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Irregularity of characteristic elements and construction of null-solutions

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Let $P(x, \partial)$ be a linear partial differential operator with real analytic coefficients. We define for each non-singular characteristic element $(x_0, \xi_0\infty)$ of $P(x, \partial)$ its multiplicity d and irregularity $\sigma \geqq 1$ so that $\sigma = 1$ if and only if $P(x, \partial)$ satisfies E. E. Levi's condition at $(x_0, \xi_0\infty)$.

Then, we construct null-solutions for each characteristic surface S of constant multiplicity. If S is regular, i.e. $\sigma = 1$, there exists a null-solution with an arbitrarily prescribed regularity or singularity. If S is of irregularity $\sigma > 1$, then for each $1 < s \leqq \sigma/(\sigma - 1)$ there exist an ultradifferentiabl null-solution of Gevrey class $\{s\}$ and an ultradistribution null-solution of Gevrey class (s). In any case there are infinitely differentiable null-solutions.

Lastly, we prove that there is a homogeneous solution whose singularity spectrum coincides with a given real bicharacteristic strip (or with a given real element of a certain type of complex bicharacteristic strip) and having a given regularity or singularity as in the case of null-solutions.

L. Hörmander [10], [11] has shown that there is always an infinitely differentiable null-solution for any linear partial differential operator with constant coefficients and any characteristic hyperplane. As far as we know, however, no proofs have been published of the existence of infinitely differentiable null-solution in the variable coefficient case even for simple characteristic surfaces.[1)]

We employ S. Mizohata's method in [22] where he constructs finitely differentiable null-solutions for simple characteristic surfaces. The method may be traced back to J. Hadamard [5], §§ 49-53, P. D. Lax [18] and D. Ludwig [20]. We construct a formal solution of the form

$$U(x) = \sum_{j=-\infty}^{\infty} u_j(x)\, \Phi_j(\varphi(x)) , \tag{0.1}$$

where $u_j(x)$ and $\varphi(x)$ are analytic functions and $\Phi_j(t)$ is a sequence of functions of one variable satisfying

$$\frac{d}{dt} \Phi_j(t) = \Phi_{j-1}(t) , \tag{0.2}$$

and then we prove the convergence of (0.1) in a suitable topology by estimating the coefficients $u_j(x)$. Interesting is the fact that $u_j(x)$ do not depend on the sequence $\Phi_j(t)$.

When $P(x, \partial)$ satisfies Levi's condition relative to the characteristic surface S (or the characteristic function $\varphi(x)$), we may put $u_j(x) = 0$ for $j < 0$ and can estimate $u_j(x)$ much more easily than the irregular case. Actually this case has been discussed by J. Vaillant [26] when the multiplicity is at most double and by J. -C. De Paris [3], [4] when the multiplicity is arbitrary and the estimates of u_j we need are already known.

Originally Levi's condition was introduced by E. E. Levi [19] and A. Lax [17] in the case of two independent variables for the purpose to characterize such weakly hyperbolic linear partial differential operators $P(x, \partial)$ that the Cauchy problem of the equation $P(x, \partial)u = 0$ is correctly posed in the category of infinitely differentiable functions. Their result has been generalized to the case of n independent variables by S. Mizohata - Y. Ohya [23] under the restriction that the multiplicity of characteristics is at most double and then by J. Chazarain [1] without the restriction as far as the sufficiency part is concerned.

In order to apply to the construction of formal solutions (0.1) we need a decomposition of the operator $P(x, \partial)$ into a polynomial of differential operators due to J. -C. De Paris [2]. The definition of irregularity relies also on the decomposition. The decomposition is global in the cotangential variables ξ. When a characteristic element $(x, \xi\infty)$, a characteristic surface S or a characteristic function φ is given, it is desirable, however, to formulate the decomposition microlocally in the sense of M. Sato - T. Kawai - M. Kashiwara [25]. J. Vaillant [26] makes an attempt by use of the algebraic localization. We prove with the aid of a version of Ritt's lemma (cf. Hervé [9]) that a microlocal decomposition implies a global one as far as the characteristic element is a non-singular point on the characteristic variety.

In the irregular case we estimate the coefficients $u_j(x)$ by Y. Hamada's ingenious method in [8].

Null-solutions are solutions with minimal supports. In the last section we construct solutions with minimal singularity spectra generalizing results by M. Zerner [28], L. Hörmander [12] and T. Kawai [13]. In particular, we obtain a necessary condition for analytic hypoellipticity.

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1. Irregularity of non-singular characteristic elements. Let

$$P(x, \partial) = \sum_{|\alpha| \leq m} a_\alpha(x)\, \partial^\alpha \tag{1.1}$$

be a linear partial differential operator of order m with coefficients $a_\alpha(x)$ real analytic on an open set Ω in $\mathbb{R}^n$. The $a_\alpha(x)$ may be complex valued. We employ the notion

$$\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n} \tag{1.2}$$

to avoid confusion with Hörmander's

$$D^\alpha = (-i\, \partial/\partial x_1)^{\alpha_1} \cdots (-i\, \partial/\partial x_n)^{\alpha_n} .$$

Since the coefficients $a_\alpha(x)$ are continued analytically to a complex neighborhood V of Ω in $\mathbb{C}^n$, $P(x, \partial)$ is also considered to be a linear partial differential operator with holomorphic coefficients in V.

The principal part of $P(x, \partial)$ is denoted by $p(x, \partial)$:

$$p(x, \partial) = \sum_{|\alpha| = m} a_\alpha(x)\, \partial^\alpha . \tag{1.3}$$

We always assume that $P(x, \partial)$ is non-degenerate or that $p(x, \xi) \not\equiv 0$ for any fixed $x \in V$.

A point $(x_0, \xi_0 \infty)$ in $S^*\Omega = (T^*\Omega \setminus \Omega)/\mathbb{R}_+$ or in

$P^*V = (T^*V \setminus V)/\mathbb{C}^\times$ is said to be a characteristic element of $P(x, \partial)$ if

$$p(x_0, \xi_0) = 0 .$$

($\xi_0 \infty$ denotes the class of ξ_0.) It is said to be non-singular if it is on the non-singular part of the characteristic variety

$$N = \{(x, \xi\infty) \in P^*V ; \quad p(x, \xi) = 0\}.$$

Then the multiplicity d is defined as usual; under a suitable coordinate system there are holomorphic functions $\rho(x, \xi)$ and $\lambda(x, \xi')$ on a neighborhood of (x_0, ξ_0) so that $p(x, \xi)$ is decomposed as

$$(1.4) \qquad p(x, \xi) = \rho(x, \xi)(\xi_1 - \lambda(x, \xi'))^d$$

with $\rho(x_0, \xi_0) \neq 0$ and $\xi_{0,1} - \lambda(x_0, \xi_0') = 0$. Here ξ' denotes $(\xi_2, \ldots, \xi_n)$. Since $p(x, \xi)$ is homogeneous in ξ, so are the factors ρ and $\xi_1 - \lambda(x, \xi')$.

