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Cohomology of vector fields on a complex manifold

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§1. Introduction

Let M be a complex manifold of complex dimension n, and let $A(M)$ denote the Lie algebra of smooth vector fields of type (1, 0) on M. Elements of $A(M)$ can be expressed locally as $\sum_{\alpha=1}^{n} \xi^{\alpha}(z, \bar{z})\frac{\partial}{\partial z^{\alpha}}$ in terms of a local holomorphic coordinate $\{z^1, \ldots, z^n\}$, $\xi^{\alpha}(1 \leqq \alpha \leqq n)$ being smooth functions, and the bracket operation is defined by

$$[\sum_{\alpha=1}^{n} \xi^{\alpha}\frac{\partial}{\partial z^{\alpha}}, \sum_{\alpha=1}^{n} \eta^{\alpha}\frac{\partial}{\partial z^{\alpha}}] = \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} (\xi^{\beta}\frac{\partial \eta^{\alpha}}{\partial z^{\beta}} - \eta^{\beta}\frac{\partial \xi^{\alpha}}{\partial z^{\beta}})\frac{\partial}{\partial z^{\alpha}} .$$

Recently it is proved that this Lie algebra determines the complex structure of M :

<u>Theorem</u> (I. Amemiya [1]). Let M and M' be complex manifolds and assume that there is an isomorphism of Lie algebras $\varphi : A(M) \xrightarrow{\approx} A(M')$. Then there is a biholomorphic mapping $\Phi : M \xrightarrow{\approx} M'$ which induces φ.

On the other hand, in these years, I. M. Gel'fand and D. B. Fuks have developed the cohomology theory for the infinite dimensional Lie algebras such as the Lie algebra of all the smooth vector fields on a smooth manifold and the Lie algebra of formal vector fields (cf. [3]). Their methods can be applied to our infinite-dimensional Lie algebra A(M). In view of Amemiya's theorem, it is expected that the cohomology of A(M) is useful in the study of the complex manifold M.

§2. Lie algebra cohomology

We shall recall briefly the definition of the Lie algebra cohomology.

Let g be a Lie algebra and W a g-module.
Put $C^p(g;W) = \{w : \underbrace{g \times \cdots \times g}_{p} \longrightarrow W \mid \omega$ is multi-linear and skew symmetric$\}$ for $p > 0$, $C^0(g;W) = W$ and $C^p(g;W) = 0$ for $p < 0$. Define the coboundary operator $d: C^p(g;W) \longrightarrow C^{p+1}(g;W)$ by the following formula :

$$(d\omega)(X_1,\ldots,X_{p+1}) = \sum_{i=1}^{p+1}(-1)^{i-1}X_i\,\omega(X_1,\ldots,\hat{X}_i,\ldots,X_{p+1})$$
$$+ \sum_{1\le i<j\le p+1}(-1)^{i+j}\omega([X_i,X_j],X_1,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_{p+1})$$

($\omega \in C^p(g;W)$, $X_1,\ldots,X_{p+1} \in g$). It is easy to check $d^2 = 0$.
The p-th cohomology of the cochain complex $C(g;W) = \bigoplus_{p=0}^{\infty} C^p(g;W)$ is denoted by $H^p(g;W)$. If W has a ring structure and every element X of g acts on W as a derivation, that is, $X(uv) = (Xu)v + u(Xv)$ for all $u, v \in W$, then the total cohomology $H^*(g;W) = \bigoplus_{p=0}^{\infty} H^p(g;W)$ has a natural graded ring structure.
(For more details, see [8] for example.)

When g is an infinite dimensional Lie algebra (e.g. $A(M)$), we often diminish the cochain space by imposing auxiliary conditions such as continuity with respect to some topology of g.

In the next section, we shall review some of the main results in the case of a smooth manifold.

§3 Lie algebra of smooth vector fields

Let N be a smooth manifold of dimension n and $a(N)$ the Lie algebra of all the smooth vector fields on N.

