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Some remarks on subvarieties of Hopf manifolds

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§ 1. Introduction

A holomorphic automorphism g of a complex space $\mathfrak{X}$ is called a contraction to a point $0 \in \mathfrak{X}$ if g satisfies the following three conditions:

(i) $g(0) = 0$,

(ii) $\lim_{\nu \to +\infty} g^{\nu}(x) = 0$ for any point $x \in \mathfrak{X}$,

(iii) for any small neighborhood U of 0 in $\mathfrak{X}$, there exists an integer ν_0 such that $g^{\nu}(U) \subset U$ for all $\nu \geq \nu_0$,

where g^{ν} is the ν-times composite of g. By [2]* the complex space $\mathfrak{X}$ which admits a contracting automorphism is holomorphically isomorphic to an algebraic subset of $\mathbb{C}^N$ for some N. We identify $\mathfrak{X}$ to the algebraic subset of $\mathbb{C}^N$. Then there exists a contracting automorphism $\tilde{g}$ of $\mathbb{C}^N$ to the origin 0 such that $\tilde{g}|_{\mathfrak{X}} = g$ ([2], [3]). Obviously the action of $\tilde{g}$ on $\mathbb{C}^N - \{0\}$ is free and properly discontinuous. Hence the quotient space $H = \mathbb{C}^N - \{0\}/\langle\tilde{g}\rangle$ is a compact complex manifold which is called a primary Hopf manifold. Sometimes we indicate by H^N a N-dimensional primary Hopf manifold. The compact complex space $\mathfrak{X} - \{0\}/\langle g\rangle$ is clearly an analytic subset of a primary Hopf manifold. A compact complex manifold X of dimension n ($n \geq 2$) is called a Hopf manifold if its universal covering is holomorphically isomorphic to $\mathbb{C}^n - \{0\}$ (Kodaira[4]).

The purpose of this paper is to show several properties of subvarieties of Hopf manifolds.

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* In [2], the condition (iii) is forgotten.

§ 2. Hopf manifolds

The following proposition shows that it is sufficient to consider only subvarieties of primary Hopf manifolds.

<u>Proposition</u> 1. <u>Any Hopf manifold is a submanifold of a (higher dimensional) primary Hopf manifold.</u>

<u>Proof</u>. Let X be any Hopf manifold. Then, by definition, there exists a group G of holomorphic transformations of $\mathbb{C}^n-\{0\}$ such that $X = \mathbb{C}^n-\{0\}/G$ ($n = \dim X$ (≥ 2)). It follows from a theorem of Hartogs that any element of G can be extended to a holomorphic transformation of $\mathbb{C}^n$. Hence we may assume that each element of G is a holomorphic transformation of $\mathbb{C}^n$ which fixes the origin $0 \in \mathbb{C}^n$. By the same argument as in [4] pp 694-695, G contains a contraction.

For each element $x \in G$, we denote by $dx(0)$ the jacobian matrix at the origin $0 \in \mathbb{C}^n$.

<u>Lemma</u> 1. <u>An element</u> $x \in G$ <u>is a contraction if and only if</u> $|\det(dx(0))| < 1$.

<u>Proof</u>. If $x \in G$ is a contraction, then any eigenvalue α of $dx(0)$ satisfies $|\alpha| < 1$ (see [3] for the detail). Hence $|\det(dx(0))| < 1$. Conversely, let x be an element of G satisfying $|\det(dx(0))| < 1$. Let g be a contraction contained in G. Since $\mathbb{C}^n-\{0\}/\langle g\rangle$ is compact, the index of the infinite cyclic subgroup $\{g\}$ generated by g is finite in G. Now assume that x is not a contraction. Then x^n is not a contraction for any integers n. Hence $x^n \neq g^m$ for any pair of integers n and m except $n = m = 0$. This implies that $\{x\} \cap \{g\} = \{1\}$.

This contradicts the fact that $\{g\}$ is of the finite index in G, q.e.d.

Let U be a subgroup of G defined by

$$U = \{ x \in G : |\det(dx(0))| = 1 \}.$$

Obviously U is a normal subgroup of G.

Lemma 2. _There exists an infinite cyclic subgroup_ Z _of_ G _such that_ G _is the semi-direct product of_ Z _and_ U ; $G = Z \cdot U$.

Proof. Define a group homomorphism $\mathfrak{l} : G \longrightarrow \mathbb{R}$ by $\mathfrak{l}(x) = -\log |\det(dx(0))|$ $(x \in G)$. Let $g_1 \in G$ be a contraction. Then the index d of the infinite cyclic group $\{\mathfrak{l}(g_1)\}$ generated by $\mathfrak{l}(g_1)$ in $\mathfrak{l}(G)$ is finite. Hence $d^{-1}\, \mathfrak{l}(g_1)$ is a minimum positive element of $\mathfrak{l}(G)$. Let g be an element of G such that $\mathfrak{l}(g) = d^{-1}\, \mathfrak{l}(g_1)$. We put $Z = \{g\}$. Then it is clear that $G = Z \cdot U$, q.e.d.

Lemma 3. U _is a finite normal subgroup of_ G.

Proof. Clear by Lemma 2.

Now continue the proof of Proposition 1. It is easy to see that any holomorphic transformation u of $\mathbb{C}^n$ which fixes the origin is linear, if u is of the finite order. Hence U is a finite subgroup of $GL(n,\mathbb{C})$. Hence, by H. Cartan [1], $\mathfrak{X} = \mathbb{C}^n/U$ is a complex space with unique possible singularity at $\overline{0}$, where $\overline{0}$ is the corresponding point to the origin $0 \in \mathbb{C}^n$. The generator g of Z induces a contracting automorphism $\overline{g}$ of $\mathfrak{X}$ such that $\overline{g}(\overline{0}) = \overline{0}$. Hence $X = \mathfrak{X} - \{\overline{0}\}/\langle \overline{g} \rangle$ is a submanifold of a primary Hopf manifold as we have seen in the introduction. Q.E.D.