In order to show that decomposition (1.4) is realized by polynomials in ξ, we make a few preparations.

We denote by $\mathcal{O}$ the ring of germs of holomorphic functions on a neighborhood of x_0 and by $\mathcal{O}[\xi]$ the ring of polynomials in $\xi = (\xi_1, \ldots, \xi_n)$ with coefficients in $\mathcal{O}$. Since $\mathcal{O}$ is a unique factorization domain, so is $\mathcal{O}[\xi]$.

Let $K(x, \xi)$ be an irreducible polynomial in $\mathcal{O}[\xi]$ which is non-degenerate in the sense that the principal part $k(x, \xi)$ does not vanish at x_0. We may assume, then, without loss of generality that the coefficient of ξ_1^{ℓ} is one. If (x_0, ξ_0) is a non-singular zero of $K(x, \xi)$, $K(x, \xi)$ is decomposed as

$$(1.5) \qquad K(x, \xi) = \mu(x, \xi)(\xi_1 - \lambda(x, \xi'))$$

with holomorphic functions μ and λ defined on a neighborhood of (x_0, ξ_0) such that $\mu(x_0, \xi_0) \neq 0$ and $\xi_{0,1} - \lambda(x_0, \xi_0') = 0$. In this case the multiplicity is always one. For, otherwise, the discriminant of K with respect to ξ_1 would be identically zero so that $K(x, \xi)$ would be divided by a polynomial of lower order.

Lemma 1.1. *Let $K(x, \xi)$ be a non-degenerate irreducible polynomial in $\mathcal{O}[\xi]$ with the decomposition* (1.5). *If a non-degenerate polynomial $A(x, \xi)$ in $\mathcal{O}[\xi]$ is divisible by $\xi_1 - \lambda(x, \xi')$ as a holomorphic function in a neighborhood of (x_0, ξ_0), then it is divisible by $K(x, \xi)$ in $\mathcal{O}[\xi]$.*

Proof. Clearly the function

$$F(x, \xi) = A(x, \xi)/K(x, \xi)$$

is defined and holomorphic outside the variety $N(K) = \{(x, \xi) \in \mathbb{C}^{2n};\ K(x, \xi) = 0\}$. By the assumption, $F(x, \xi)$ is holomorphic on a neighborhood of (x_0, ξ_0) which lies on the non-singular part of $N(K)$. The analytic continuations of the numerator and the donominator prove that $F(x, \xi)$ is holomorphic on the connected component of (x_0, ξ_0) in the non-singular part.

On the other hand, since $K(x, \xi)$ is irreducible, it follows that the non-singular part of $N(K)$ is connected when x is restricted to a suitable neighborhood V of x_0 (cf. Hervé [9]). Hence $F(x, \xi)$ is holomorphic on $V \times \mathbb{C}^n$ outside an exceptional set of codimension 2 and hence on $V \times \mathbb{C}^n$ by Hartogs' continuation theorem. Then it is easy to see that $F(x, \xi) \in \mathcal{O}[\xi]$.

Proposition 1.2. If $(x_0, \xi_0 \infty)$ is a non-singular characteristic element of multiplicity d of $P(x, \partial)$, then there are a homogeneous polynomial $Q(x, \xi)$ and an irreducible homogeneous polynomial $K(x, \xi)$ in $\mathcal{O}[\xi]$ such that

$$(1.6) \qquad p(x, \xi) = Q(x, \xi) K(x, \xi)^d$$

with $Q(x_0, \xi_0) \neq 0$.

Proof. Let

$$p(x, \xi) = \prod_{j=1}^{r} Q_j(x, \xi)^{\nu_j}$$

be the irreducible decomposition in $\mathcal{O}[\xi]$. Since $p(x, \xi)$ is non-degenerate and homogeneous, so are the irreducible factors $Q_j(x, \xi)$. The factor $(\xi_1 - \lambda(x, \xi'))$ in (1.4) divides the right hand side and hence a factor $K(x, \xi) = Q_j(x, \xi)$ as a holomorphic function in a neighborhood of (x_0, ξ_0). Applying Lemma 1.1, we find that $K(x, \xi)^d$ divides $p(x, \xi)$ in $\mathcal{O}[\xi]$. Let $Q(x, \xi)$ be the quotient.

We call $K(x, \xi)$ the irreducible factor associated with the characteristic element $(x_0, \xi_0 \infty)$.

Remark. Our method applies also to the proof of equivalence of two definitions of hyperbolic operators of constant multiplicity, which was originally proved by S. Matsuura [21] by making full use of the hyperbolicity.

Theorem 1.3 (Cf. De Paris [3] [4]). Let $(x_0, \xi_0 \infty)$ be a non-singular characteristic element of multiplicity d of $P(x, \partial)$

and let $K(x, \xi)$ *be the associated factor of* $p(x, \xi)$. *Then there are non-negative integers or* $+\infty$ $d_0, d_1, \ldots, d_m = d$ *and linear homogeneous differential operators* $Q_i(x, \partial)$ *with holomorphic coefficients on a neighborhood* U *of* x_0 *such that*

$$(1.7) \qquad P(x, \partial) = \sum_{i=0}^{m} Q_i(x, \partial) K(x, \partial)^{d_i}$$

and that

$$Q_i(x, \partial) = 0 \quad \text{if} \quad d_i = +\infty ,$$

and

$$\operatorname{ord}(Q_i(x, \partial) K(x, \partial)^{d_i}) = i$$

and

$$Q_i(x, \xi) \not\equiv 0$$

on a neighborhood of (x_0, ξ_0) *in the zeros* $N(K)$ *of* $K(x, \xi)$ *if* $d_i < \infty$.

Proof. Using $Q(x, \xi)$ in (1.6), we define $Q_m(x, \partial) = Q(x, \partial)$. Then

$$R(x, \partial) = P(x, \partial) - Q_m(x, \partial) K(x, \partial)^d$$

is an operator of order $\leq m-1$. If the homogeneous part $R^{m-1}(x, \partial)$ of order $m-1$ is zero, we define $d_{m-1} = \infty$ and $Q^{m-1}(x, \partial) = 0$. Otherwise, let d_{m-1} be the largest integer such that $(\xi_1 - \lambda(x, \xi'))^{d_{m-1}}$ divides $R_{m-1}(x, \xi)$ as a holomorphic function on a neighborhood of (x_0, ξ_0). Then it follows from Lemma 1.1 that there is a homogeneous polynomial $Q_{m-1}(x, \xi) \in \mathcal{O}[\xi]$ such that

$$R_{m-1}(x, \xi) = Q_{m-1}(x, \xi) K(x, \xi)^{d_{m-1}} .$$

If $Q_{m-1}(x, \xi)$ vanishes on a neighborhood of (x_0, ξ_0) in the zeros of $K(x, \xi)$, we can raise the exponent d_{m-1} by 1 contrarily to the definition of d_{m-1}. Hence it does not vanish identically. We can repeat the same procedure m times to obtain d_i and $Q_i(x, \partial)$.

