $y\in R$. This implies that $\rho_1(A)\times R\subset\rho(A\times R)$ for every $A\in\mathcal{B}_X$. Similarly $R\times\rho_1(B)\subset\rho(R\times B)$ for every $B\in\mathcal{B}_Y$. By Theorem 1 we conclude ρ is of product form $\rho_1\times\rho_1$.

§3. The conditions of the existence of liftings of product form

In §2 we have shown that a lifting of product form $\rho_X\times\rho_Y$ is strong on $(Z,\mathcal{T}_{\rho_X}\times\mathcal{T}_{\rho_Y})$, so the existence of a lifting of product form $\rho_x \times \rho_y$ is equivalent to the existence of a strong lifting on $(Z, \mathcal{T}_{\rho_x} \times \mathcal{T}_{\rho_y})$. Therefore some results given in [I.T], [4] and [5] are useful to show the conditions of the existence of a lifting of product form. For this we give a new definition. Let $(X, \mathcal{B}, \mu)$ be a measure space and X is a topological space with a topology $\mathcal{T}$. A (linear)lifting ρ of $M_R^\infty(X)$ is called almost strong on $(X, \mathcal{T})$ if $C_R^\infty(X, \mathcal{T}) \subset M_R^\infty(X)$ and there exists $N_0 \in \mathcal{N}_{\mathcal{B}}$ such that

(VII′) $\rho(f) = f$ on $X - N_0$ for every $f \in C_R^\infty(X, \mathcal{T})$.

Let X, Y, Z, ρ_x and ρ_y be same in §2. Then we prove;

THEOREM 2. The following statements are equivalent to each others.

i) There exists a lifting on Z of product form $\rho_x \times \rho_y$,

ii) There exists an almost strong lifting on $(Z, \mathcal{T}_{\rho_x} \times \mathcal{T}_{\rho_y})$,

iii) There exists an almost strong linear lifting on $(Z, \mathcal{T}_{\rho_x} \times \mathcal{T}_{\rho_y})$,

iv) There exists a lower density θ of $\mathcal{B}_z$ such that for some $N_0 \in \mathcal{N}_{\mathcal{B}_z}$ $(\rho_x(A) \times Y) - N_0 \subset \theta(A \times Y)$ and $(X \times \rho_y(B)) - N_0 \subset \theta(X \times B)$ for every $A \in \mathcal{B}_x$ and $B \in \mathcal{B}_y$,

v) There exists a weak derivation basis $\mathcal{F}$ on Z such that for some $N_0 \in \mathcal{N}_{\mathcal{B}_z}$, $D(A \times Y, \mathcal{F}) = 1_{\rho_x(A) \times Y}$ and $D(X \times B, \mathcal{F}) = 1_{X \times \rho_y(B)}$ on $Z - N_0$ for every $A \in \mathcal{B}_x$ and $B \in \mathcal{B}_y$.

Proof. i) → ii) → iii) is obvious. iii) → iv) Let ρ_z be an almost strong linear lifting on $(Z, \mathcal{T}_{\rho_x} \times \mathcal{T}_{\rho_y})$ and $N_0 \in \mathcal{N}_{\mathcal{B}_z}$ such that $\rho_z(f) = f$ on $Z - N_0$ for every $f \in C_R^\infty(Z, \mathcal{T}_{\rho_x} \times \mathcal{T}_{\rho_y})$. For $A \in \mathcal{B}_z$ put $\theta(A) = \{z \in Z : \rho_z(1_A)(z) = 1\}$ then θ is a lower density of $\mathcal{B}_z$ ([I. T]). Since $1_{\rho_x(A)} \times 1$ is $\mathcal{T}_{\rho_x} \times \mathcal{T}_{\rho_y}$-continuous we have $1_{\theta(A \times Y)} = \rho_z(1_A \times 1) = \rho_z(\rho_x(1_A) \times 1) = \rho_x(1_A) \times 1$ on $Z - N_0$ for every $A \in \mathcal{B}_x$. Consequently $(\rho_x(A) \times Y) - N_0 \subset \theta(A \times Y)$ for every $A \in \mathcal{B}_x$. Similarly $(X \times \rho_y(B)) - N_0 \subset \theta(X \times B)$ for every $B \in \mathcal{B}_y$. iv) → i) Let θ be a lower density of $\mathcal{B}_z$ which satisfies the conditions in iv). For