§ 3. Line bundles defined by divisors

Let M be an arbitrary compact complex manifold and N be a divisor of M. The line bundle [N] defined by N is an element of $H^1(M, O^*)$. There is a natural homomorphism $i : H^1(M, \mathbb{C}^*) \longrightarrow H^1(M, O^*)$ induced by the natural injection $\mathbb{C}^* \longrightarrow O^*$. If [N] is in the image of i, then [N] is called a <u>locally flat</u> line bundle. In other words, [N] is locally flat if and only if its transition functions can be written by constant functions.

Now let $\tilde{g}$ be any contracting automorphism of $\mathbb{C}^N$ which fixes the origin $0 \in \mathbb{C}^N$. Then, by L. Reich ([6], [7]), we can choose a system of coordinates of $\mathbb{C}^N$ such that $\tilde{g}$ can be written in the following form:

$$
(1)\qquad
\begin{aligned}
z_1' &= \alpha_1 z_1 \\
z_2' &= z_1 + \alpha_2 z_2 \\
&\;\vdots \\
z_{r_1}' &= z_{r_1-1} + \alpha_{r_1} z_{r_1} \\
z_{r_1+1}' &= \alpha_{r_1+1} z_{r_1+1} + P_{r_1+1}(z_1, \ldots, z_{r_1}) \\
&\;\vdots \\
z_{r_1+r_2}' &= z_{r_1+r_2-1} + \alpha_{r_1+r_2} z_{r_1+r_2} + P_{r_1+r_2}(z_1, \ldots, z_{r_1}) \\
z_{r_1+r_2+1}' &= \alpha_{r_1+r_2+1} z_{r_1+r_2+1} + P_{r_1+r_2+1}(z_1, \ldots, z_{r_1+r_2}) \\
&\;\vdots \\
z_N' &= z_{N-1} + \alpha_N z_N + P_N(z_1, \ldots, z_{r_1+r_2+\cdots+r_{\mu-1}}),
\end{aligned}
$$

where $1 > |\alpha_1| \geq \cdots \geq |\alpha_N| > 0$, μ is the number of Jordan blocks of the linear part, P_j $(r_1+\cdots+r_s < j \leq r_1+\cdots+r_{s+1})$ are finite sums of monomials $z_1^{m_1} \cdots z_{r_s}^{m_{r_s}}$ which satisfy

$$(2) \qquad \alpha_j = \alpha_1^{m_1} \cdots \alpha_{r_s}^{m_{r_s}},$$

$$m_1 + \cdots + m_{r_s} \geq 2 \quad (\text{ all } m_i > 0).$$

Let $\tilde{\omega}: \mathbb{C}^N - \{0\} \longrightarrow H = \mathbb{C}^N - \{0\}/\langle\tilde{g}\rangle$ be the covering projection. For any analytic subset X in H, the set $\tilde{\omega}^{-1}(X)$ is an analytic subset in $\mathbb{C}^N - \{0\}$. If $\dim X \geq 1$, then by a theorem of Remmert-Stein, $\mathcal{X} = \tilde{\omega}^{-1}(X) \cup \{0\}$ is an analytic subset of $\mathbb{C}^N$. In what follows, we indicate by the script letters the analytic subsets in $\mathbb{C}^N$ corresponding in the above manner to the analytic subsets of H written by the Roman letters. An analytic subset is called a variety if it is irreducible.

Assume that X is an analytic subvariety in H of $\dim X \geq 2$ and that D is an analytic subvariety of codimension 1 in X. It is clear that $\mathcal{X}$ and $\mathcal{D}$ are both $\tilde{g}$-invariant in $\mathbb{C}^N$, i.e. $g(\mathcal{X}) = \mathcal{X}$ and $g(\mathcal{D}) = \mathcal{D}$.

Lemma 4 ([2]). *There exists a non-constant holomorphic function f on $\mathcal{X}$ such that $g^*f = \alpha f$ for some constant α $(0 < |\alpha| < 1)$ and that $f_{|\mathcal{D}} = 0$.*

Remark 1. In [2], the word "variety" is used as "analytic set".

Let X be a non-singular manifold. Consider f of Lemma 4 as a multiplicative multi-valued holomorphic function on X (K. Kodaira [4] pp 701). The divisor $D_1 = (f)$ is well-defined. The equation $g^*f = \alpha f$ implies that the line bundle $[D_1]$ is locally flat of which the transition functions are some powers of α. We summarize these facts as follows.

Theorem 1. *Let X be a submanifold of H and D an effective divisor on X. Assume that* $\dim X \geq 2$. *Then there exists an effective divisor E on X such that the line bundle* $[D + E]$ *is locally flat of which the transition functions are some powers of a certain constant* $\alpha \in \mathbb{C}^*$ $(0 < |\alpha| < 1)$.

Remark 2. The following example shows that there are cases such that the "additional" effective divisor E of Theorem 1 is indispensable.

Let (x_0, x_1, x_2, x_3) be a standard system of coordinates of $\mathbb{C}^4$. Fix a complex number α such that $0 < |\alpha| < 1$. Let $\tilde{g}$ be a contracting holomorphic automorphism of $\mathbb{C}^4$ defined by

$$\tilde{g} : (x_0, x_1, x_2, x_3) \longmapsto (\alpha x_0, \alpha x_1, \alpha x_2, \alpha x_3).$$

Define $\tilde{g}$-invariant subvarieties of $\mathbb{C}^4$ by

$$\mathfrak{X} : \ x_0 x_1 = x_2 x_3$$

and

$$\mathcal{A} : \ x_3 = 0$$

Denote the intersection $\mathfrak{X} \cap \mathcal{A}$ by $\mathcal{S}$. Then $\mathcal{S} = \{x_0 = x_3 = 0\} \cup \{x_1 = x_3 = 0\}$. We put

$$\mathcal{S}_1 = \{x_0 = x_3 = 0\}$$

and

$\mathcal{S}_2 = \{ x_1 = x_3 = 0 \}$.