Definition 1.4. We define the irregularity σ of the characteristic element $(x_0, \xi_0\infty)$ (or the associated factor $K(x, \xi)$) of $P(x, \partial)$ by

$$(1.8) \qquad \sigma = \max\left\{1,\ (d - d_i)/(m - i);\quad 0 \leqq i < m\right\}$$

that is, the maximal slope of the Newton polygon associated with the graph $\{(j, d_i);\ i = 0, 1, \ldots, m\}$ (cf. the definition of the irregularity of singular points of an ordinary differential operator in [15]).

d_i and $Q_i(x, \partial)$ in the decomposition (1.7) depend on the coordinate system but it is easy to show that the irregularity σ does not.

When $\sigma = 1$, we say that the characteristic element $(x_0, \xi_0\infty)$ (or the associated factor $K(x, \xi)$) is regular or that $P(x, \partial)$ satisfies Levi's condition at $(x_0, \xi_0\infty)$. (Cf. E. E. Lévi [19], A. Lax [17], S. Mizohata - Y. Ohya [23], J. Vaillant [26], J. -C. De Paris [3], [4] and J. Chazarain [1].)

When $\sigma > 1$, we say that $(x_0, \xi_0\infty)$ (or $K(x, \xi)$) is irregular.

We consider only non-singular real analytic characteristic surfaces S in Ω. Namely S is defined by $\varphi(x) = 0$ with a real valued real analytic function $\varphi(x)$ defined on a neighborhood of S such that grad $\varphi(x) \neq 0$ on S, and $(x, \text{grad}\, \varphi(x)\infty)$, $x \in S$, are non-singular characteristic elements of $P(x, \partial)$. It is easy to see that the multiplicity d and the irregularity σ of $(x, \text{grad}\, \varphi(x)\infty)$ are constants on each connected component of S. We call them the multiplicity and the irregularity of the component.

The surface $S = \{x \in \Omega\ ;\ \varphi(x) = 0\}$ is characteristic if

(1.9) $$p(x, \text{grad}\, \varphi(x)) = 0 \quad \text{whenever} \quad \varphi(x) = 0\ .$$

If we can choose more strongly a real valued real analytic function φ with $S = \{\varphi(x) = 0\}$ such that

(1.10) $$p(x, \text{grad}\, \varphi(x)) \equiv 0\ ,$$

then we say that S is a <u>totally real</u> characteristic surface. This means that S is imbedded in the one-parameter family $S_t = \{x;\ \varphi(x) = t\}$ of characteristic surfaces.

A function φ satisfying (1.10) will be called a <u>characteristic function</u> of $P(x, \partial)$.

We admit that the coefficients $a_\alpha(x)$ are complex valued. Therefore, a real characteristic surface S is not necessarily totally real. We have, however, the following.

<u>Proposition 1.5</u>. <u>For each non-singular point</u> x_0 <u>in a real characteristic surface</u> S <u>we can find a holomorphic charateristic function</u> φ <u>defined on a complex neighborhood</u> V <u>of</u> x_0 <u>such that</u> S <u>is the zeros of</u> φ <u>in</u> V <u>and that</u> grad φ <u>never vanishes on</u> S. <u>Moreover</u>, <u>for each</u> $\theta > 0$ <u>we can find</u> V <u>and</u> φ <u>such that either</u> $|\arg \varphi(x)| < \theta$ <u>or</u> $|\arg \varphi(x) - \pi| < \theta$ <u>whenever</u>

x is in the real neighborhood $V \cap \Omega$ of x_0.

Proof. For the sake of simplicity we assume that $x_0 = 0$. Suppose that S is defined by $\psi(x) = 0$ with a real valued real analytic function ψ on a neighborhood of S such that $\operatorname{grad}\psi(x) \neq 0$ on S.

Since $(0, \operatorname{grad}\psi(0))$ is a non-singular characteristic element of $P(x, \partial)$, we have the decomposition (1.4) of $p(x, \xi)$ such that $(x, \operatorname{grad}\psi(x))$ are zeros of the factor $\xi_1 - \lambda(x, \xi')$. Clearly a solution $\varphi(x)$ of the first order non-linear differential equation

$$(1.11) \qquad \frac{\partial\varphi(x)}{\partial x_1} - \lambda\left(x, \frac{\partial\varphi(x)}{\partial x'}\right) = 0$$

is a characteristic function. We solve this under the initial condition

$$(1.12) \qquad \varphi(0, x') = \psi(0, x').$$

As is well known the solution is obtained by integrating Hamilton's canonical equations

$$(1.13) \quad \begin{cases} \dfrac{dx_1}{dt} = 1, \quad \dfrac{dx_j}{dt} = -\dfrac{\partial\lambda(x, p')}{\partial\xi_j}, \quad j = 2, \ldots, n; \\ \dfrac{dp_j}{dt} = \dfrac{\partial\lambda(x,p')}{\partial x_j}, \quad j = 1, 2, \ldots, n; \\ \dfrac{d\varphi}{dt} = p_1 - \lambda(x, p'), \end{cases}$$

under the initial conditions

$$(1.14) \quad \begin{cases} x_1(0) = 0; \quad x_j(0) = y_j, \quad j = 2, 3, \ldots, n; \\ p_1(0) = \lambda(0, y'; p'(0)); \\ p_j(0) = \dfrac{\partial\psi}{\partial x_j}(0, y'), \quad j = 2, 3, \ldots, n; \\ \varphi(0) = \psi(0, y') \end{cases}$$

and then eliminating t and y'. It is known that $p_1 - \lambda(x, p')$ and $\frac{\partial \varphi}{\partial y_j} - \sum_{i=1}^{n} p_i \frac{\partial x_i}{\partial y_j}$, $j = 2, \ldots, n$, are constants on each trajectory of (1.13). Since the initial values are chosen so that they vanishes on the initial surface $t = 0$, it follows that $p_j = \partial\varphi/\partial x$ and that φ is a solution of (1.11) and (1.12). Then $\varphi(x)$ is constant along each trajectory. Hence the zeros of $\varphi(x)$ are exactly the union of all trajectories of (1.13) passing through elements $(0, y', \text{grad}\,\psi(0, y'))$ with $\psi(0, y') = 0$.

On the other hand, suppose for the sake of simplicity that $\partial\psi(0)/\partial x_1 \neq 0$. Then the equation $\psi(x) = 0$ of S can be solved with respect to x_1, so that we have $x_1 = \chi(x')$. It is easy to see that χ satisfies the equation

$$1 + \lambda(\chi, x', \partial\chi/\partial x') = 0 . \tag{1.15}$$

Since the "momenta" q_j associated with this equation may be written $-p_j/p_1$, it is easily shown that every trajectory of (1.13) passing through an element on S satisfies the canonical equations of (1.15):

$$(1.16)\quad \begin{cases} \dfrac{dx_j}{dt} = -\dfrac{\partial\lambda(\chi, x', q')}{\partial \xi_j'}, \quad j = 2, 3, \ldots, n ; \\[2ex] \dfrac{dq_j}{dt} = \dfrac{\partial\lambda(\chi, x', q')}{\partial x_j'} + q_j \dfrac{\partial\lambda(\chi, x', q')}{\partial x_1}, \quad j = 2, \ldots, n; \\[2ex] \dfrac{d\chi}{dt} = -\sum_{j=2}^{n} q_j \dfrac{\partial\lambda(\chi, x', q')}{\partial \xi_j} = -\lambda(\chi, x', q'). \end{cases}$$

with $\chi = x_1$.