1°. $W = C(N)$. Here $C^{\infty}(N)$ denotes the ring of smooth functions on N and elements of $a(N)$ acts on $C^{\infty}(N)$ as derivations. (Recall that we can identify the Lie algebra $a(N)$ with the Lie algebra of all the derivations of the ring $C^{\infty}(N)$.) Deline the space of support preserving cochains $C^p_{\Delta}(a(N);C^{\infty}(N))$ $=\{\omega\in C^p(a(N);C^{\infty}(N)) \mid \operatorname{supp}\omega(X_1,\dots,X_p)\subset\bigcap_{i=1}^{p}\operatorname{supp} X_i\,(X_1,\dots,X_p\in a(N))\}$. Then $C_{\Delta}(a(N);C^{\infty}(N)) = \bigoplus_{p=0}^{\infty} C^p_{\Delta}(a(N);C(N))$ forms a subcomplex of $C(a(N);C^{\infty}(N))$, whose p-th cohomology group will be denoted by $H^p_{\Delta}(a(N);C^{\infty}(N))$. Note that both

$$H^*(a(N), C^{\infty}(N)) \quad \text{and} \quad H^*_{\Delta}(a(N);C^{\infty}(N)) = \bigoplus_{p=0}^{\infty} H^p_{\Delta}(a(N);C^{\infty}(N))$$

have a natural graded ring structure.

<u>Theorem</u> (Losik [6]). We have an isomorphism of graded rings: $H^*_{\Delta}(a(N);C^{\infty}(N)) \cong H^*(B(\tau^{\mathbb{C}}), \mathbb{R})$. Here τ denotes the real tangent bundle of N and $B(\tau^{C})$, the principal $U(n)$- bundle associated with the complexification $\tau^{\mathbb{C}}$ of τ.

2^o. $W = a(N)$. Here $a(N)$ is regarded as an $a(N)$-module by the adjoint representation : an element X of $a(N)$ acts on $a(N)$ by $Y \longmapsto [X, Y]$ $(Y \in a(N))$. In this case also, we define $C^p_\Delta(a(N)); a(N))$ to be the subspace of $C^p(a(N); a(N))$ consisting of the support preserving cochains, and the "diagonal cohomology" $H^*_\Delta(a(N); a(N)) = \bigoplus_{p=0}^{\infty} H^p_\Delta(a(N); a(N))$ is defined. Then we have

Theorem S (K. Shiga [8]). $H^*_\Delta(a(N); a(N)) = 0$.

Corollary. Every derivation of the Lie algebra $a(N)$ is inner.

3^o. $W = \mathbb{R}$. Here $\mathbb{R}$ is regarded as a trivial $a(N)$-module. In this case, we give $a(N)$ the smooth jet topology (that is, the uniform convergence topology of all derivatives on each compact subset of N) and we put

$C^p_\Delta(a(N); \mathbb{R}) = \{\omega \in C^p(a(N); \mathbb{R}) | \omega$ is continuous and $\omega(X_1,..,X_p) = 0$ if $\bigcap_{i=1}^{p} \mathrm{Supp}\, X_i = \emptyset$ $(X_1,...,X_p \in a(N))$. Then it follows that $d(C^p_\Delta(a(N); \mathbb{R})) \subset C^{p+1}_\Delta(a(N)); \mathbb{R})$ and the p-th cohomology of the complex $\bigoplus_{p=0}^{\infty} C^p_\Delta(a(N); \mathbb{R})$ is denoted by $H^p_\Delta(a(N); \mathbb{R})$.

Theorem (Gel'fand-Fucks, Losik, Guillemin).
Let N be a compact oriented smooth manifold. Then $H^p_\Delta(a(N); \mathbb{R}) \cong H^{p+n}(L, \mathbb{R})$ for all p.

Here L is a fiber bundle over N associated to the principal $U(n)$-bundle $B(\tau^{\mathbb{C}})$ with the $U(n)$-space X_n as the fiber; X_n is defined as the inverse image of the 2n-skelton of the Grassmann manifold $Gr(m, n)$ $(m \geqslant 2n)$ with the usual structure of CW-complex by the natural projection map of the complex Stiefel manifold $E(m, n)$ onto $Gr(m, n)$. (It turns out that X_n does not depend on the number m.)

(For more details, see Guillemin [5], Losik [7], etc.)

§4. Complex analytic case

We return to the case of complex manifolds. The notations are the same as in §1.

1°. $W = C^{\infty}(M)$. Here $C^{\infty}(M)$ is the ring of $\mathbb{C}$-valued smooth functions on M and $C^{\infty}(M)$ is regarded as an $A(M)$-module in the following way: $X \in A(M)$ acts on $C^{\infty}(M)$ by $Xf = \sum_{\alpha=1}^{n} \xi^{\alpha} \frac{\partial f}{\partial z^{\alpha}}$, where $X = \sum_{\alpha=1}^{n} \xi^{\alpha} \frac{\partial}{\partial z^{\alpha}}$ in terms of a local holomorphic coordinates $\{z^1, \ldots, z^n\}$. Note that X acts on $C^{\infty}(M)$ as a derivation. Denote by $C^p_{\Delta}(A(M); C^{\infty}(M))$ the subspace of $C^p(A(M); C^{\infty}(M))$ consisting of support preserving cochains, and by $C^p_{\partial}(A(M); C^{\infty}(M))$ the subspace of $C^p_{\Delta}(A(M); C^{\infty}(M))$ consisting of the elements ω such that, if $f \in C^{\infty}(M)$ is anti-holomorphic on an open subset U of M, then $\omega(fX_1, X_2, \ldots, X_p) = f\omega(X_1, X_2, \ldots, X_p)$ on U for all $X_1, \ldots, X_p \in A(M)$. The cohomology of the complex

