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Spectral theory and eigenfunction expansions for uniformly propagative systems

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1. Introduction.

In a recent year C.H.Wilcox showed that linear partial differential systems describing the wave propagation phenomona of classical physics in inhomogeneous anisotropic media filling the whole space can be written in the form

$$\frac{\partial u}{\partial t} = M(x)^{-1}\sum_{j=1}^{n} A_j \frac{\partial u}{\partial x_j} + f(x,t), \tag{1.1}$$

where $t \in \mathbb{R}^1$, $x \in \mathbb{R}^n$, $u(x,t)$ is $\mathbb{C}^m$-valued function, $M(x)$ is an $m \times m$ positive definite hermitian matrix depending on $x \in \mathbb{R}^n$ and A_j's are $m \times m$ constant hermitian matrices.

In this lecture we shall study some spectral properties, especially eigenfunction expansions , for the operator

$$L = M(x)^{-1}\sum_{j=1}^{n} A_j D_j \qquad \left(D_j = \frac{1}{i}\frac{\partial}{\partial x_j} \right) \tag{1.2}$$

in a suitable Hilbert space and scattering theory between system (1.2) and system

$$L_0 = \sum_{j=1}^{n} A_j D_j . \tag{1.3}$$

2. Preliminaries.

Our assumptions imposed on the operators are as follows:

(A.1) L_0 is uniformly propagative system in the sense of Wilcox, i.e, the roots of the characteristic equation $p(\lambda,\xi) = \det(\lambda I - \sum_{j=1}^{n} A_j \xi_j) = 0$ have constant multiplicities and never vanish unless vanish identically;

(A.2) There exist constants $C_1, C_2 > 0$ and $\delta > 1$ such that

(2.1) $C_1|\xi|^2 \leqq (\xi, M(x)\xi) \leqq C_1^{-1}|\xi|^2$ for all x and $\xi \in \mathbb{R}^n$,

(2.2) $\sup_{1\leqq i,j\leqq m} | m_{ij}(x) - \delta_{ij}| \leqq C_2(1 + |x|^2)^{-\frac{\delta}{2}}$ for all $x \in \mathbb{R}^n$,

where $m_{ij}(x)$ is the (i,j)-component of $M(x)$.

By assumption (A.1) the roots $\lambda_l(\xi)$ of $p(\lambda,\xi) = 0$ can be enumerated as $\lambda_{-\mu}(\xi) < \cdots < \lambda_{-1}(\xi) < \lambda_0(\xi) < \lambda_1(\xi) < \cdots < \lambda_\mu(\xi)$, where $\lambda_0(\xi) \equiv 0$ if it exists and will be omitted otherwise. Let us take $\delta_l(\xi) > 0$ so small that $\Gamma_l(\xi) = \{\zeta \in \mathbb{C}^1 : |\zeta - \lambda_l(\xi)| = \delta_l(\xi)\}$ does not enclose any root of $p(\lambda,\xi) = 0$ except $\lambda_l(\xi)$ and put

(2.3) $$P_l(\xi) = -\frac{1}{2\pi i}\oint_{\Gamma_l(\xi)} (\sum A_j\xi_j - \zeta I)^{-1} d\zeta .$$

Let $\hat{P}_l$ be the operator determined by the multiplication by $P_l(\xi)$ and put $P_l = \mathcal{F}^{-1}\hat{P}_l\mathcal{F}$. We put $S_l = \{\xi \in \mathbb{R}^n ; \lambda_l(\xi) = \operatorname{sign} l\}$. Let ds_l be the surface element of S_l and $d\sigma_l = ds_l/|\nabla\lambda_l|$.

We need some auxiliary spaces. For $\sigma \in \mathbb{R}^1$ and $s \in \mathbb{R}^1$ we put

$$H^s_{0,\sigma} = \{ f \in \mathcal{S}'(\mathbb{R}^n, \mathbb{C}^m) : \|f\|^2_{H^s_{0,\sigma}} = \int_{\mathbb{R}^n} |\mathcal{F}^{-1}((1+|\xi|^2)^{\frac{\sigma}{2}}(\mathcal{F}u)(\xi))|^2 (1+|x|^2)^{s} dx < \infty \},$$