Since S is covered by those trajectories, we find that both of the holomorphic functions φ and ψ have S as simple zeros. Hence $\mu(x) = \varphi(x)/\psi(x)$ is a holomorphic function on a neighborhood of S. By (1.12) we have $\mu(0, x') = 1$. Consequently, for each $\theta > 0$ we can find a complex neighborhood V of 0 on which φ is defined and such that

$$|\arg \varphi(x)| = |\arg \mu(x)| < \theta$$

on $V \cap \Omega$.

Remark. If $\lambda(x, p')$ is a real valued function, the characteristic function φ constructed above is also real valued. Hence it follows that S is a totally real characteristic surface.

A solution $(x(t), p(t))$ of the first two sets of equations in (1.13) such that $p_1 - \lambda(x, p') = 0$ is said to be a bicharacteristic strip of (the factor $\xi_1 - \lambda(x, \xi')$ of) the operator $P(x, \partial)$.

2. Construction of formal solutions. Let $A(x, \partial)$ be a linear partial differential operator of order m and let φ be a smooth function of n variables. Then, we can find by Leibniz' formula linear partial differential operators $A_{\varphi}^{j}(x, \partial)$, $j = 0, 1, \ldots, m$, of order $\leqq j$ such that for any smooth functions u and Φ of n variables and one variable, respectively, we have

$$(2.1) \qquad A(x, \partial)(u(x)\,\Phi(\varphi(x))) = \sum_{j=0}^{m} A_{\varphi}^{j}(x, \partial)\, u(x) \cdot \Phi^{(m-j)}(\varphi(x)) .$$

A simple calculation shows that

$$(2.2) \qquad A_{\varphi}^{0}(x, \partial) = a(x, \operatorname{grad} \varphi(x)) ;$$

$$(2.3) \qquad A_{\varphi}^{1}(x, \partial) = \sum_{k=1}^{n} \frac{\partial a(x, \operatorname{grad} \varphi)}{\partial \xi_k} \frac{\partial}{\partial x_k} + [A^{m-1}(x, \operatorname{grad} \varphi) + \frac{1}{2} \sum_{k,\ell=1}^{n} \frac{\partial^2 a(x, \operatorname{grad} \varphi)}{\partial \xi_k \partial \xi_\ell} \frac{\partial^2 \varphi}{\partial x_k \partial x_\ell}] ,$$

where $a(x, \partial)$ is the principal part and $A^{m-1}(x, \partial)$ is the homogeneous part of order $m-1$ of $A(x, \partial)$.

Proposition 2.1 (De Paris [3]). Let $B(x, \partial)$ and $C(x, \partial)$ be linear partial differential operators of orders m and n respectively. Then for the product

$$A(x, \partial) = B(x, \partial)\, C(x, \partial)$$

we have

$$(2.4) \qquad A_{\varphi}^{j}(x, \partial) = \sum_{k+\ell=j} B_{\varphi}^{k}(x, \partial)\, C_{\varphi}^{\ell}(x, \partial) .$$

Proof. We have

$$B(x, \partial)C(x, \partial)(u\,\Phi(\varphi))$$

$$= B(x, \partial)\Big(\sum_{\ell=0}^{n} C_\varphi^{\ell}(x, \partial)\, u \cdot \Phi^{(n-\ell)}(\varphi)\Big)$$

$$= \sum_{k=0}^{m} \Big(B_\varphi^{k}(x, \partial) \sum_{\ell=0}^{n} C_\varphi^{\ell}(x, \partial)u\Big) \Phi^{(m+n-k-\ell)}(\varphi).$$

Theorem 2.2. If φ is a regular characteristic function of multiplicity d of a linear partial differential operator $P(x, \partial)$, we have

$$P_\varphi^0(x, \partial) \equiv P_\varphi^1(x, \partial) \equiv \cdots \equiv P_\varphi^{d-1}(x, \partial) \equiv 0 \tag{2.5}$$

and $P_\varphi^d(x, \partial)$ is an ordinary differential operator of order d along the bicharacteristics on the characteristic surfaces $\varphi(x) =$ const.

Proof. Let $K(x, \xi)$ be the irreducible factor of $p(x, \xi)$ associated with $(x_0, \text{grad}\,\varphi(x_0)\infty)$. It is decomposed as (1.5). Hence we have by (2.2) and (2.3)

$$K_\varphi^0(x, \partial) = 0 \;; \tag{2.6}$$

$$K_\varphi^1(x, \partial) = b(x)\left(\frac{\partial}{\partial x_1} - \sum_{k=2}^{n} \frac{\partial \lambda}{\partial \xi_k}(x, \text{grad}\,\varphi(x)) \frac{\partial}{\partial x_k}\right) + c(x), \tag{2.7}$$

where $b(x)$ and $c(x)$ are holomorphic functions and $b(x)$ never vanishes on a neighborhood of x_0. In view of (1.13) we find that the differential operator in the parenthesis is exactly differentiation d/dt with respect to the parameter t along the bicharacteristics.

Let (1.7) be a decomposition of $P(x, \partial)$. We have by Proposition 2.1

$$P_\varphi^j(x, \partial) = \sum_{k=m-j}^{m} (Q_k(x, \partial)K(x, \partial)^{d_k})_\varphi^{j+k-m}$$

(2.8)

$$= \sum_{k=m-j}^{m} \sum_{\ell+|\mu|=j+k-m} Q_{k,\varphi}^{\ell}(x, \partial)K_\varphi^{\mu_1}(x,\partial) \cdots K_\varphi^{\mu_{d_k}}(x, \partial) .$$

Since $d_k \geq d - m + k$ and $K_\varphi^0(x, \partial) = 0$, we have (2.5) and

$$P_\varphi^d(x, \partial) = \sum_{d_k=d-m+k} Q_k(x, \text{grad } \varphi(x)) K_\varphi^1(x, \partial)^{d_k}.$$

By (2.7) this is an ordinary differential operator along the bicharacteristics. Since $Q_m(x_0, \text{grad } \varphi(x_0)) \neq 0$, the order is equal to d.