$$C_{\Delta}(A(M); C^{\infty}(M)) = \bigoplus_{p=0}^{\infty} C^p_{\Delta}(A(M); C^{\infty}(M)) \qquad \text{(resp.}$$

$C_{\partial}(A(M); C^{\infty}(M)) = \bigoplus_{p=0}^{\infty} C^p_{\partial}(A(M); C^{\infty}(M)))$ is indicated by $H^*_{\Delta}(A(M);$

$C^{\infty}(M)) = \bigoplus_{p=0}^{\infty} H^p_{\Delta}(A(M); C^{\infty}(M))$ (resp. $H^*_{\partial}(A(M); C^{\infty}(M)) = \bigoplus_{p=0}^{\infty} H^p_{\partial}(A(M);$

$C^{\infty}(M))$. Note that both $H^*_{\Delta}(A(M);\, C^{\infty}(M))$ and $H^*_{\partial}(A(M);\, C^{\infty}(M))$ have natural structures of graded rings.

We remark that the anti-Dolbeault complex $\Gamma(\Lambda T') = \bigoplus_{p=0}^{\infty} \Gamma(\Lambda^p T')$ is a subcomplex of $C_{\partial}(A(M);\, C^{\infty}(M))$ since $\Gamma(\Lambda^p T') = \mathrm{Hom}_{C^{\infty}(M)}(\Lambda^p(A(M));\, C^{\infty}(M))$ can be identified with the subspace of $C^p_{\partial}(A(M);\, C^{\infty}(M))$ consisting of the elements ω such that $\omega(fX_1, X_2, \ldots, X_p) = f\omega(X_1, X_2, \ldots, X_p)$ for all $f \in C^{\infty}(M)$, $X_1, \ldots, X_p \in A(M)$. (Here T is the holomorphic tangent bundle of M, T' is the dual bundle of T, $\Gamma(\Lambda^p T')$ denotes the space of the smooth cross-sections of the vector bundle $\Lambda^p T'$) The inclusion holomorphisms

$$\Gamma(\Lambda T') \overset{\lambda}{\hookrightarrow} C_{\partial}(A(M);\, C^{\infty}(M)) \overset{\mu}{\hookleftarrow} C_{\Delta}(A(M);\, C^{\infty}(M))$$

induce homomorphisms of graded rings :

$$H^*(M, \bar{0}) \xrightarrow{\lambda^*} H^*_{\partial}(A(M);\, C^{\infty}(M)) \xrightarrow{\mu^*} H^*_{\Delta}(A(M);\, C^{\infty}(M)).$$

(Note that the cohomology of the complex $\Gamma(\Lambda T')$ conincides with $H^*(M, \bar{0})$, since $\Gamma(\Lambda T')$ is a fine resolution of the sheaf $\bar{0}$ of germs of anti-holomorphic functions on M.

Now we can state the main theorems.

<u>Theorem 1</u>. We have a commutative diagram of graded ring holomorphisms :

$$\begin{array}{ccc} & H^*(M, \bar{0}) & \\ \swarrow \lambda^* & & \searrow \mathrm{id} \otimes 1 \\ H^*_{\partial}(A(M);\, C^{\infty}(M)) & \xrightarrow{\sim} & H^*(M, \bar{0}) \otimes H^*(gl(n, \mathbb{C});\, \mathbb{C}). \end{array}$$

Here the horizontal map is an isomorphism and $\mathbb{C}$ is regarded as the trivial $gl(n, \mathbb{C})$-module.

Theorem 2. $\lambda^p : H^p_\partial(A(M); C^\infty(M)) \longrightarrow H^p_\Delta(A(M); C^\infty(M))$ is an isomorphism for $p \leqq n$.

2°. $W = A(M)$. Here $A(M)$ is regarded as an $A(M)$-module by the adjoint action of $A(M)$. We can define in the same way as before two cohomology groups $H^*_\partial(A(M); A(M))$ and $H^*_\Delta(A(M); A(M))$. First we have

Theorem 3. $H^*_\partial(A(M); A(M)) = 0$.