$z \in N_0$ denote by $\mathcal{J}'(z)$ the ideal of $\mathbb{B}(Z)$ generated by $\{\dot{A}\times\dot{Y},\ \dot{X}\times\dot{B} :$ $z\in\rho_x(A^c)\times Y$ and $z\in X\times\rho_y(B^c)\}$. Let $\mathcal{J}(\cdot)$ be a mapping from Z into $\breve{Z}$ such that $\mathcal{J}(z)\supset\mathcal{J}_\theta(z)$ for every $z\in Z - N_0$ and $\mathcal{J}(z)\supset\mathcal{J}'(z)$ for every $z\in N_0$. Then by Proposition 1 $\rho_{\mathcal{J}}$ is a lifting on Z and for every $C\in\mathcal{B}_z$ $\theta(C) - N_0\subset\rho_{\mathcal{J}}(C)$. For $A\in\mathcal{B}_x$ and $z\in(\rho_x(A)\times Y)\cap N_0$ we have $\dot{A}^c\times\dot{Y}\in\mathcal{J}'(z)\subset\mathcal{J}(z)$ and then $z\in\rho_{\mathcal{J}}(A\times Y)$. Consequently $\rho_x(A)$ $\times Y\subset\rho_{\mathcal{J}}(A\times Y)$ for every $A\in\mathcal{B}_x$. Similarly $X\times\rho_y(B)\subset\rho_{\mathcal{J}}(X\times B)$ for every $B\in\mathcal{B}_y$. By Theorem 1 $\rho_{\mathcal{J}}$ is a lifting of product form $\rho_x\times\rho_y$.
i) → v) Let ρ_z be a lifting on Z of product form $\rho_x\times\rho_y$. Put $N_0 = (\cup\{\rho_z(A) : A\in\mathcal{B}_z{}^*_0\})^c$ then $N_0\in\mathcal{N}_{\mathcal{B}_z}$. For $z\in N_0^c$ denote by $\mathcal{F}_\rho(z)$ the filter basis consisting of the sections of the family $\{\rho_z(A) : z\in$ $\rho_z(A)$ and $A\in\mathcal{B}_z{}^*_0\}$ and for $z\in N_0$ denote by $\mathcal{F}_\rho(z)$ an arbitrary filter basis on $\{\rho_z(A) : A\in\mathcal{B}_z{}^*_0\}$. Now for $A\in\mathcal{B}_z{}^*$ and $z\in\rho_z(A) - N_0$ there exists $C_0\in\mathcal{B}_z{}^*_0$ such $z\in\rho_z(C_0)$. Since the section $\mathcal{A} = \{\rho_z(A$ $\cap C) : z\in\rho_z(A\cap C)$ and $\dot{C}\subset\dot{C}_0\}$ belongs to $\mathcal{F}_\rho(z)$ and $\mu_z(A\cap C)/\mu_z(C) =$ 1 for every $C\in\mathcal{A}$, $D(A,\mathcal{F}_\rho)(z) = 1$ on $\rho_z(A) - N_0$. Similarly $D(A^c,\mathcal{F}_\rho)$ $(z) = 1$ on $\rho_z(A^c) - N_0$ and hence $D(A,\mathcal{F}_\rho)(z) = 1 - D(A^c,\mathcal{F}_\rho)(z) = 0$ on $\rho_z(A^c) - N_0$. We have then $D(A,\mathcal{F}_\rho)(z) = 1_{\rho_z(A)}$ on $Z - N_0$. This implies $N^A_{\mathcal{F}_\rho}\in\mathcal{N}_{\mathcal{B}_z}$ and therefor $\mathcal{F}_\rho$ is weak. Moreover it satisfies the conditions in v). Finally we shall prove v) → iv). Let $\mathcal{F}$ be a derivation basis on Z which satisfies the conditions in v). Let $\theta_{\mathcal{F}}$ be the lower density of $\mathcal{B}_z$ defined in §1. For $A\in\mathcal{B}_x$ and $z\in(\rho_x(A)\times Y) - N_0$ $D(A\times Y,\mathcal{F})(z) = 1$, so $z\in\theta_{\mathcal{F}}(A\times Y)$. We have then $(\rho_x(A)\times Y) - N_0\subset\theta_{\mathcal{F}}(A\times Y)$. Similarly $(X\times\rho_y(B)) - N_0\subset\theta_{\mathcal{F}}(X\times B)$.