<u>Remark</u>. Suppose that the irregularity σ of the irreducible factor $K(x, \xi)$ is greater than 1. If we choose the least k with $(d - d_k)/(m - k) = \sigma$ and let $e = m - k + d_k$, then we have $e < d$ and

$$P_\varphi^e(x, \partial) = Q_k(x, \text{grad } \varphi(x))K_\varphi^1(x, \partial)^{d_k} + \text{lower order term.}$$

Here $Q_k(x, \xi) \not\equiv 0$ in the zeros of $K(x, \xi)$. Hence if we choose a suitable characteristic function φ we obtain a converse of Theorem 2.2. We note, however, that for a fixed φ (2.5) does not imply the regularity. For example, ((0,0,0); (0,1,1)) is an irregular characteristic element of multiplicity 2 of $(\partial_1^2 + \partial_2^2 - \partial_3^2)^2 + \partial_1^3$ but we have $P_\varphi^0 \equiv P_\varphi^1 \equiv 0$ for $\varphi(x) = x_2 + x_3$.

Now let $\Phi_j(s)$, $j \in \mathbb{Z}$, be a sequence of (generalized) functions of one variable satisfying

$$\frac{d\, \Phi_j(s)}{ds} = \Phi_{j-1}(s) . \tag{2.9}$$

We want to find a solution $U(x)$ of

$$P(x, \partial)U(x) = 0 \tag{2.10}$$

of the form

$$U(x) = \sum_{j=-\infty}^{\infty} u_j(x)\, \Phi_j(\varphi(x)). \tag{2.11}$$

First we consider the case where φ is a regular characteristic function of multiplicity d. In this case P. D. Lax [18], D. Ludwig [20], S. Mizohata [22] and C. De Paris [3], [4] constructed a formal solution (2.11) in the following way. Applying (2.1) formally, we have by Theorem 2.2

$$\begin{aligned} P(x, \partial) \sum_{j=0}^{\infty} u_j(x)\, \Phi_j(\varphi(x)) &= P_\varphi^d(x, \partial)\, u_0(x)\, \Phi_{-m+d}(\varphi(x)) \\ &+ [P_\varphi^d(x, \partial)u_1(x) + P_\varphi^{d+1}(x, \partial)u_0(x)]\, \Phi_{-m+d+1}(\varphi(x)) \\ &+ \cdots\cdot \\ &+ [P_\varphi^d(x, \partial)\, u_j(x) + P_\varphi^{d+1}(x, \partial)u_{j-1}(x) \\ &\qquad + \cdots + P_\varphi^m(x, \partial)u_{j-m+d}]\, \Phi_{j-m+d}(\varphi(x)) \\ &+ \cdots . \end{aligned}$$

Hence $U(x)$ is a formal solution of (2.1) if $u_j(x)$ satisfy the following equations:

$$\left\{ \begin{aligned} & P_\varphi^d(x, \partial)\, u_0(x) = 0 , \\ & P_\varphi^d(x, \partial)u_1(x) = - P_\varphi^{d+1}(x, \partial)u_0(x) , \\ & \cdots\cdot \\ & P_\varphi^d(x, \partial)\, u_j(x) = - P_\varphi^{d+1}(x, \partial)\, u_{j-1}(x) - \cdots \\ & \qquad\qquad - P_\varphi^m(x, \partial)u_{j-m+d}(x). \end{aligned} \right. \tag{2.12}$$

Since $P_\varphi^d(x, \partial)$ is an ordinary differential operator of order d with non-degenerate principal part, these equations have certainly holomorphic solutions $u_j(x)$ in a certain neighborhood V of x_0 independent of j. We may also impose d initial conditions. We adopt the following initial conditions

$$(2.13)\qquad \begin{cases} u_0(0, x') = 1, \\ \partial_1^k u_0(0, x') = 0, \quad k = 1, \ldots, d-1, \\ \partial_1^k u_j(0, x') = 0, \quad j = 1, 2, \ldots, \quad k = 0, 1, \ldots, d-1. \end{cases}$$

We have therefore the following theorem.

Theorem 2.3. If φ is a regular characteristic function of of $P(x, \partial)$ on a neighborhood of x_0, then there is a formal solution

$$(2.14)\qquad U(x) = \sum_{j=0}^{\infty} u_j(x)\Phi_j(\varphi(x))$$

of (2.10) with holomorphic coefficients $u_j(x)$ on a neighborhood of x_0 which are independent of the sequence of functions $\Phi_j(s)$ satisfying (2.9).

We note that the same method applies to the equation

$$(2.15)\qquad P(x, \partial)U(x) = V(x),$$

where $V(x)$ has the formal expansion

$$(2.16)\qquad V(x) = \sum_{j=-m+d}^{\infty} v_j(x)\,\Phi_j(\varphi(x)).$$

In the case where φ is an irregular characteristic function we employ Hamada's method in [8]. Namely we write

$$(2.17)\qquad P(x, \partial) = p(x, \partial) - R(x, \partial).$$

Since φ is a regular characteristic function of $p(x, \partial)$, there is a formal solution $U^0(x)$ of

$$p(x, \partial)U^0(x) = 0 \tag{2.18}$$

of the form

$$U^0(x) = \sum_{j=0}^{\infty} u_j^0(x)\, \Phi_j(\varphi(x)) . \tag{2.19}$$

Let $c = \max\{k - d_k - m + d\}$. Then we can solve the equations

$$p(x, \partial)U^k(x) = R(x, \partial)U^{k-1}(x) \tag{2.20}$$

recursively in the form

$$U^k(x) = \sum_{j=-ck}^{\infty} u_j^k(x)\Phi_k(\varphi(x)) . \tag{2.21}$$

We will impose the following initial conditions on $u_j^k(x)$:

$$\begin{cases} u_0^0(0, x') = 1 ; \\ \partial_1^{\ell} u_0^0(0, x') = 0 , \quad \ell = 1, 2, \ldots, d\text{-}1 ; \\ \partial_1^{\ell} u_j^k(0, x') = 0 , \quad (k, j) \neq (0, 0), \quad \ell = 0, 1, \ldots, d\text{-}1. \end{cases} \tag{2.22}$$

As we will prove in the next section, the coefficients of the formal sum

$$U(x) = \sum_{k=0}^{\infty} U^k(x)$$

converge and this gives a formal solution of (2.10). We remark again that the coefficients $u_j^k(x)$ do not depend on the sequence of functions $\Phi_j(s)$.

3. Estimates of the coefficients of formal solutions. To estimate the coefficients $u_j(x)$ in (2.14) and $u_j^k(x)$ in (2.21), we employ the majorants of C. Wagschal [27] and Y. Hamada [8]. Our estimates are no more than a slight improvement of Hamada's. However, since the details of [8] have not been published yet, we give all the proofs for the sake of the reader's convenience.

We always assume that $0 < r < R' < R$. When $a(x)$ and $b(x)$ are formal power series, $a(x) \ll b(x)$ means that each Taylor coefficient of $b(x)$ bounds the absolute value of the corresponding coefficient of $a(x)$.