This result corresponds to the Theorem S. The following result is outstanding compared with the smooth case.

Theorem 4. We have an isomorphism :

$$H^p_\Delta(A(M); A(M)) \cong \bigoplus_{r+s=p-1} H^r(M, \bar{\Theta}) \otimes H^s(gl(n;\mathbb{C});\mathbb{C})$$

for $p \leqq n$. Here $\bar{\Theta}$ donotes the sheaf of germs of anti-holomorphic vector fields of type $(0, 1)$ on M.

As a corollary, we can determine the space of outer derivations of A. Let $Der(A(M))$ be the Lie algebra of all the derivations of $A(M)$ and let $ad(A(M))$ be the ideal of $Der(A)$ spanned by the inner derivations. Then we have

Corollary to Theorem 4. We have an isomorphism of Lie algebras : $Der(A(M))/ad(A(M)) \cong H^0(M, \bar{\Theta})$, where $H^0(M, \bar{\Theta})$ is regarded as a Lie algebra by the usual bracket operation of vector fields.

Proof of Corollary. By definition $\mathrm{Der}(A(M)) = \{\omega: A(M) \longrightarrow A(M) \mid \omega$ is $\mathbb{C}$-linear and $\omega([X, Y]) = [\omega(X), Y] + [X, \omega(y)]$ for all $X, Y \in A(M)\}$, $\mathrm{ad}(A(M)) = \{\omega \in \mathrm{Der}(A(M)) \mid$ There is some $Y \in A(M)$ such that $\omega(X) = [Y, X]$ for all $X \in A(M)\}$. Then, from the definition of the coboundary operator and from the fact that every derivation is support preserving (cf. [2]), it follows that $\mathrm{Der}(A(M)) = \{\omega \in C^1_\Delta(A(M); A(M)) \mid d\omega = 0\}$ and $\mathrm{ad}(A(M)) = d(C^0_\Delta(A(M); A(M)))$. Hence we have $\mathrm{Der}(A(M))/\mathrm{ad}(A(M)) \cong H^1_\Delta(A(M); A(M))$, which is isomorphic to $H^0(M\ \bar{\Theta})$ by Theorem 4. It is easily verified that this is in fact an isomorphism of Lie algebras. Q. E. D.

3^o. $W = \mathbb{C}$. Here $\mathbb{C}$ is regarded as the trivial A-module. $A(M) = \Gamma(T)$ is given the smooth jet topology and we define $H^*_\Delta(A(M); \mathbb{C})$ just as in the smooth case (cf 3^o in §3). What we know is that $H^0_\Delta(A(M); \mathbb{C}) = \mathbb{C}$ (by definition) and $H^1_\Delta(A(M); A(M)) = 0$ since $[A(M), A(M)] = A(M)$ (cf. [2]).

<u>Problem</u>. If M is compact, is the total cohomology $H^*_\Delta(A; C)$ finite-dimensional? Calculate $H^*_\Delta(A; \mathbb{C})$.

§5. Outline of the proofs of the theorems

We shall sketch the proof of Theorem 1.

Lemma 1. (cf. [10]). $C^p_\partial(A(M);\ C^\infty(M)) \cong \Gamma(\Lambda^p(J_\partial T)')$.

Here $J_\partial T = \varprojlim_k J^{k,0}(T)$ and $J^{k,0}(T)$ is the k-th holomorphic jet bundle of T, whose fiber over a point x of M is $\Gamma(T)/(I_x^{k+1} + \bar{I}_x)\Gamma(T)$, I_x (resp. $\bar{I}_x$) being the ideal of $C^\infty(M)$ generated by the smooth functions f such that $\frac{\partial f}{\partial z^\alpha}(x)=0$ $\left(\text{resp. } \frac{\partial f}{\partial \bar{z}^\alpha}(x) = 0\right)$ for $\alpha = 1,\dots,n$ with respect to a holomorphic coordinate system $\{z^1,\dots,z^n\}$ around x. (For more details, see [9]).

We introduce a filtration due to Gel'fand and Fuks in the cochain complex $C_\partial(A(M);\ C^\infty(M))$: We put

$$F_p C^{p+q}_\partial = \Gamma((\Lambda^p T') \wedge (\Lambda^q (JT)')),$$

where $(\Lambda^p T') \wedge (\Lambda^q (J_\partial T)')$ is the subbundle of $\Lambda^{p+q}(J_\partial T)'$ whose fiber over x is spanned by the elements of the form $\omega_1 \wedge \dots \wedge \omega_p \wedge \eta_1 \wedge \dots \wedge \eta_q$ with $\omega_i \in T'_x$ and $\eta_j \in (J_\partial T)'_x$.