$$H^s_{1,\sigma} = \{ f \in \mathcal{S}'(\mathbb{R}^n, \mathbb{C}^m) : \|f\|^2_{H^s_{1,\sigma}} = \int_{\mathbb{R}^n} |\mathcal{F}^{-1}((1+|\xi|^2)^{\frac{\sigma}{2}}(\mathcal{F}u)(\xi))|^2_M (1+|x|^2)^{s} dx < \infty \},$$

where we put $(u(x),v(x))_M = (u(x),M(x)v(x))$;

$$H_0 = H^0_{0,0} \quad , \qquad H_1 = H^0_{1,0} .$$

By assumption (A.2) $H^s_{0,\sigma} = H^s_{1,\sigma}$ as a set. We define the identification operator $J: H^s_{1,\sigma} \to H^s_{0,\sigma}$ by the equation $Ju(x) = u(x)$.

We can define the selfadjoint realization L_0 naturally since it has constant coefficiets. We put $L = J^* L_0 J$. L is obviously a selfadjoint operator in H_1.

3. Theorems.

Our results are summarized in the following theorems.

Theorem 1. (Limiting absorption principle for L_0)

Let assumption (A.1) be satisfied. Let $I_0 = R^1 \setminus \{0\}$ and let $\Pi^{\pm} = \{ \zeta \in C^1 : \operatorname{Im}\zeta \gtrless 0 \}$. Let ε be any positive constant. Then the following statements hold:

(1) The resolvent $R_0(\zeta)$ ($\operatorname{Im}\zeta \neq 0$) of L_0 can be extended to $\Pi^{\pm} \cup I_0$ as a $B(H_{0,\frac{1+\varepsilon}{2}}, H_{0,-\frac{1+\varepsilon}{2}})$-valued locally Hölder continuos function. Moreover $R_0(\zeta)(I-P_0)$ can be extended to the same regions as a $B(H_{0,\frac{1+\varepsilon}{2}}, H^1_{0,-\frac{1+\varepsilon}{2}})$-valued locally Hölder continuous function. We denote their boundary values on I_0

as $R_0(\lambda \pm i0)$ and $R_0'(\lambda \pm i0)$, respectively.

(2) For any $u \in H_{0,(1+\varepsilon)/2}$ and $\lambda \in I_0$, $(L_0 - \lambda) R_0(\lambda \pm i0) u = u$.

<u>Theorem 2.</u> (limiting absorption principle for L) Let assumptions (A.1) and (A.2) be satisfied. Let $I_1 = \mathbb{R}^1 \setminus (\sigma_p(L) \cup \{0\})$, where $\sigma_p(L)$ is the point spectrum of L. Then the following statements hold:

(1) The resolvent $R(\zeta)$ of L $(\operatorname{Im}\zeta \neq 0)$ can be extended to $\Pi^{\pm} \cup I_1$ as a $B(H_{1,\delta/2}, H_{1,-\delta/2})$-valued locally Hölder continuous function.

(2) For any $u \in H_{1,\delta/2}$ and $\lambda \in I_1$, $(L-\lambda) R(\lambda \pm i0) u = u$.

<u>Theorem 3.</u> (Discreteness of the point spectrum) Let assumptions (A.1) and (A.2) be satisfied. Then $\sigma_p(L) \setminus \{0\}$ is discrete and only possible accumulation point is the origin.

<u>Lemma 4.</u> For each $\ell = 0$, let $\varphi_\ell^{(k)}(\omega_\ell)$, k = 1,2,3,... be a C^∞-class complete orthonormal system of $L^2(S_\ell, C^1, d\sigma_\ell)$ and let $h_\ell(x,\lambda,k)$ be the matrix depending on $x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}_{\operatorname{sign}\ell}$ and $k \in N$ (= the set of all integers) defined by

$$(3.1) \qquad h_\ell(x,\lambda,k) = (2\pi)^{-\frac{n}{2}} \int_{S_\ell} e^{i|\lambda|\omega_\ell \cdot x} \varphi_\ell^{(k)}(\omega_\ell) \hat{P}_\ell(\omega_\ell)\, d\sigma_\ell(\omega_\ell).$$

Then $h_\ell(x,\lambda,k)$ is a bounded function of all variables. Furthermore for each fixed $\lambda \in \mathbb{R}_{\mathrm{sign}\, \ell}$, $k \in N$ and for any $\varepsilon > 0$ each column $h_\ell^{(i)}(x,\lambda,k)$ of $h_\ell(x,\lambda,k)$ satisfies the following relations:

(3.2) For any n-tuple of nonnegative integers $(\alpha_1, \alpha_2, \cdots, \alpha_n)$,

$$D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n} h_\ell^{(i)}(x,\lambda,k) \in H_{0,-(1+\varepsilon)/2};$$

(3.3) $(L_0 - \lambda I) h_\ell^{(i)}(x,\lambda,k) = 0.$

<u>Theorem 5.</u> (Eigenfunction expansion for the operator L_0)

Let $h_\ell(x,\lambda,k)$ $(\ell \neq 0)$ be the function defined in Lemma 4.

Then for any $f \in H_0$ and $K > 0$, $\int_{|x|<K} \bar{h}_\ell(x,\lambda,k)^* f(x)dx$ belongs to $L^2(\mathbb{R}_{\mathrm{sign}\, \ell}, \ell^2(\mathbb{C}^m), |\lambda|^{\frac{n-1}{2}} d\lambda)$ and

$$(T_j f)(\lambda,k) = \underset{K\to\infty}{\mathrm{l.i.m.}} \int_{|x|<K} h_j(x,\lambda,k)^* f(x)dx$$

exists. Let

$$T : H_0 \longrightarrow \sum_{\ell \neq 0} \oplus L^2(\mathbb{R}_{\mathrm{sign}\, \ell}, \ell^2(\mathbb{C}^m), |\lambda|^{\frac{n-1}{2}} d\lambda)$$

be the operator defined by

$$Tf = \sum_{\ell \neq 0} \oplus T_\ell f \quad , \quad f \in H_0.$$

The operator $T' : \sum_{\ell \neq 0} \oplus L^2(\mathbb{R}_{\mathrm{sign}\, \ell}, \ell^2(\mathbb{C}^m), |\lambda|^{\frac{n-1}{2}} d\lambda) \to H_0$ can be defined by

$$T'(\sum_{\ell \neq 0} \oplus \hat{f}_\ell)(x) = \sum_{\ell \neq 0} \underset{K\to\infty}{\mathrm{l.i.m.}} \int_{\mathbb{R}_{\mathrm{sign}\, \ell} \cap \{|\lambda|<K\}} \sum_{k=1}^{K} \bar{h}_j(x,\lambda,k) \hat{f}_\ell(\lambda,k) |\lambda|^{\frac{n-1}{2}} d\lambda .$$

Furthermore the following statements hold:

(1) T is a partially isometric operator from H_0 into $\sum_{\ell \neq 0} \oplus L^2(\mathbb{R}_{\mathrm{sign}\, j}, \ell^2(\mathbb{C}^m), |\lambda|^{\frac{n-1}{2}} d\lambda)$ and T' is its adjoint operator;

(2) (Expansion formula) $(I-P_0)f = T'T f$ for all $f \in H_0$;

(3) (Diagonal representation formula) $f \in D(L_0)$ if and only if

$\lambda(T_\ell f)(\lambda,k) \in L^2(\mathbb{R}_{\mathrm{sign}\,\ell}, \ell^2(\mathbb{C}^m), |\lambda|^{\frac{n-1}{2}} d\lambda)$ for all $\ell \neq 0$. If

$f \in D(L_0)$, $(T_\ell L_0 f)(\lambda,k) = \lambda (T_\ell f)(\lambda,k)$.

Lemma 6. Put $G(\lambda \pm i0) = J^{*-1} + \lambda(J^{*-1} - J) R(\lambda \pm i0) \in B(H_{1,\delta/2}, H_{0,+\delta/2})$. Let $\alpha_\ell^\pm(x,\lambda,k)$ ($\ell \neq 0$) be an $m \times m$-matrix depending on $x \in R^n$, $\lambda \in I_1$ and $k \in N$ defined by

$$\alpha_\ell^\pm(x,\lambda,k) = G(\lambda \pm i0)^* h_\ell(x,\lambda,k), \tag{3.4}$$

where $G(\lambda \pm i0)^*$ is applied to $h_\ell(x,\lambda,k)$ by matrix multiplication rule. Then the following statements hold:

(1) Each column $\alpha_\ell^{\pm(i)}(x,\lambda,k)$ of $\alpha_\ell^\pm(x,\lambda,k)$ is an $H_{1,-\delta/2}$-valued locally Hölder continuous function of $\lambda \in I_1$ for each fixed $k \in N$;

(2) $\alpha_\ell^\pm(x,\lambda,k)$ can be decomposed into three parts as

$$\alpha_\ell^\pm(x,\lambda,k) = J^{-1} h_\ell(x,\lambda,k) + t_\ell^\pm(x,\lambda,k) + w_\ell^\pm(x,\lambda,k) \tag{3.5}$$

where $h_\ell(x,\lambda,k)$ is the matrix defined by (3.1), $t_\ell^\pm(x,\lambda,k)$ is an $H_{1,-(1+\varepsilon)/2}$-valued continuous function of $\lambda \in I_1$ for each fixed $k \in N$ and any $\varepsilon > 0$, and is given by

$$t_\ell^\pm(x,\lambda,k) = -\lambda J^{-1} \sum_{\mathrm{sign}\,k = \mathrm{sign}\,\ell} \mathcal{F}^{-1}\left\{ \lim_{\eta \downarrow 0} \frac{[\hat{P}_\ell(\xi)\mathcal{F}((1-M(y))\alpha_\ell^\pm(y,\lambda,k))(\xi)]\big|_{\lambda_\ell(\xi)=\lambda}}{\lambda_k(\xi) - (\lambda \pm i\eta)} \right\} \tag{3.6}$$

using $\alpha_\ell^\pm(x,\lambda,k)$, and $w_\ell^\pm(x,\lambda,k)$ is an $H_{1,(\delta-2)/2}$-valued continuous function of $\lambda \in I_1$ for each fixed $k \in N$;

(3) $L(D)\alpha_{\mathfrak{g}}^{\pm}(x,\lambda,k) = \lambda\alpha_{\mathfrak{g}}^{\pm}(x,\lambda,k)$, where $L(D)$ is applied by matrix multiplication rule and the differentiation is in the sense of distributions.

Remark. In the decomposition (3.5) of the function $\alpha_{\mathfrak{g}}^{\pm}(x,\lambda,k)$ $J^{-1}h_{\mathfrak{g}}(x,\lambda,k)$, $t_{\mathfrak{g}}^{\pm}(x,\lambda,k)$ and $w_{\mathfrak{g}}^{\pm}(x,\lambda,k)$ are considered to describe the state of the incident wave, outgoing (imcoming) spherical wave, and the wave damping rapidly at infinity, respectively, in the description of the stationary scattering process. Furthermore $t_{\mathfrak{g}}^{\pm}(x,\lambda,k)$'s are the quatities which are connected with the scattering amplitudes. (See furmulas (3.6) and (3.8).)

Theorem 7. Let $\sigma_p^K(L) = \{\lambda \in \mathbb{R}^1 : |\lambda-\mu| < K^{-1}$ for some $\mu \in \sigma_p(L)\}$ and let $I_{\mathfrak{g},K} = \mathbb{R}_{\mathrm{sign}\,\mathfrak{g}} \setminus [\{|\lambda| > K\} \cup \{|\lambda| < K^{-1}\} \cup \sigma_p^K(L)]$. Let $\alpha_{\mathfrak{g}}^{\pm}(x,\lambda,k)$ be the matrix-valued function defined in Lemma 6 . Then for any $f \in H_1$ and $K>0$, $\int_{|x|<K} \alpha_{\mathfrak{g}}^{\pm}(x,\lambda,k)^* M(x) f(x)dx$ belongs to $L^2(\mathbb{R}_{\mathrm{sign}\,\mathfrak{g}}, \ell^2(\mathbb{C}^m), |\lambda|^{\frac{n-1}{2}}d\lambda)$ and

$$(Z_{\mathfrak{g}}^{\pm}f)(\lambda,k) = \underset{K\to\infty}{\mathrm{l.i.m.}} \int_{|x|<K} \alpha_{\mathfrak{g}}^{\pm}(x,\lambda,k)^* M(x) f(x)dx$$