Proposition 3.1 (Wagschal, Hamada [8]). Let $\Theta(t)$ be a formal power series in one variable t such that $\Theta(t) \gg 0$ and

(3.1) $$(R' - t)\Theta(t) \gg 0 .$$

Then for the derivatives $\Theta^{(j)}(t) = (d/dt)^j \Theta(t)$, $j = 0, 1, 2, \ldots,$ we have

(3.2) $$\Theta^{(j)}(t) \ll R'\Theta^{(j+1)}(t) ,$$

(3.3) $$\frac{1}{R-t}\Theta^{(j)}(t) \ll \frac{1}{R-R'}\Theta^{(j)}(t) .$$

Proof. Differentiating (3.1), we have

$$0 \ll \Theta(t) \ll (R' - t)\Theta'(t) \ll R'\Theta'(t) .$$

This shows that (3.2) holds for $j = 0$ and that $\Theta'(t)$ satisfies the same assumptions as $\Theta(t)$. Hence we have (3.2) for any j.

(3.3) for $j = 0$ follows from

$$\frac{1}{R-R'}\Theta(t) - \frac{1}{R-t}\Theta(t) = \frac{(R'-t)\Theta(t)}{(R-R')(R-t)} \gg 0 .$$

From now on we put

$$t = \rho x_1 + x_2 + \cdots + x_n \tag{3.4}$$

with a constant $\rho \geq 1$ to be determined later and assume that $\Theta(t)$ satisfies conditions of Proposition 3.1.

Proposition 3.2 (Wagschal [27]). Let

$$B(x, \partial) = \sum_{\substack{\alpha_1 \leq m_1 \\ |\alpha| \leq m}} b_\alpha(x) \partial^\alpha \tag{3.5}$$

be a linear partial differential operator with coefficients $b_\alpha(x)$ holomorphic on a neighborhood of the polydisk $\{x \in \mathbb{C}^n;\ |x_i| \leq R\}$. Then there is a constant B independent of $\Theta(t)$ and $\rho \geq 1$ such that if

$$u(x) \ll \Theta^{(j)}(t) , \tag{3.6}$$

then

$$B(x, \partial)u(x) \ll B \rho^{m_1} \Theta^{(j+m)}(t) . \tag{3.7}$$

Proof. Let $\alpha = (\alpha_1, \alpha')$ with $\alpha_1 \leq m_1$ and $|\alpha| \leq m$. We have by (3.4) and (3.2)

$$\partial^\alpha u(x) \ll \partial^\alpha \Theta^{(j)}(t) = \rho^{\alpha_1} \Theta^{(j+|\alpha|)}(t)$$

$$\ll \rho^{m_1} (R')^{m-|\alpha|} \Theta^{(j+m)}(t) .$$

Since there is a constant M such that $b_\alpha(x) \ll M(R - t)^{-1}$, we obtain by (3.3)

$$b_\alpha(x) \partial^\alpha u(x) \ll \rho^{m_1} (R')^{m-|\alpha|} M(R - R')^{-1} \Theta^{(j+m)}(t).$$

Thus it is sufficient to take

$$(3.8) \qquad B = \sum_{\substack{\alpha_1 = m_1 \\ |\alpha| \leq m}} (R')^{m-|\alpha|}(R - R')^{-1}M .$$

Proposition 3.3 (De Paris [4]). Let

$$(3.9) \qquad C(x, \partial) = \sum_{\substack{\alpha_1 < d \\ |\alpha| \leq d}} c_\alpha(x)\, \partial^\alpha$$

be a linear partial differential operator with coefficients $c_\alpha(x)$ holomorphic on a neighborhood of the polydisk $\{x \in \mathbb{C}^n;\ |x_i| \leq R\}$. Then there are constants $\rho \geq 1$ and B_1 independent of $\textcircled{H}(t)$ such that if

$$(3.10) \qquad v(x) \ll \textcircled{H}^{(j+d)}(t) ,$$

then the solution $u(x)$ of the initial value problem

$$(3.11) \qquad \begin{cases} \partial_1^d u(x) = C(x, \partial)u(x) + v(x) , \\ \partial_1^\ell u(0, x') = 0, \quad \ell = 0, 1, \dots, d-1 , \end{cases}$$

satisfies

$$(3.12) \qquad u(x) \ll B_1 \textcircled{H}^{(j)}(t) .$$

Proof. We choose a constant M so that $c_\alpha(x) \ll M(R - t)^{-1}$. Then it suffices to find constants ρ and B_1 such that

$$(3.13) \qquad \begin{aligned} &\partial_1^d (B_1 \textcircled{H}^{(j)}(t)) \\ &\gg M(R - t)^{-1} \sum_{\substack{\alpha_1 < d \\ |\alpha| \leq d}} \partial^\alpha (B_1 \textcircled{H}^{(j)}(t)) + \textcircled{H}^{(j+d)}(t) . \end{aligned}$$

For, we can then prove that

$$\partial_1^\ell u(0, x') \ll B_1\, \partial_1^\ell \textcircled{H}^{(j)}(t)$$

inductively by the initial conditions and the equations obtained from (3.11) by differentiation.

It follows from Propositions 3.1 and 3.2 that there is a constant $M_1 \geq 1$ such that the right hand side of (3.13) is majorized by $(M_1 B_1 \rho^{d-1} + 1)\Theta^{(j+d)}(t)$. Hence if we choose ρ and B_1 so that $\rho > M_1$ and $B_1 \geq \rho^{-d+1}(\rho - M_1)^{-1}$, we obtain (3.13).

The following power series were introduced by C. Wagschal [27] for $k \geq 0$ and by Y. Hamada [8] for $k < 0$:

$$(3.14) \qquad \theta^{(k)}(t) = \frac{k!}{(r-t)^{k+1}} = \sum_{j=0}^{\infty} \frac{(j+k)!}{j!} \frac{t^j}{r^{j+k+1}}, \quad k \geq 0;$$

$$(3.15) \qquad \theta^{(k)}(t) = \frac{1}{(-k-1)!} \int_0^t (t-s)^{-k-1} \theta^{(0)}(s)\, ds$$

$$= \sum_{j=-k}^{\infty} \frac{(j+k)!}{j!} \frac{t^j}{r^{j+k+1}}, \quad k < 0.$$

The notation is compatible with the fact that

$$(3.16) \qquad \frac{d^\ell}{dt^\ell} \theta^{(k)}(t) = \theta^{(k+\ell)}(t).$$

If $k \gtrless 0$, $\theta^{(k)}(t)$ satisfies the conditions of Proposition 3.1 and therefore may be used as $\Theta(t)$ in Propositions 3.2 and 3.3. Actually Wagschal and De Paris proved the propositions in that form. In the case where $k < 0$ we have the following important inequality.