Then $S = \mathcal{S} - \{0\}/\langle\tilde{g}\rangle$, $S_1 = \mathcal{S}_1 - \{0\}/\langle\tilde{g}\rangle$ and $S_2 = \mathcal{S}_2 - \{0\}/\langle\tilde{g}\rangle$ are subvarieties of a compact complex manifold $X = \mathfrak{X} - \{0\}/\langle\tilde{g}\rangle$. It is clear that $[S_1 + S_2] = [S]$ is locally flat. We shall prove that either $[S_1]$ or $[S_2]$ is not locally flat. Assume that both $[S_1]$ and $[S_2]$ are locally flat. Let $\mathcal{U} = \{U_\lambda\}$ be a sufficiently fine finite open covering of X. We represent $[S_1]$ as a 1-cocycle $\{c_{1\lambda\mu}\} \in Z^1(\mathcal{U}, \mathbb{C}^*)$. Since $\dim H^0(X, \mathcal{O}[S_1]) > 0$, there exists a non-zero section φ_1 which vanishes exactly on S_1. Let $\varphi_{1\lambda} = c_{1\lambda\mu}\varphi_{1\mu}$ on $U_\lambda \cap U_\mu$. As we can easily see,

$$\eta_1 = \frac{d\varphi_{1\lambda}}{\varphi_{1\lambda}} = \frac{d\varphi_{1\mu}}{\varphi_{1\mu}} = \cdots$$

is a meromorphic 1-form on X. Since $\mathfrak{X} - \{0\}$ is simply connected,

$$f_1(x) = \exp \int^x \eta_1$$

is a holomorphic function on $\mathfrak{X} - \{0\}$ such that $\tilde{g}^* f_1 = \beta_1 f_1$ ($\beta_1 \in \mathbb{C}^*$, $0 < |\beta_1| < 1$) which vanishes exactly on $\mathcal{S}_1 - \{0\}$ with multiplicity 1. Since $\mathfrak{X}$ is normal at 0, f_1 uniquely extends to a holomorphic function on $\mathfrak{X}$. Comparing the initial terms of $\tilde{g}^* f_1$ and f_1 at 0, we see that β_1 is some power of α, i.e. $\beta_1 = \alpha^{m_1}$ $(m_1 \geq 1)$. By the same manner, we construct f_2 for a non-zero section $\varphi_2 \in H^0(X, \mathcal{O}[S_2])$ such that $\tilde{g}^* f_2 = \alpha^{m_2} f_2$ $(m_2 \geq 1)$. Let f_0 be a restriction of a holomorphic function x_3 to $\mathfrak{X} - \{0\}$. Then $\tilde{g}^* f_0 = \alpha f_0$. It is easy to see that $f = f_1 \cdot f_2 \cdot f_0^{-1}$ is a non-vanishing holomorphic function on $\mathfrak{X} - \{0\}$ such that $\tilde{g}^* f = \alpha^{m_1 + m_2 - 1} f$ $(m_1 + m_2 - 1 \geq 1)$. But this does not occur if $\dim X > 1$. In fact, using the non-vanishing holomorphic function f, we get the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{X}-\{0\} & \xrightarrow{\quad\tilde{g}\quad} & \mathfrak{X}-\{0\}, \\ \big\downarrow f & & \big\downarrow f \\ \mathbb{C}^* & \xrightarrow{\times\, \alpha^{m_1+m_2-1}} & \mathbb{C}^*. \end{array}$$

Then f induces a proper surjective holomorphic mapping $\bar{f} : X \longrightarrow \mathbb{C}^*/\langle\alpha^{m_1+m_2-1}\rangle$. For any point $\tau \in \mathbb{C}^*/\langle\alpha^{m_1+m_2-1}\rangle$, $\bar{f}^{-1}(\tau) = X_\tau$ is a compact subvariety in X. Hence $\tilde{\omega}^{-1}(X_\tau)$ is a complex analytic subset in $\mathbb{C}^4-\{0\}$ whose connected components are compact,where $\tilde{\omega}$ is the covering map $\mathbb{C}^4-\{0\}\longrightarrow\mathbb{C}^4-\{0\}/\langle\tilde{g}\rangle$. This implies that $\tilde{\omega}^{-1}(X_\tau)$ is a countable union of points. Hence $\dim X_\tau = 0$. This contradicts $\dim X > 1$. This implies that either $[S_1]$ or $[S_2]$ is not locally flat.

Remark 3. If dim X = 2, then [D] is always locally flat ([3]).

§ 4. Some properties of subvarieties

By Lemma 5 in [2], we have easily

Proposition 2. *Let* Y_1 *and* Y_2 *be subvarieties of a (primary) Hopf manifold (H) such that* $Y_1 \subset Y_2$ *and* $0 < n_1 = \dim Y_1 < n_2 = \dim Y_2$. *Then there exists a sequence of subvarieties* $W_0, W_1, \ldots, W_p$ $(p = n_2 - n_1)$ *in* H *with following properties:*

(i) $W_0 = Y_1$, $W_p = Y_2$,

(ii) $W_i \subset W_{i+1}$ $(i = 0,\ldots,p-1)$, $\dim W_i + 1 = \dim W_{i+1}$.

Proposition 3. Let $H^N = \mathbb{C}^N - \{0\}/\langle\tilde{g}\rangle$ be a primary Hopf manifold. Then

(a) any positive dimensional subvariety in H^N contains a curve,

(b) any irreducible curve in H^N is non-singular elliptic,

(c) for any elliptic curve C in H^N, there exist an eigenvalue α of $\tilde{g}$, a constant β and certain positive integers m, n with $\alpha^m = \beta^n$ such that C is isomorphic to $\mathbb{C}^*/\langle\beta\rangle$.

Proof. (a) Let Y be a n-dimensional subvariety in H^N $(n \geq 1)$. For any integer k $(1 \leq k \leq N)$, the (N-k)-dimensional subspace $\mathbb{C}^{N-k}$ defined by $z_1 = \cdots = z_k = 0$ is $\tilde{g}$-invariant. There exists an integer k such that $\dim(\mathbb{C}^{N-(k-1)} \cap \mathcal{Y}) = 1$. Then $\tilde{\omega}((\mathbb{C}^{N-(k-1)} \cap \mathcal{Y}) - \{0\})$ is a 1-dimensional analytic subset of Y.