exists. For any $\hat{f}_{\mathfrak{g}}(\lambda,k) \in L^2(\mathbb{R}_{\mathrm{sign}\,\mathfrak{g}}, \ell^2(\mathbb{C}^m), |\lambda|^{\frac{n-1}{2}}d\lambda)$ and $K>0$, $\int_{I_{j,k}} \sum_{k=1}^{K} \alpha_{\mathfrak{g}}^{\pm}(x,\lambda,k)\hat{f}_j(\lambda,k)|\lambda|^{\frac{n-1}{2}}d\lambda$ belongs to H_1 and

$$(Z_{\ell}^{\prime\pm} \hat{f}_{\ell})(x) = \underset{K\to\infty}{\text{l.i.m.}} \int_{I_{j,k}} \sum_{k=1}^{K} \alpha_{\ell}^{\pm}(x,\lambda,k)\hat{f}_{\ell}(\lambda,k)|\lambda|^{\frac{n-1}{2}} d\lambda$$

exists, where l.i.m. stands for the convergence in H_1 and the integration is Bochner integral. Let

$$Z^{\pm} : H_1 \longrightarrow \sum_{\ell\neq 0} \oplus L^2(\mathbb{R}_{\text{sign } \ell}, \ell^2(\mathbb{C}^m), |\lambda|^{\frac{n-1}{2}} d\lambda)$$

and

$$Z^{\prime\pm} : \sum_{\ell\neq 0} \oplus L^2(\mathbb{R}_{\text{sign } \ell}, \ell^2(\mathbb{C}^m), |\lambda|^{\frac{n-1}{2}} d\lambda) \longrightarrow H_1$$

be the operators defined by

$$Z^{\pm}f = \sum_{\ell\neq 0} \oplus Z_{\ell}^{\pm} f, \qquad f \in H_1$$

and

$$Z'(\hat{f}_{-\mu},\ldots,\hat{f}_{-1},\hat{f}_1,\ldots,\hat{f}_{\mu}) = \sum_{\ell\neq 0} Z_j^{\pm} \hat{f}_j,$$
$$(\hat{f}_{-\mu},\ldots,\hat{f}_{-1},\hat{f}_1,\ldots,\hat{f}_{\mu}) \in \sum_{\ell=0} \oplus L^2(\mathbb{R}_{\text{sign } \ell}, \ell^2(\mathbb{C}^m), |\lambda|^{\frac{n-1}{2}} d\lambda),$$

respectively. Then the following statements hold:

(1) Let P_{ac} be the projection operator in H_1 onto the absolutely continuous subspace with respect to L. Then $Z^{\pm}$ is a partially isometric operator from H_1 into $\sum_{\ell\neq 0} \oplus L^2(\mathbb{R}_{\text{sign } \ell}, \ell^2(\mathbb{C}^m), |\lambda|^{\frac{n-1}{2}} d\lambda)$ with initial set $P_{ac}H_1$ and $Z^{\prime\pm}$ is its adjoint operator;

(2) (Expansion formuras for L) For any $f \in H_1$

$$P_{ac}f = Z^{\prime\pm} Z^{\pm} f$$
$$= \sum_{\ell\neq 0} \underset{K\to\infty}{\text{l.i.m.}} \Big[\int_{I_{j,k}} \sum_{k=1}^{K} \alpha_{\ell}^{\pm}(x,\lambda,k) \times$$
$$\Big\{ \underset{K'\to\infty}{\text{l.i.m.}} \int_{|y|<K'} \alpha_{\ell}(y,\lambda,k)^{*} M(y) f(y)\, dy \Big\} |\lambda|^{\frac{n-1}{2}} d\lambda \Big];$$

(3) (Diagonal representations) For $f \in H_1$, $P_{ac} f \in D(L)$ if and only if $\lambda Z_{\ell}^{\pm} f(\lambda, \cdot) \in L^2(\mathbb{R}_{\mathrm{sign}\,\ell}, \ell^2(\mathbb{C}^m), |\lambda|^{\frac{n-1}{2}} d\lambda)$ for each $\ell \neq 0$. Furthermore for $f \in H_1$ with $P_{ac} f \in D(L)$

$$(Z_{\ell}^{\pm} Lf)(\lambda, k) = \lambda (Z_{\ell} f)(\lambda, k) \quad , \quad \lambda \in I_1 .$$

Theorem 8. (Orthogonality of eigenfunctions) The range $R(Z^{\pm})$ of $Z^{\pm}$ is equal to the range $R(T)$ of T, where T is the operator defined in Theorem 5.