<u>Proposition</u> 3.4 (<u>Hamada</u> [8]). <u>If</u> $R > 2r$ <u>and</u> $k < 0$, <u>then we have</u>

$$(3.17) \qquad \frac{1}{R-t} \theta^{(k)}(t) \ll \frac{2^{-k}}{R-2r} \theta^{(k)}(t).$$

<u>Proof.</u> To make the computation easy we write $-k$ instead of

k. Then

$$\frac{R}{R-t}\,\theta^{(-k)}(t) = \sum_{i=0}^{\infty}\frac{t^i}{R^i}\sum_{j=k}^{\infty}\frac{(j-k)!}{j!}\,\frac{t^j}{r^{j-k+1}}$$

$$= \sum_{\ell=k}^{\infty}\Big(\sum_{i=0}^{\ell-k}\frac{(\ell-i-k)!}{(\ell-i)!}\,\frac{r^i}{R^i}\Big)\frac{t^\ell}{r^{\ell-k+1}} .$$

Writing $\ell = k+h$, we have for $0 \leqq i \leqq h$

$$\frac{(\ell-k-i)!}{(\ell-i)!}\,\frac{\ell!}{(\ell-k)!} = (1+\frac{k}{h})(1+\frac{k}{h-1})\cdots(1+\frac{k}{h-i+1})$$

$$\leqq (1+\frac{k}{i})(1+\frac{k}{i-1})\cdots(1+\frac{k}{1})$$

$$= \binom{i+k}{i} \leqq 2^{i+k} .$$

Hence

$$\frac{R}{R-t}\,\theta^{(-k)}(t) \ll 2^k\sum_{\ell=k}^{\infty}\sum_{i=0}^{\infty}\Big(\frac{2r}{R}\Big)^i\,\frac{(\ell-k)!}{\ell!}\,\frac{t^\ell}{r^{\ell-k+1}}$$

$$= \frac{2^k R}{R-2r}\,\theta^{(-k)}(t) .$$

Since $\theta^{(k)}(t)$ itself does not fulfill the conditions of Proposition 3.1, we employ

$$\Theta_k(t) = \frac{R'}{R'-t}\,\theta^{(k)}(t) \tag{3.18}$$

instead according to Hamada.

We note the following facts :

<u>Proposition 3.5.</u> (a) <u>If</u> $k < \ell$,

$$\Theta_\ell^{(j)}(t) \ll \Theta_k^{(j-k+\ell)}(t) ; \tag{3.19}$$

(b) <u>If</u> $k \geqq 0$,

$$\theta^{(j+k)}(t) \ll \Theta_k^{(j)}(t) \ll \frac{R'}{R'-r}\,\theta^{(j+k)}(t) ; \tag{3.20}$$

(c) <u>If</u> $k < 0$ <u>and</u> $R' > 2r$,

$$\theta^{(j+k)}(t) \ll \Theta_k^{(j)}(t) \ll \frac{2^{-k}R'}{R'-2r}\,\theta^{(j+k)}(t). \tag{3.21}$$

Now let $P(x, \partial) = p(x, \partial) - R(x, \partial)$, $\varphi(x)$ and $\Phi_j(s)$ be as in §2. We choose $R > 0$ so small that the coefficients of $P(x, \partial)$ and $p_\varphi^i(x, \partial)$ and the inverse of the coefficient of ∂_1^d in $p_\varphi^d(x, \partial)$ are holomorphic on a neighborhood of the polydisk $\{x \in \mathbf{C}^n;\ |x_i| \leq R\}$. Except for the first stage the coefficients of the formal solution of (2.10) are obtained by solving the equation

$$p(x, \partial) \sum_{j=c}^{\infty} u_j(x)\,\Phi_j(\varphi(x)) = \sum_{j=d-m+c}^{\infty} v_j(x)\,\Phi_j(\varphi(x)) \tag{3.22}$$

under the initial conditions

$$\partial_1^{\ell} u_j(0, x') = 0, \qquad 0 \leq \ell < d. \tag{3.23}$$

<u>Proposition 3.6</u>. <u>There are positive constants</u> $\rho \geq 1$, C_0 <u>and</u> C_1 <u>independent of</u> $\Theta(t)$ <u>such that if</u>

$$v_j(x) \ll C^{j-d+m-c}\,\Theta^{(j+m-c)}(t), \quad j \geq d-m+c, \tag{3.24}$$

<u>with a constant</u> $C \geq C_0$, <u>then the solutions</u> $u_j(x)$ <u>of</u> (3.22) <u>and</u> (3.23) <u>satisfy</u>

$$u_j(x) \ll C_1 C^{j-c}\,\Theta^{(j-c)}(t), \quad j \geq c. \tag{3.25}$$

<u>Proof</u>. Without loss of generality we may assume that $c = 0$. As we have seen in §2, $u_j(x)$ are solutions of the equations

$$\begin{cases} p_{\varphi}^{d}(x,\partial)\,u_0(x) = v_{d-m}(x) \\ p_{\varphi}^{d}(x,\partial)\,u_1(x) = -p_{\varphi}^{d+1}(x,\partial)\,u_0(x) + v_{d-m+1}(x) \\ \cdots\cdot \\ p_{\varphi}^{d}(x,\partial)\,u_j(x) = -p_{\varphi}^{d+1}(x,\partial)\,u_{j-1}(x) - \cdots \\ \qquad\qquad \cdots - p_{\varphi}^{m}(x,\partial)\,u_{d-m+j}(x) + v_{d-m+j}(x) \end{cases}$$

under the initial conditions (3.23).

It follows from Proposition 3.3 that there are constants $\rho \geqq 1$ and B_1 independent of $\Theta(t)$ such that

$$u_0(x) \ll B_1\,\Theta^{(0)}(t)\ .$$

Hence (3.25) holds for $j = 0$ if $C_1 \geqq B_1$.

Suppose that (3.25) holds for $u_0(x), \ldots, u_{j-1}(x)$. Then we have by Proposition 3.2

$$-p_{\varphi}^{d+1}(x,\partial)\,u_{j-1}(x) - \cdots - p_{\varphi}^{m}(x,\partial)\,u_{d-m+j}(x) + v_{d-m+j}(x)$$
$$\ll B\rho^{d+1}C_1C^{j-1}\Theta^{(j+d)}(t) + \cdots + B\rho^{m}C_1C^{d-m+j}\Theta^{(j+d)}(t) + C^{j}\Theta^{(j+d)}(t)$$

Consequently we have

$$u_j(x) \ll B_1(B\rho^{m}C_1C^{j-1}(1-C^{-1})^{-1} + C^{j})\,\Theta^{(j)}(t)\ .$$

Let $C_1 = 2B_1$ and $C_0 = \max\{2B\rho^{m}C_1,\ 2\}$. Then (3.25) holds for $u_j(x)$. This completes the proof.

We write the decomposition (1.7) of $P(x,\partial)$ as

$$P(x,\partial) = p(x,\partial) - R^{m-1}(x,\partial) - \cdots - R^{0}(x,\partial), \tag{3.26}$$

where

$$R^{i}(x,\partial) = -Q_i(x,\partial)\,K(x,\partial)^{d_i} \tag{3.27}$$

is of order i or identically zero. To make the computation easy, we multiply $Q_i(x,\partial)$ by $K(x,\partial)^{d_i - d}$ if necessary and assume that $d_i \leq d$ for all i.