(b) Let C be any irreducible curve in H^N. Then $\mathcal{C}$ is a 1-dimensional analytic subset of $\mathbb{C}^N$. Let $\mathcal{C}_0$ be one of the irreducible components of $\mathcal{C}$. Then, for some positive integer n_0, g^{n_0} acts on $\mathcal{C}_0$ as a contracting automorphism of $\mathcal{C}_0$. Let $\lambda: \mathcal{C}_0^* \longrightarrow \mathcal{C}_0$ be the normalization of $\mathcal{C}_0$. Then g^{n_0} naturally induces a contracting automorphism of $\mathcal{C}_0^*$. By [2], $\mathcal{C}_0^* \simeq \mathbb{C}$. It is clear that $\lambda^{-1}(0)$ consists of one point 0^*. Hence $\mathcal{C}_0 - \{0\} \simeq \mathcal{C}_0^* - \{0^*\} \simeq \mathbb{C}^*$. Thus $\mathbb{C}^*$ is an infinite cyclic unramified covering of C. Therefore C is a non-singular elliptic curve.

(c) Consider the $\tilde{g}$-invariant subspaces $\mathbb{C}^{N-k}$ defined in (a). For $k = 0$, $\mathbb{C}^{N-k}$ is the total space. Fix the integer k $(0 \leq k \leq N-1)$

such that $\mathcal{C} \subset \mathbb{C}^{N-k}$ and $\mathcal{C} \not\subset \mathbb{C}^{N-k-1}$. If $\mathcal{C} \cap \mathbb{C}^{N-k-1}$ contains a point p other than 0, then $\mathcal{C} \cap \mathbb{C}^{N-k-1}$ contains an infinite sequence of points $\tilde{g}^n(p) \longrightarrow 0$ $(n = 1,2,\ldots)$. Hence one of the irreducible components of $\mathcal{C}$ is contained in $\mathbb{C}^{N-k-1}$. Since $\tilde{g}$ is transitive over all the irreducible components of $\mathcal{C}$, this implies that $\mathcal{C} \subset \mathbb{C}^{N-k-1}$, (contradiction.) Therefore $\mathcal{C} \cap \mathbb{C}^{N-k-1} = \{0\}$. Hence $f = z_{k+1}|_{\mathbb{C}^{N-k}}$, the restriction of z_{k+1} to $\mathbb{C}^{N-k}$, vanishes nowhere on $\mathcal{C} - \{0\}$. Moreover f satisfies the equation $g^*f = \alpha_{k+1} f$. Hence we get the following commutative diagram:

$$\begin{array}{ccc} \mathcal{C}-\{0\} & \xrightarrow{\ g\ } & \mathcal{C}-\{0\} \\ \downarrow f & & \downarrow f \\ \mathbb{C}^* & \xrightarrow{\ \alpha_{k+1}\ } & \mathbb{C}^*. \end{array}$$

This induces a covering $\bar{f} : C \longrightarrow \mathbb{C}^*/\langle \alpha_{k+1} \rangle$. Since both C and $\mathbb{C}^*/\langle \alpha_{k+1} \rangle$ are non-singular elliptic curves, $\bar{f}$ has no branch points by the Hurwitz's formula. Hence there exist $\beta \in \mathbb{C}^*$ and positive integers m, n such that $C \simeq \mathbb{C}^*/\langle \beta \rangle$ and $\alpha_{k+1}^m = \beta^n$. Q.E.D.

Remark 4. By Propositions 2 and 3 (a), it follows that any n-dimensional subvariety of a Hopf manifold contains subvarieties of arbitrary dimensions less than n.

§ 5. Subvarieties of algebraic dimension 0

In general, let M be a compact complex analytic subvariety. Then the field $\mathfrak{M}(M)$ of all meromorphic functions on M has the finite transendental degree a(M) over C. We call a(M) the algebraic dimension of M. It is well known that $a(M) \leq \dim M$. The number $\dim M - a(M)$ is called the algebraic codimension of M.

Theorem 2. *Let Y be a subvariety of dimension k in N-dimensional primary Hopf manifold H^N. Assume that $a(Y) = 0$. Then the number of (k-1)-dimensional subvarieties in Y is at most N.*

Before proving the theorem, we shall make some preparations.

Let $\alpha_1, \ldots, \alpha_N$ be the eigenvalues of $\tilde{g}((1))$. Put $\theta_j = \log \alpha_j$ ($0 \leq \arg \theta_j < 2\pi$, $j = 1, 2, \ldots, N$). Let K be a vector space over the field of rational numbers Q generated by the elements $2\pi\sqrt{-1}$, $\theta_1, \ldots, \theta_N$. Choose a basis $\tau_0, \tau_1, \ldots, \tau_\lambda$ of K so that the following conditions may be satisfied:

(i) $\tau_0 = 2\pi\sqrt{-1}$,

(ii) $\{\tau_1, \ldots, \tau_\lambda\}$ is a subset of $\{\theta_1, \ldots, \theta_N\}$,

(iii) for any $\nu \geq 1$, τ_ν is linearly independent to

$$Q\tau_0 + Q\tau_1 + \cdots + Q\tau_{\nu-1},$$

(iv) if $\tau_\nu = \theta_j$, $\tau_\mu = \theta_k$ and $\nu < \mu$, then $j < k$.