Next we shall give the theorems concerning the applications of the previous theorems.

Theorem 9. (Existence of the wave operator and its completeness) Let assumptions (A.1) and (A.2) be satisfied. Then the wave operators

$$W_{\pm}(L, L_0, J^*) = \operatorname*{s-lim}_{t \to \pm\infty} e^{itL} J^* e^{-itL_0}(I - P_0) \quad ,$$

$$W_{\pm}(L_0, L, J) = \operatorname*{s-lim}_{t \to \pm\infty} e^{itL_0} J e^{-itL} P_{ac}$$

exist and therefore they are complete.

Theorem 10. Let assumptions (A.1) and (A.2) be satified.

Let $\Delta \subset I_0$ (or $\Delta \subset I_1$) , then

$$W_{\pm}(L, L_0, J^*) E_0(\Delta) = Z^{\pm *} T E_0(\Delta) \quad , \Delta \subset I_0 \qquad \text{and}$$

$$W_{\pm}(L_0, L, J) E(\Delta) = T^* Z^{\pm} E(\Delta) \quad , \Delta \subset I_1 \quad .$$

Scattering operator S is custumarily defined by the formula

(3.7) $\quad S = W_+(L, L_0, J^*)^* W_-(L, L_0, J^*)$.

Put $\hat{S} = TST^*$. Then S is a unitary operator $(I-P_0)H_0 \longrightarrow P_{ac}H_1$ and $\hat{S}$ is a unitary operator $M \longrightarrow M$, where $M = R(T) = R(Z^{\pm})$.

<u>Theorem 11.</u> (Representation formula for the scattering operator) For $\mathfrak{l} \neq 0$, let $F_{\mathfrak{l},\mathfrak{m}}(\lambda, k, k')$ be the matrix depending on $\lambda \in R_{\mathrm{sign}\,\mathfrak{l}} \cap I_1$, $k \in N$, $k' \in N$, and $\mathfrak{m}$ (sign$\mathfrak{m}$ = sign $\mathfrak{l}$, $\mathfrak{m} = -\mu, -\mu+1, \ldots, -1, 1, \ldots, \mu$) defined by

(3.8) $\quad F_{\mathfrak{l},\mathfrak{m}}(\lambda, k, k')$

$$= \int_{S_m} \varphi_m^{(k')}(\omega_m) P_\ell(\omega_m) [\mathcal{F}(|\lambda|^{\frac{n-1}{2}} \cdot \lambda (M(x)-1)\alpha_\ell^+(x, \lambda, k)]\Big|_{\lambda_m(\xi)=\lambda} d\sigma_m(\omega_m) ,$$

Then for any fixed λ and k , each column of $F_{j,\mathfrak{m}}(\lambda, k, k')$ belongs to $\ell^2(\mathbf{C}^m)$ with respect to $k' \in N$ and for any $\hat{f}(k') \in \ell^2(\mathbf{C}^m)$ and any fixed $\lambda \in R_{\mathrm{sign}\,\mathfrak{l}} \cap I_1$,

$$\sum_{k'=1}^{\infty} F_{\mathfrak{l},m}(\lambda, k, k')\hat{f}(k') \in \ell^2(\mathbf{C}^m)$$

with respect to k. Let

$$\hat{t}(\lambda) : \sum_{\mathrm{sign}\,\mathfrak{l} = \mathrm{sign}\lambda} \oplus\ \ell^2(\mathbf{C}^m) \longrightarrow \sum_{\mathrm{sign}\,\mathfrak{l} = \mathrm{sign}\lambda} \oplus\ \ell^2(\mathbf{C}^m)$$

be the operator defined by

$$(\hat{t}(\lambda)(\sum_{\mathrm{sign}\,\mathfrak{l}=\mathrm{sign}\lambda} \oplus\ \hat{f}_j))(k)$$
$$= \sum_{\mathrm{sign}\,\mathfrak{l}=\mathrm{sign}\lambda} \oplus\ (\sum_{\mathrm{sign}\mathfrak{m}=\mathrm{sign}\lambda} \sum_{k=1}^{\infty} F_{\mathfrak{l},m}(\lambda, k, k')\hat{f}_m(k')).$$