Then we have

$$0 = F_{m+1}C^m_\partial \subset F_m C^m_\partial = \Gamma(\Lambda^m T') \subset \dots \subset F_0 C^m_\partial = C^m_\partial(A(M);\ C^\infty(M)),$$

and $d(F_p C^m_\partial) \subset F_p C^{m+1}$. We denote by $\{E_r^{p,q},\ d_r^{p,q}\}$ the spectral sequence induced by the filtration, which is obviously convergent to $H^*_\partial(A(M);\ C^\infty(M))$. Then

Lemma 2. $E_2^{p,q} \cong H^p(M, \bar{0}) \otimes H^q(gl(n, \mathbb{C});\ \mathbb{C})$.

Using a formula in the Chern-Weil theory, we can prove

Lemma 3. $d_r^{p,q} = 0$ for $r \geq 2$.

These lemmas imply Theorem 1.

In order to prove Theorem 2, we introduce a similar filtration in the complex $C_\Delta(A(M); C^\infty(M))$ and compare the E_1-term of the spectral sequence induced by this filtration with that of the previous spectral sequence. Then the proof is reduced to the calculation of cohomology of formal vector fields.

Theorem 3 and Theorem 4 are proved in similar ways. (For more details, see [11]).

§6. An application

Let E be a smooth complex vector bundle over M and $\Gamma(E)$ the space of all the smooth cross-sections of E.

Definition. $\Gamma(E)$ is called a differential $A(M)$-module if $\Gamma(E)$ is an $A(M)$-module such that

(1) $\operatorname{supp}(Xs) \subset \operatorname{supp} X \cap \operatorname{supp} s$ for all $X \in A(M)$ and $s \in \Gamma(E)$.

(This definition is due to K. Shiga [8]).

Definition. A differential $A(M)$-module $\Gamma(E)$ is of connection-type if

(2) $X(fs) = (Xf)s + f(Xs)$ for all $X \in A(M)$, $f \in \mathbb{C}^\infty(M)$, $s \in \Gamma(E)$.

Chern(E) is meant by the subalgebra of $H^*(M, \mathbb{C})$ generated by the Chern classes $c_i(E)$ $(i \geq 1)$. Then we have

Theorem 5. If $\Gamma(E)$ is a differential $A(M)$-module of connection type, then $i^*(\text{Chern }(E)) = 0$. Here $i^* : H^*(M, \mathbb{C}) \longrightarrow H^*(M, \bar{O})$ comes from $i : \mathbb{C} \hookrightarrow \bar{O}$.

Outline of the proof. A $\mathbb{C}$-linear map $\varphi : A(M) \longrightarrow \mathrm{Hom}_{\mathbb{C}}(\Gamma(E); \Gamma(E))$ satisfying (1) and (2) is called a generalized connection of type (1,0) on E. From any generalized connection φ of type (1,0) on E, we can construct a generalized Chern class $\tilde{c}_i(E) \in H^{2i}_{\Delta}(A(M); C^{\infty}(M))$, $i \geqslant 1$ (cf. [10]). $\tilde{c}_i(E)$ vanishes if φ is a Lie algebra homomorphism. Moreover it turns out that $\kappa(c_i(E)) = \tilde{c}_i(E)$ $(i \geqslant 1)$, where $\kappa = \mu^* \circ \lambda^* \circ i^*$ (see §4). If $\Gamma(E)$ is a differential A-module, then $\tilde{c}_i(E) = 0$ $(i \geqslant 1)$, whence $(\mu^* \circ \lambda^* \circ i^*)(c_i(E)) = \tilde{c}_i(E) = 0$ $(i \geqslant 1)$. Since $\mu^* \circ \lambda^*$ is an injection map by Theorem 1 and Theorem 2, we have $i^*(c_i(E)) = 0$, $i \geqslant 1$. Q.E.D.

Corollary. Suppose that M is connected and L is a smooth complex line bundle over M. Then $\Gamma(L)$ is a non-trivial $A(M)$-module if and only if L has a structure of holomorphic line bundle.

Proof. By a direct calculation, we can show that, if L is a line bundle, then a non-trivial differential $A(M)$-module $\Gamma(L)$ turns out to be of connection type. Hence by Theorem 4, we have $i^*(c_1(L)) = 0$. This implies that L has a structure of holomorphic line bundle. The other implication is well-known. Q.E.D.

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