It is easy to check that we can choose such a basis. We denote by α_{i_ν} the element of $\{\alpha_1, \ldots, \alpha_N\}$ corresponding to τ_ν. Note that $\tau_\nu = \theta_{i_\nu} = \log \alpha_{i_\nu}$ ($\nu = 1, 2, \ldots, \lambda$). If the equation

$$\alpha_{i_\nu} = \alpha_1^{a_1} \cdots \alpha_l^{a_l} \qquad (l < i_\nu)$$

holds for some integers $a_1, \ldots, a_l$, then

$$\tau_\nu = \theta_{i_\nu} = \sum_{j=1}^{\ell} a_j\theta_j + p\tau_0 \qquad (p \in \mathbb{Z}).$$

Since $\sum_{j=1}^{\ell} a_j\theta_j$ is written by a linear combination of $\tau_0, \tau_1, \ldots, \tau_{\nu-1}$, this is absurd. Therefore α_{i_ν} has no such relations. Hence by (1),

$$z_i' = \alpha_{i_\nu} z_i \qquad (\nu = 1, 2, \ldots, \lambda).$$

Proof of Theorem 2. We may assume that Y can't be contained any primary Hopf manifold of dimension less than N. Let D be a subvariety of codimension 1 in Y. By Lemma 4, $\mathcal{D}$ is contained in the zero locus of a non-constant holomorphic function f on $\mathcal{Y}$ such that $\tilde{g}^*f = \alpha f$ ($0 < |\alpha| < 1$). There exist some integers m, $m_1, \ldots, m_\lambda$ such that

$$\alpha^m = \alpha_{i_1}^{m_1} \cdots \alpha_{i_\lambda}^{m_\lambda}.$$

Put

$$h = z_{i_1}^{m_1} \cdots z_{i_\lambda}^{m_\lambda}.$$

Since Y is not contained in any lower dimensional primary Hopf manifold, h is not equal to zero on $\mathcal{Y}$. Hence both f^m and h are eigenfunctions of $\tilde{g}^*$ of which the eigenvalues are the same α^m. Then h/f^m defines a non-zero meromorphic function on Y. By the assumption a(Y) = 0, h/f^m = constant = $c \neq 0$. Hence we get

(3) $\quad h = cf^m.$

Let Z_{i_ν} ($\nu = 1, \ldots, \lambda$) be analytic subsets of Y corresponding to $\{z_{i_\nu} = 0\} \cap \mathcal{Y}$. The equation (3) implies that D is contained in

$\bigcup_{\nu=1}^{\lambda} Z_{i_\nu}$. Since $\lambda \leq N$, this proves the theorem. Q.E.D.

§ 6. $\mathbb{C}^*$-actions

Proposition 4. There exists a holomorphic mapping

$$\tilde{\varphi}: \mathbb{C} \times \mathbb{C}^N \longrightarrow \mathbb{C}^N$$
$$(t, z) \longmapsto \tilde{\varphi}_t(z)$$

which satisfies the following properties:

(i) for every $t \in \mathbb{C}$, $\tilde{\varphi}_t$ is a holomorphic automorphism of $\mathbb{C}^N$ which fixes the origin,

(ii) $\tilde{\varphi}_{t+s} = \tilde{\varphi}_t \circ \tilde{\varphi}_s$,

(iii) there exists an integer n_0 such that $\tilde{\varphi}_1 = \tilde{g}^{n_0}$,

(iv) every $\tilde{g}$-invariant subvarieties in $\mathbb{C}^N$ is $\tilde{\varphi}_t$-invariant for all $t \in \mathbb{C}$.

We say that an analytic subset of $\mathbb{C}^N$ is $\tilde{\varphi}$-invariant, if it is $\tilde{\varphi}_t$-invariant for all $t \in \mathbb{C}$.

Proof. Let $\alpha_{i_1}, \ldots, \alpha_{i_\lambda}$ be the eigenvalues of $\tilde{g}$ considered in § 5. For any eigenvalue α_j of $\tilde{g}$, there exist some integers m_j, $m_{j_1}, \ldots, m_{j_\lambda}$ such that

$$\alpha_j^{m_j} = \alpha_{i_1}^{m_{j_1}} \cdots \alpha_{i_\lambda}^{m_{j_\lambda}} \qquad (j = 1, 2, \ldots , N).$$

Put $n_0 = m_1 \cdots m_N$ and $g_0 = g^{n_0}$. We define

$$(4) \quad \alpha_{i_\nu}^t = \exp t\tau_\nu \qquad (t \in \mathbb{C}, \ \nu = 1,2,\ldots,\lambda),$$

and

$$(5) \quad \alpha_j^{n_0 t} = \exp \left(t n_j \sum_{\nu=1}^{\lambda} m_{j_\nu} \tau_\nu \right) \qquad (n_j = n_0 m_j^{-1}, \ j = 1,2,\ldots,N).$$

Let $R(\alpha_1^{n_0},\ldots,\alpha_N^{n_0}) = 1$ be any relation among the eigenvalues of g_0, where $R(u_1,\ldots,u_N)$ is a product of some (possibly negative) powers of u_j $(j = 1,2,\ldots,N)$, u_j being indeterminates. Now let $R(u_1,\ldots,u_N) = u_1^{a_1}\cdots u_N^{a_N}$ $(a_j \in \mathbb{Z})$. Then, for $t \in \mathbb{C}$,

$$(6) \quad R(\alpha_1^{n_0 t},\ldots,\alpha_N^{n_0 t}) = \alpha_1^{a_1 n_0 t}\cdots\alpha_N^{a_N n_0 t}$$

$$= \exp\left(t\sum_{j=1}^{N} a_j n_j \sum_{\nu=1}^{\lambda} m_{j\nu}\tau_\nu\right)$$

$$= \exp\left(t\sum_{\nu=1}^{\lambda}\left(\sum_{j=1}^{N} a_j n_j m_{j\nu}\right)\tau_\nu\right).$$

Put $t = 1$ in (6). Then we get

$$\sum_{\nu=1}^{\lambda}\left(\sum_{j=1}^{N} a_j n_j m_{j\nu}\right)\tau_\nu = p\tau_0 \qquad (p\in\mathbb{Z}).$$

Hence we get $p = 0$ and $\sum_{j=1}^{N} a_j n_j m_{j\nu} = 0$ $(\nu = 1,2,\ldots,\lambda)$. Therefore

$$(7) \quad R(\alpha_1^{n_0 t},\ldots,\alpha_N^{n_0 t}) = 1$$

for all $t\in\mathbb{C}$. Put $\beta_j = \alpha_j^{n_0}$. By (1), the j-th coordinate of the point $g_0^n(z)$ is given by

$$(8) \quad (g_0^n(z))_j = \beta_j^n\left\{z_j + Q_j(n, z_1,\ldots,z_{j-1})\right\},$$

where Q_j is a polynomial of $n, z_1,\ldots,z_{j-1}$. Replace n and β_j^n of (8) by t and $\alpha_j^{n_0 t}$ ($= \beta_j^t$), respectively. Then we get a holomorphic automorphism $\tilde{\varphi}_t$ of $\mathbb{C}^N$ defined by

$$(\tilde{\varphi}_t(z))_j = \beta_j^t\left\{z_j + Q_j(t, z_1,\ldots,z_{j-1})\right\}.$$

We shall prove that $\tilde{\varphi} = \{\tilde{\varphi}_t\}_{t\in\mathbb{C}}$ satisfies the desired conditions.