Q.E.D.

We shall give some definitions and results for another conditions of the existence of a lifting of product form. Let $(X,$ $\mathcal{B},\mu)$ be a measure spce. We denote by $\mathcal{B}^\sigma$ the σ-ring of all σ-finite measurable subsets of X. For $A\in\mathcal{B}^*$ denote by $(A,\mathcal{B}\cap A,\mu|_A)$ the measure space with σ-algebra $\mathcal{B}\cap A = \{B\cap A : B\in\mathcal{B}\}$ and the measure $\mu|_A(B) = \mu(B)$ for $B \in \mathfrak{B} \cap A$. It is clear that $\mathfrak{B} \cap A$ is identical to the σ-algebra of all $\mu|_A$-measurable subsets of A and $\mu|_A$ is semi-finite. Since $\mathbb{B}(A)$ is isomorphic to $\mathbb{B}(X) \cap A$, $\mathbb{B}(A)$ is complete and A is localizable. Now let X, Y and Z be same in §2. We have then $\mathfrak{B}_z = \{A \subset Z : A \cap (E\times F) \in \mathfrak{B}_z$ for every $E \in \mathfrak{B}_x^{\sigma}$ and $F \in \mathfrak{B}_y^{\sigma}$ and $N \in \mathcal{N}_{\mathfrak{B}_z}$ iff $N \cap (E\times F) \in \mathcal{N}_{\mathfrak{B}_z}$ for every $E \in \mathfrak{B}_x^{\sigma}$ and $F \in \mathfrak{B}_y^{\sigma}$. Moreover $(E\times F, \mathfrak{B}_z \cap E\times F, \mu_z|_{E\times F}) = (E\times F, \widehat{\mathfrak{B}_x \cap E \times \mathfrak{B}_y \cap F}, \widehat{\mu_x|_E \times \mu_y|_F})$ for every $E \in \mathfrak{B}_x^{\sigma}$ and $F \in \mathfrak{B}_y^{\sigma}$, where $(E\times F, \widehat{\mathfrak{B}_x \cap E \times \mathfrak{B}_y \cap F}, \widehat{\mu_x|_E \times \mu_y|_F})$ is the completion of the usual product measure space of $(E, \mathfrak{B}_x \cap E, \mu_x|_E)$ and $(F, \mathfrak{B}_y \cap F, \mu_y|_F)$. Let $(X, \mathfrak{B}, \mu)$ and $(X', \mathfrak{B}', \mu')$ be measure spaces. A mapping Φ from X onto X′ is called non-singular when Φ is invertible bi-measurable and for $N \in \mathfrak{B}$ $N \in \mathcal{N}_{\mathfrak{B}}$ iff $\Phi(N) \in \mathcal{N}_{\mathfrak{B}'}$. Since non-singular mapping preserves σ-finiteness we have the following by above statements.

LEMMA 4. Let X, X′, Y and Y′ be measure spaces and Φ_x (respectively Φ_y) be a non-singular mapping from X onto X′ (respectivly Y onto Y′). Let Z (respectively Z′) be the product measure space of X and Y (respectively X′ and Y′), which is constracted in §2. Put $\Phi_z((x,y)) = (\Phi_x(x), \Phi_y(y))$ for $(x,y) \in Z$ then Φ_z is a non-singular mapping from Z onto Z′.

DEFINITION. Let X and X′ be measure spaces with liftings ρ_x and ρ'_x respectively. We call ρ_x and ρ'_x are weakly equivalent and write $\rho_x \simeq \rho'_x$ when there exist $N_x \in \mathcal{N}_{\mathfrak{B}_x}$, $N'_x \in \mathcal{N}_{\mathfrak{B}'_x}$ and a non-singular mapping Φ from N_x^c onto $N_x'^c$ such that

$$\text{(W.E)} \qquad \Phi(\rho_x(A) - N_x) = \rho'_x(\Phi(A - N_x)) - N'_x \quad \text{for every } A \in \mathfrak{B}.$$

THEOREM 3. Let X, Y, Z, ρ_x and ρ_y be same in §2. Let X′ and Y′ be measure spaces with liftings ρ'_x and ρ'_y such that $\rho_x \simeq \rho'_x$ and $\rho_y \simeq \rho'_y$ respectively. Let Z′ be the product measure space of X′ and Y′ constracted like as Z. If there exists a lifting

ρ_z on Z of product form $\rho_x \times \rho_y$ then there exists a lifting ρ'_z on Z' of product form $\rho'_x \times \rho'_y$ and $\rho_z \simeq \rho'_z$.