Then $\hat{t}(\lambda)$ is a compact operator on $\sum_{\text{sign}\, \ell = \text{sign}\, \lambda} \oplus\, \ell^2(\mathbb{C}^m)$ and the scattering operator $\hat{S}$ can be written in terms of the operator $t(\lambda)$ as follows: For any $f \in M = R(T) = R(Z^{\pm})$

$$(\hat{S}f)(\lambda,k) = f(\lambda,k) - 2\pi i[\hat{t}(\lambda)f(\lambda)](k) \qquad \text{a.e. } \lambda \in \mathbb{R}^1.$$

Remark 1. $F_{\ell,m}(\lambda,k,k')$ is the k'-th Fourier coefficient of the numerator in (3.6) with respect to the complete orthonormal basis $\varphi_m^{(k')}(\omega_m)$ as a element of $L^2(S_m)$.

Remark 2. The operator

$$I - 2\pi i\hat{t}(\lambda) : \sum_{\text{sign}\, \ell = \text{sign}\, \lambda} \oplus\, \ell^2(\mathbb{C}^m) \longrightarrow \sum_{\text{sign}\, \ell = \text{sign}\, \lambda} \oplus\, \ell^2(\mathbb{C}^m)$$

is a strongly continuos unitary operator valued function of $\lambda \in R^+ \cap I_1$ (or $R^- \cap I_1$).

Remark 3. If $M(x) - 1$ decreases sufficiently rapidly at infinity we can prove that the operator $\hat{t}(\lambda)$ is Hilbert-Schmit type.

4. Concluding remark.

Our method for proving limiting absoption principles is the method of Agmon and the method for proving the expansion formulas are the method of Kato-Kuroda. In using these methods the following lemma proves to play a essential role:

<u>Lemma 12.</u> Let K be an arbitrary Hilbert space and let $H^s(\mathbb{R}^1,K)$ be K-valued Sobolev space of order s in the usual sense. Let $u \in H^s(\mathbb{R}^1,K)$ with $s > 1/2$. For $\mu \in R^1$ put

$$u_1(\lambda,\mu) = \frac{u(\lambda)-u(\mu)}{\lambda - \mu} \quad \text{and} \quad u_2^{\pm}(\lambda,\mu) = \lim_{\eta \downarrow 0} \frac{u(\mu)}{\lambda-(\mu \pm i\eta)} .$$

We fix μ and regard u_1 and u_2 as K-valued distributions of the variable λ. Then $u_1 \in H^{s-1}(\mathbb{R}^1,K)$ and $u_2^{\pm} \in H^{-(1+\varepsilon)/2}(\mathbb{R}^1,K)$ for any $\varepsilon > 0$, and there exist constants C_1 and C_2 such that

$$\| u_1(\cdot,\mu) \|_{H^{s-1}} \leqq C_1 \| u \|_{H^s} ,$$

$$\| u_2^{\pm}(\cdot,\mu) \|_{H^{-(1+\varepsilon)/2}} \leqq C_2 \| u \|_{H^s} .$$

$u_1(\cdot,\mu)$ (or $u_2^{\pm}(\cdot,\mu)$) are $H^{s-1}(\mathbb{R}^1,K)$-valued (or $H^{-(1+\varepsilon)/2}$-valued) continuous function of μ.

Furthermore the following equation holds in $H^{-(1+\ )/2}$:

$$\lim_{\eta \downarrow 0} \frac{u(\lambda)}{\lambda-(\mu \pm i\eta)} = \frac{u(\lambda)-u(\mu)}{\lambda - \mu} + \lim_{\eta \downarrow 0} \frac{u(\mu)}{\lambda-(\mu \pm i\eta)} .$$

For the proof of the theorems and lemmas we refer the readers to the papers [1] and [2].

5. References.

[1] Yajima, K., The limiting absorption principle for uniformly propagative systems., J.Fac.Sci.Univ.Tokyo, Sec.1A, 21, No.1, (1974), 119-131.

[2] Yajima,K., Eigenfunction expansions associated with uniformly propagative systems and their applications to scattering theory, (to appear).

[3] Ikebe,T., Remarks on non-elliptic stationary wave propagation problems, Trabalho de Mathematica, No.79, Univ. Brasilia, 1974.