The condition (i) and (iii) are clearly satisfied. To prove the condition (ii) is satisfied we put

$$z = \begin{pmatrix} z_1 \\ \vdots \\ z_N \end{pmatrix}, \quad Q(t,z) = \begin{pmatrix} Q_1(t,z) \\ \cdots \\ Q_N(t,z) \end{pmatrix} \quad \text{and } A^t = \begin{pmatrix} \beta_1^t & & 0 \\ & \ddots & \\ 0 & & \beta_N^t \end{pmatrix}.$$

We write $\tilde{\varphi}_t(z)$ as

(9) $\tilde{\varphi}_t(z) = A^t(z + Q(t,z)).$

Again we put

(10) $d(t,s,z) = \tilde{\varphi}_{t+s}(z) - \tilde{\varphi}_t \circ \tilde{\varphi}_s(z).$

It is sufficient to prove that $d(t,s,z)$ vanishes identically. By (9),

$$(11) \quad \begin{aligned} d(t,s,z) &= A^{t+s}(z+Q(t+s,z)) - A^t(A^s(z+Q(s,z))+Q(t,A^s(z+Q(s,z)))) \\ &= A^{t+s}Q(t+s,z) - A^{t+s}Q(s,z) - A^tQ(t,A^s(z+Q(s,z))). \end{aligned}$$

Let $Q_j(s,z) = \sum q_{i_1\cdots i_{j-1}}(s)z_1^{i_1}\cdots z_{j-1}^{i_{j-1}}$ be the j-th component of $Q(s,z)$, where $i_1,\ldots,i_{j-1}$ satisfy $\beta_1^{i_1}\cdots\beta_{j-1}^{i_{j-1}} = \beta_j$ and $i_\ell > 0$.

Then, by (7),

$$\begin{aligned} & Q_j(t,A^s(z+Q(s,z))) \\ &= \sum q_{i_1\cdots i_{j-1}}(t)\{\beta_1^s(z_1+Q_1(s,z))\}^{i_1}\cdots\{\beta_{j-1}^s(z_{j-1}+Q_{j-1}(s,z))\}^{i_{j-1}} \\ &= \beta_j^s \sum q_{i_1\cdots i_{j-1}}(t)(z_1+Q_1(s,z))^{i_1}\cdots(z_{j-1}+Q_{j-1}(s,z))^{i_{j-1}}. \end{aligned}$$

Hence we get

(12) $A^tQ(t,A^s(z+Q(s,z))) = A^{t+s}Q(t,z+Q(s,z)).$

Combining (11) with (12), we obtain

$$d(t,s,z) = A^{t+s}(Q(t+s,z)-Q(s,z)-Q(t,z+Q(s,z))).$$

Hence it is sufficient to show that

$$d_1(t,s,z) = Q(t+s,z) - Q(s,z) - Q(t,z+Q(s,z))$$

vanishes identically. Note that every component of $d_1(t,s,z)$ is a polynomial of t, s and z.

Fix any integer $t = m$. Since $d_1(m,n,z)$ vanishes identically for any $n \in \mathbb{Z}$, the algebraic subset in $\mathbb{C}^{N+1}$ defined by

$$\{(s,z) \in \mathbb{C}^{N+1} : d_1(m,s,z) = 0\}$$

contains infinitely many N-dimensional subspaces of $\mathbb{C}^{N+1}$. Hence we infer that $d_1(m,s,z)$ vanishes identically for any integer m. Again, since $d_1(m,s,z) = 0$ for any $m \in \mathbb{Z}$, the algebraic subset in $\mathbb{C}^{N+2}$ defined by $d_1(t,s,z) = 0$ contains infinitely many (N+1)-dimensional subspaces of $\mathbb{C}^{N+2}$. Hence we conclude that d_1 vanishes identically on $\mathbb{C}^{N+2}$. Therefore the condition (ii) is satisfied.

Next we prove that the condition (iv') is satisfied. We need the following

Lemma 5. *Let $\mathcal{Y}$ be a ($\tilde{g}$- and) $\tilde{\varphi}$-invariant analytic subvariety in $\mathbb{C}^N$. Let $\mathcal{Z}$ be a pure 1-codimensional $\tilde{g}$-invariant analytic subset of $\mathcal{Y}$. Then each irreducible component of $\mathcal{Z}$ is $\tilde{\varphi}$-invariant.*

Proof. By Lemma 4, there exists a holomorphic function f on $\mathcal{Y}$ such that $\tilde{g}^*f = \alpha f$ $(0<|\alpha|<1)$ and that $f|_{\mathcal{Z}} = 0$. Here we shall prove the following equation:

(13) $\tilde{\varphi}_t^* f = \alpha^t f.$

Once the equation (13) is proved, the lemma is clear. In fact, each irreducible component of $\mathcal{Z}$ is an irreducible component of the zero locus of f. Since everything continuously varies depending on t, (13) implies that the irreducible components of $\mathcal{Z}$ is $\widetilde{\varphi}$-invariant.