Proof. Let $N_x \in \mathcal{N}_{\mathcal{B}x}$, $N'_x \in \mathcal{N}_{\mathcal{B}'x}$, $N_y \in \mathcal{N}_{\mathcal{B}y}$, $N'_y \in \mathcal{N}_{\mathcal{B}'y}$, Φ_x and Φ_y be sets and mappings by which $\rho_x \simeq \rho'_x$ and $\rho_y \simeq \rho'_y$ are defined. In the following denote by N_z (respectively N'_z) instead of $(N_x^c \times N_y^c)^c$ (respectively $(N_x'^c \times N_y'^c)^c$). We have then $N_z \in \mathcal{N}_{\mathcal{B}_z}$ and $N'_z \in \mathcal{N}_{\mathcal{B}'_z}$. Put $\Phi((x,y)) = (\Phi_x(x), \Phi_y(y))$ for $(x,y) \in N_z^c$ then by Lemma 4 Φ is a non-singular mapping from N_z^c onto $N_z'^c$. For $z' \in N'_z$ denote by $\mathcal{J}(z')$ the maximal ideal in $\mathbb{B}(Z')$ which contains the family $\{(\dot{A}' \times \dot{Y}'), (\dot{X}' \times \dot{B}') : z' \in \rho'_x(A'^c) \times Y$ and $z' \in X' \times \rho'_y(B'^c)\}$. For $C' \in \mathcal{B}'_z$ put $\rho'_z(C') = \Phi(\rho_z(\Phi^{-1}(C' - N'_z)) - N'_z) \cup \{z' \in N'_z : \dot{C}'^c \in \mathcal{J}(z')\}$. It is easy to show that ρ'_z is a lifting on Z' of product form $\rho'_x \times \rho'_y$ and $\rho_z \simeq \rho'_z$.

Q.E.D.

To show the corollaly of this theorm and the following theorem we introduce the following notations. Let $(X, \mathcal{B}, \mu)$ be a measure space with a lifting ρ. For $A \in \mathcal{B}^*$ we denote by $\{\rho|_A\}$ the family of all liftings on $(A, \mathcal{B} \cap A, \mu|_A)$ such that if $\rho' \in \{\rho|_A\}$ then $\rho(B) \cap A = \rho'(B) \cap A$ for every $B \in \mathcal{B} \cap A$. It is clear that for $A, B \in \mathcal{B}^*$ such that $A \equiv B$, we have $\rho_1 \simeq \rho_2$ for every $\rho_1 \in \{\rho|_A\}$ and $\rho_2 \in \{\rho|_B\}$. We note that $\{\rho|_A\} \neq \phi$ because for $B \in \mathcal{B} \cap A$ put $\rho'(B) = (\rho(B) \cap A) \cup \{x \in A \cap \rho(A)^c : \dot{B}^c \in \mathcal{J}\}$, where $\mathcal{J}$ is an arbitrary but fixed maximal ideal of $\mathbb{B}(A)$, then $\rho' \in \{\rho|_A\}$. Moreover if $A \subset \rho(A)$ then $\{\rho|_A\}$ is singleton and in this case we write it $\rho|_A$. When there is no ambiguity we shall write A instead of $(A, \mathcal{B} \cap A, \mu|_A)$.

COROLLARY. Let X, Y, Z, ρ_x and ρ_y be same in §2. For $A \in \mathcal{B}_x^*$ and $B \in \mathcal{B}_y^*$ let ρ_A and ρ_B be liftings on A and B respectively. If $\rho_A \in \{\rho_x|_A\}$ and $\rho_B \in \{\rho_y|_B\}$ and there exists a lifting ρ_z on Z of product form $\rho_x \times \rho_y$ then there exists a lifting $\rho_{A \times B}$ on $A \times B$ of product form $\rho_A \times \rho_B$ and $\rho_{A \times B} \in \{\rho_z|_{A \times B}\}$.