We put

$$M(\alpha) = \{ h \in \mathcal{O}_{\mathcal{Y}} \ : \ g^*h = \alpha h \}.$$

Then $M(\alpha)$ is a finite dimensional vector space over $\mathbb{C}$ (cf.[2]). Let $\sigma_1, \ldots, \sigma_s$ be a basis of $M(\alpha)$. Put $\sigma_i^t(z) = \sigma_i(\widetilde{\varphi}_t(z))$ $(i = 1, 2, \ldots, s)$. Since $\mathcal{Y}$ is $\widetilde{\varphi}_t$-invariant, the elements $\sigma_1^t, \ldots, \sigma_s^t$ form another basis of $M(\alpha)$. Hence there exist some constants $c_{ij}(t)$ depending on t such that

$$\sigma_i^t = \sum_{j=1}^{s} c_{ij}(t)\, \sigma_j.$$

We claim that $C(t) = (c_{ij}(t))$ is holomorphically dependent on t. In fact, we can choose points $z_1, \ldots, z_s \in \mathcal{Y}$ such that

$$S = \begin{pmatrix} \sigma_1(z_1) & \cdots & \sigma_1(z_s) \\ \vdots & & \vdots \\ \sigma_s(z_1) & \cdots & \sigma_s(z_s) \end{pmatrix}$$

is a non-singular matrix. Then,

$$\begin{pmatrix} \sigma_1^t(z_1) & \cdots & \sigma_1^t(z_s) \\ \vdots & & \vdots \\ \sigma_s^t(z_1) & \cdots & \sigma_s^t(z_s) \end{pmatrix} S^{-1} = C(t). \tag{14}$$

Since the left hand side of (14) is holomorphically dependent on t, C(t) is holomorphic.

It is easy to see that $\{C(t)\}_{t\in\mathbb{C}}$ is a 1-parameter subgroup of $GL(s,\mathbb{C})$, satisfying the equality,

(15) $\quad C(n) = \alpha^n I \qquad (n\in\mathbb{Z}).$

Hence there exist a matrix A which has the Jordan canonical form and a non-singular matrix P such that

$$C(t) = P^{-1}\exp(tA)P.$$

By (15), A is a diagonal matrix. Put $P^{-1}\sigma_j = \tau_j$ $(j = 1,2,\dots,s)$. Then,

(16) $\quad \tau_j^t = (\exp ta_j)\,\tau_j \qquad (j = 1,2,\dots,s),$

where $a_1,\dots,a_s$ are the diagonal components of A. Comparing the initial terms of the both sides of (16), we get

(17) $\quad \exp ta_j = \exp\sum_{\nu=1}^{\lambda} tn_{j\nu}\tau_\nu \qquad (j = 1,2,\dots,s),$

for some integers $n_{j\nu}$. Letting $t = 1$, we get

$$\alpha = \exp a_j = \exp\sum_{\nu=1}^{\lambda} n_{j\nu}\tau_\nu \quad (j = 1,2,\dots,s).$$

Hence for any i and j,

$$\sum_{\nu=1}^{\lambda}(n_{j\nu} - n_{i\nu})\tau_\nu = p_{ij}\,\tau_0,$$

choosing some integers p_{ij}. Since $\tau_0, \tau_1,\dots,\tau_\lambda$ are linearly independent over $\mathbb{Q}$, this implies that $n_{j\nu} = n_{i\nu}$ and $p_{ij} = 0$. Hence $\exp ta_j = \exp ta_i$ for any i and j. Therefore C(t) is a scalar matrix:

$$C(t) = \alpha^t I \qquad (\alpha^t = \exp ta_j).$$

Since $f\in M(\alpha)$, f can be expressed as

$$f = c_1\tau_1 + \cdots + c_s\tau_s \qquad (c_j \in \mathbb{C}).$$

Then $\tilde{\varphi}_t^* f = \sum_j c_j \tilde{\varphi}_t^* \tau_j = \alpha^t \sum c_j \tau_j = \alpha^t f,$ q.e.d.

Proof of (iv). By Lemma 5 [2], there exists a sequence $\{\mathcal{W}_j : j = 0,1,\ldots,p\}$ of $\tilde{g}$-invariant subvarieties of $\mathbb{C}^N$ such that $\mathcal{W}_0$ = a given $\tilde{g}$-invariant subvariety $(\mathcal{W})$, $\mathcal{W}_j \subset \mathcal{W}_{j+1}$, $\dim \mathcal{W}_j + 1 = \dim \mathcal{W}_{j+1}$ and $\mathcal{W}_p = \mathbb{C}^N$ $(p = N - \dim \mathcal{W}_0)$. Since $\mathbb{C}^N$ is obviously $\tilde{g}$- and $\tilde{\varphi}$-invariant, we infer that $\mathcal{W}$ is $\tilde{\varphi}$-invariant by the previous lemma. Q.E.D.

As a corollary, we obtain

Theorem 3. For any primary Hopf manifold H^N, there exists another primary Hopf manifold H'^N with following properties:

(i) H'^N is a finite cyclic unramified covering of H^N,

(ii) H'^N has a free $\mathbb{C}^*$-action $\varphi = \{\varphi_\tau\}_{\tau \in \mathbb{C}^*}$ such that every positive dimensional subvariety in H'^N is φ-invariant.

Proof. Let $H' = \mathbb{C}^N - \{0\}/\langle \tilde{g}^{n_0} \rangle$. Then everythig is clear from Proposition 4.

Corollary. The Euler number of a submanifold of a Hopf manifold is equal to 0.

Proof. By Theorem 3, every submanifold of a Hopf manifold has a finite unramified covering which admits a free S^1-action. Hence the Euler number vanishes. Q.E.D.

§ 7. Subvarieties of algebraic codimension 1

Let Y be a n-dimensional ($n \geq 2$) subvariety of a primary Hopf manifold H^N. Take another primary Hopf manifold H'^N of Theorem 3. Let $\omega : H'^N \longrightarrow H^N$ be the covering map. We denote by Y' a connected component of $\omega^{-1}(Y)$.

Theorem 4. The algebraic dimension of Y is n-1 if and only if the $\mathbb{C}^*$-action φ on Y' reduces to a complex torus action.

Proof. Assume that $a(Y) = n-1$. Since $a(Y') = a(Y) = n-1$, Y' has an (n-1)-dimensional algebraic family of elliptic curves. The moduli of curves depens continuously on the parameters. Hence, by Proposition 3, the moduli are constant. Since every curve in Y is φ-invariant, the $\mathbb{C}^*$-action reduces to a complex torus action on the open dense subset of Y' and therefore on the whole Y'.

Conversely, assume that φ reduces to a complex torus action ψ on Y'. Then $\mathcal{Y}'$ is an affine variety in $\mathbb{C}^N$ with the $\mathbb{C}^*$-action $\tilde{\psi}$ induced by $\tilde{\varphi}$. Moreover the action $\tilde{\psi}$ is compatible with g', where g' is a contracting automorphism to 0 of $\mathbb{C}^N$ defining H'^N. It is not difficult to check that the $\mathbb{C}^*$-action $\tilde{\psi}$ on $\mathcal{Y}'$ is algebraic. (Construct a contracting automorphism on $\mathbb{C} \times \mathcal{Y}' \times \mathcal{Y}'$ which leaves invariant the closure $\bar{\Gamma}$ of the graph Γ of $\tilde{\psi}$, where $\bar{\Gamma}$ is an analytic subset of $\mathbb{C} \times \mathcal{Y}' \times \mathcal{Y}'$. Use the result of [2] to show that $\bar{\Gamma}$ is an algebraic subset of $\mathbb{C} \times \mathcal{Y}' \times \mathcal{Y}'$.) Hence, by Proposition (1.1.3) in Orlik-Wagreich [5], there is an embedding $j : \mathcal{Y}' \longrightarrow \mathbb{C}^{N'}$ for some N' and a $\mathbb{C}^*$-action $\tilde{\psi}'$ on $\mathbb{C}^{N'}$ such that $j(\mathcal{Y}')$ is $\tilde{\psi}'$-invariant and that $\tilde{\psi}'$ induces $\tilde{\psi}$ on $\mathcal{Y}'$. Moreover, by a suitable choice of coordinates $(z_1, \ldots, z_N)$ on $\mathbb{C}^{N'}$, the action $\tilde{\psi}'$ on $\mathbb{C}^{N'}$ can be written as

$$\tilde{\varphi}'(\rho,(z_1,\ldots,z_{N'})) = (\rho^{q_1}z_1,\ldots,\rho^{q_{N'}}z_{N'}),$$

where the q_i's are positive integers. There exists a constant α such that $\tilde{\varphi}'_\alpha$ induces g' on $\mathcal{Y}'$. Then $Y' = \mathcal{Y}'-\{0\}/\langle g'\rangle$ can be considered as a submanifold of $\mathbb{C}^{N'}-\{0\}/\langle\tilde{\varphi}'_\alpha\rangle$.

The following theorem is known.

Theorem (Ueno [8]). Let M_1 be a subvariety of a compact complex variety M_0. Then

(18) $\dim M_1 - a(M_1) \leq \dim M_0 - a(M_0)$.

Now it is clear that $a(\mathbb{C}^{N'}-\{0\}/\langle\tilde{\varphi}'_\alpha\rangle) = N'-1$. Hence, by (18), we get $a(Y') \geq \dim Y' - 1$. Since $a(Y') < \dim Y'$, we obtain $a(Y') = a(Y) = n-1$. Q.E.D.

Remark 5. Topologically, <u>any submanifold of a Hopf manifold is diffeomorphic to a fibre bundle over a 1-dimensional circle of which the transition function has a finite order as an element of the diffeomorphism group of the fibre</u>. This can be seen without difficulty from Theorem 3.

Remark 6. <u>A compact complex surface</u> S <u>is a submanifold of a Hopf manifold if and only if</u> S <u>is a relatively minimal surface of class</u> VI_0, VII_0-<u>elliptic or a Hopf surface</u>. (See [3] for the proof of the "if" part.) Let S be a submanifold of a Hopf manifold. It is clear by Proposition 3 that S is relatively minimal. By Theorem 1, S is not algebraic. Hence $a(S) \leq 1$. Assume that $a(S) = 1$. Then, by Theorem 1, there exists a locally flat line bundle L on S such that the mapping $\Phi_L : S \longrightarrow \mathbb{P}^n$ defined by the linear system $|L|$ gives an algebraic reduction of S which is defined everywhere. Put $\Delta = \Phi_L(S)$. Let η be the line bundle on Δ associated to a hyperplane section of Δ. Then we have $\Phi_L^* \eta = L$. We note that every fibre of $\Phi_L : S \longrightarrow \Delta$ is a non-singular elliptic curve (Proposition 3). We indicate by $b_i(M)$ the i-th Betti number of a manifold M. It is clear that $b_1(\Delta) \leq b_1(S) \leq b_1(\Delta) + 2$. Assume first that $b_1(\Delta) = b_1(S)$. Since L is a locally flat line bundle on S, L is raised from a group representation ρ of $H_1(S, \mathbb{Z})$ into $\mathbb{C}^*$. Let m be a certain positive integer such that ρ^m is trivial on the torsion part of $H_1(S, \mathbb{Z})$. Then, in view of $b_1(\Delta) = b_1(S)$, there exists a locally flat line bundle ξ on Δ such that $\Phi_L^* \xi = L^m$. Hence we get $\Phi_L^* \xi = \Phi_L^* \eta^m$.

Since $\Phi_L^*: H^1(\Delta, O^*) \longrightarrow H^1(S, O^*)$ is an injection, this implies that the ample line bundle η on Δ is locally flat. This is absurd. Hence we get $b_1(\Delta) < b_1(S)$. Next assume that $b_1(S) = b_1(\Delta) + 2$. By Corollary to Theorem 3, we get $b_2(S) = 2b_1(\Delta) + 2$. This implies that the dual of the homology class represented by a general fibre is a Betti base of $H^2(S, \mathbb{Z})$. This contradicts Theorem 1. Hence we conclude that $b_1(S) = b_1(\Delta) + 1$. Therefore $b_1(S)$ is odd. Hence S is either a surface of VI_0 or VII_0-elliptic. Consider the case $a(S) = 0$. By the classification theory of surfaces [4], a relatively minimal surface with no non-constant meromorphic functions and vanishing Euler number is either a complex torus or a surface of VII_0. A complex torus has a positive algebraic dimension if it contains a divisor. Hence by Proposition 3 we infer that S is of VII_0-class. Moreover $b_1(S) = 1$ and $b_2(S) = 0$. Hence, by Theorem 34 [4], S is a Hopf surface.

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