Proof. Since $\rho_A \triangleq \rho_x|_{\rho_x(A)}$ and $\rho_B \triangleq \rho_y|_{\rho_y(B)}$, it is enough to show that there exists a lifting on $\rho_x(A)\times \rho_y(B)$ of product form $\rho_x|_{\rho_x(A)}\times \rho_y|_{\rho_y(B)}$. It is clear that $\rho_z|_{\rho_x(A)\times\rho_y(B)} = \rho_z|_{\rho_z(A\times B)}$ is the desirous lifting on $\rho_x(A)\times \rho_y(B)$.

q.e.d.

Let $(X,\mathcal{B},\mu)$ be a measure space with a lifting ρ. A subfamily $\{X_\iota\}_{\iota\in I}\subset\mathcal{B}$ is called μ-dence if $X_\iota\cap X_{\iota'}\in\mathcal{N}_{\mathcal{B}}$ for $\iota\neq\iota'$ and for every $A\in\mathcal{B}^*$ there exists $\iota\in I$ such that $A\cap X_\iota\in\mathcal{B}^*$. If $\{X_\iota\}_{\iota\in I}$ is μ-dence then $\{\rho(X_\iota)\}_{\iota\in I}$ is also μ-dence. Moreover put $N=(\bigcup_{\iota\in I} X_\iota)^c$ then $N\in\mathcal{N}_{\mathcal{B}}$ and for $A,\ B\in\mathcal{B}$ $A\equiv B$ iff $A\cap X_\iota\equiv B\cap X_\iota$ for every $\iota\in I$.

THEOREM 4. Let X, Y, Z, ρ_x and ρ_y be same in §2. If there exists a μ_x-dence family $\{X_\iota\}_{\iota\in I}$ (respectivery μ_y-dence family $\{Y_\kappa\}_{\kappa\in K}$) such that for every $\iota\in I$ there exists a lifting ρ_ι on X_ι which satisfies $\rho_\iota\in\{\rho_x|_{X_\iota}\}$ (respectively for every $\kappa\in K$ there exists a lifting ρ_κ on K_κ which satisfies $\rho_\kappa\in\{\rho_y|_{Y_\kappa}\}$) and if there exists a lifting $\rho_{(\iota,\kappa)}$ on $X_\iota\times Y_\kappa$ of product form $\rho_\iota\times\rho_\kappa$. Then there exists a lifting ρ_z on Z of product form $\rho_x\times\rho_y$ and $\rho_{(\iota,\kappa)}\in\{\rho_z|_{X_\iota\times Y_\kappa}\}$.

Proof. By the assumption, Theorem 3 and statements following to it there exists a lifting $\rho'_{(\iota,\kappa)}$ on $\rho_x(X_\iota)\times\rho_y(Y_\kappa)$ of product form $\rho_x|_{\rho_x(X_\iota)}\times\rho_y|_{\rho_y(Y_\kappa)}$. Put $N=((\bigcup_{\iota\in I}\rho_x(X_\iota))\times(\bigcup_{\kappa\in K}\rho_y(Y_\kappa)))^c$ then $N\in\mathcal{N}_{\mathcal{B}_z}$. Now for every $z\in N$, we denote by $\mathcal{J}(z)$ the maximal ideal of $\mathbb{B}(Z)$ which contains the family $\{\dot{A}\times\dot{Y},\ \dot{X}\times\dot{B} : z\in\rho_x(A^c)\times Y$ and $z\in X\times\rho_y(B^c)\}$. For $A\in\mathcal{B}_z$, put $\rho_z(A)=\bigcup_{(\iota,\kappa)\in I\times K}\rho'_{(\iota,\kappa)}(A\cap(\rho_x(X_\iota)\times\rho_y(Y_\kappa)))$ $\cup\{z\in N : \dot{A}^c\in\mathcal{J}(z)\}$, then ρ_z is a lifting on Z of product form $\rho_x\times\rho_y$ and $\rho_{(\iota,\kappa)}\in\{\rho_z|_{X_\iota\times Y_\kappa}\}$.

Q.E.D.

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