First we consider the solution $U^0(x) = \sum_{j=0}^{\infty} u_j^0(x)\Phi_j(\varphi(x))$ of (2.18). It follows from the Cauchy - Kowalevsky theorem that if r is sufficiently small, there is a holomorphic solution $u_0^0(x)$ satisfying

$$u_0^0 \ll A\Theta_0^{(0)}(t) ,$$

where we choose R' so that $2r < R' < R$. Then we can prove in the same was as above that

$$u_j^0(x) \ll A\,C^j\Theta_0^{(j)}(t) . \tag{3.28}$$

Next we consider the solution $U^1(x)$ of

$$p(x, \partial)U^1(x) = R(x, \partial)U^0(x). \tag{3.29}$$

The right hand side may be written

$$R(x, \partial)U^0(x) = \sum_{i=0}^{m-1} R^i(x, \partial)U^0(x)$$

$$= \sum_{i=0}^{m-1} \sum_{j=-i+d_i}^{\infty} v_{i,j}^1(x)\Phi_j(\varphi(x)) ,$$

where

$$v_{i,j}^1 = R_{\varphi}^{i,i}(x, \partial)u_j^0(x) + \cdots + R_{\varphi}^{i,d_i}(x, \partial)u_{j+i-d_i}^0(x).$$

We have by Proposition 3.2

$$v_{i,j}^1(x) \ll BAC^j\Theta_0^{(i+j)}(t) + \cdots + BAC^{j+i-d_i}\Theta_0^{(i+j)}(t)$$

$$\ll B'AC^{j+i-d_i}\Theta_0^{(i+j)}(t), \quad j \geq -i+d_i ,$$

with a constant B'. Hence it follows from Proposition 3.6 that the solutions $u_{i,j}^1(x)$ of

$$p(x, \partial) \sum_{\substack{j=-i+d_i\\ +m-d}}^{\infty} u_{i,j}^1(x)\Phi(\varphi(x)) = \sum_{j=-i+d_i}^{\infty} v_{i,j}^1(x)\Phi_j(\varphi(x))$$

under the initial conditions

$$\partial_1^{\ell} u^1_{i,j}(0, x') = 0 \quad \text{for all} \quad j \quad \text{and} \quad 0 \leqq \ell < d$$

are estimated as

$$(30) \qquad u^1_{i,j}(x) \ll C_1 B' A C^{j+i-d_i-m+d} \mathcal{H}^{(j+i-d_i-m+d)}_{d_i-d}(t)$$
$$= A C_2 C^{j+i-d_i-m+d} \mathcal{H}^{(j+i-d_i-m+d)}_{d_i-d}(t) .$$

Here we employed Proposition 3.5 (a).

For a sequence $I = (i_1, \ldots, i_k)$ with $0 \leqq i_\ell \leqq m-1$ we define $u^k_{I,j}(x)$ as the coefficients of the solution of

$$p(x, \partial) \sum u^k_{I,j}(x) \Phi_j(\varphi(x))$$
$$= R^{i_k}(x, \partial) \sum u^{k-1}_{I',j}(x) \Phi_j(\varphi(x))$$

with the zero initial conditions, where $I' = (i_1, \ldots, i_{k-1})$. Then we can prove by induction that

$$u^k_{i,j}(x) \ll A C_2^k C^{j+|I|-|d_I|-k(m-d)} \mathcal{H}^{(j+|I|-|d_I|-k(m-d))}_{|d_I|-kd}(t) ,$$

where $|I| = i_1 + \cdots + i_k$ and $|d_I| = d_{i_1} + \cdots + d_{i_k}$. Thus it follows from Proposition 3.5 (c) that

$$(3.31) \qquad u^k_{I,j}(x) \ll A C_3^k C^{j+|I|-|d_I|-k(m-d)} \theta^{(j+|I|-km)}(t)$$

for a constant C_3.

<u>Theorem 3.7</u>. <u>Let</u> $\varphi(x)$ <u>be a non-singular characteristic function of a linear partial differential operator</u> $P(x, \partial)$ <u>and let</u> d <u>and</u> σ <u>be the multiplicity and the irregularity of</u> $\varphi(x)$.

For each x_0 we can find a sequence of holomorphic functions $u_j(x)$ defined on a common neighborhood V of x_0 such that

$$(3.32)\qquad |u_j(x)| \leqq C^{j+1}\, j! \quad \text{for } j \geqq 0,$$

$$(3.33)\qquad |u_j(x)| \begin{cases} \leqq C^{-j+1}\left(|x - x_0|^{-j}/(-j)!\right)^{\sigma/(\sigma-1)}, & \sigma > 1, \\ = 0, \quad \sigma = 1, & \text{for } j < 0, \end{cases}$$

on V with a constant $C > 0$ and that

$$(3.34)\qquad U(x) = \sum_{j=-\infty}^{\infty} u_j(x)\Phi_j(\varphi(x))$$

is a formal solution of $P(x, \partial)U(x) = 0$ for any sequence of functions $\Phi_j(s)$ satisfying (2.9). Under a suitable coordinate system $u_j(x)$ are so chosen that they satisfy the initial conditions:

$$(3.35)\qquad \begin{cases} u_0(x_{0,1}, x') = 1, \\ \partial_1^{\ell} u_j(x_{0,1}, x') = 0, \quad (\ell, j) \neq (0, 0),\ 0 \leqq \ell < d, \end{cases}$$

as functions of x'.

Proof. First we note that

$$(3.36)\qquad d - d_i \leqq \sigma(m - i), \quad i = 0, 1, \ldots, m-1.$$

We use the notation $[a]$ to denote the greatest integer less than or equal to a.

We have to prove that

$$(3.37)\qquad u_j(x) = \sum u^k_{I,j}(x)$$

converge and satisfy (3.32) or (3.33). Here the summation ranges over all sequences $I = (i_1, \ldots, i_k)$ of $0 \leqq i_\ell \leqq m-1$.

We consider all terms with a fixed $|I| - km = -p$. From the proof of (3.31) it follows that $u^k_{I,j}(x)$ vanishes unless $j+|I|-|d_I|-k(m-d) \geqq 0$. Therefore the terms with a fixed p contribute to the sum (3.37) only if

$$j \geqq p+|d_I| - kd = p + \sum_{\ell=1}^{k} (d_{i_\ell} - d)$$

$$\geqq p + \sigma \sum_{\ell=1}^{k} (i_\ell - m) = (1-\sigma)p .$$

In particular, we have $u_j(x) = 0$ for $j < 0$ if $\sigma = 1$. Since j is an integer, we have actually the inequality

$$j \geqq -[(\sigma - 1)p]$$

and for such a term $u^k_{I,j}(x)$ we have the majorant

$$u^k_{I,j}(x) \ll A\, C_3^k\, C^{j+[(\sigma-1)p]}\, \theta^{(j-p)}(t) .$$

If we consider all sequences $I = (i_1, \ldots, i_k)$ of $-\infty \leq i_\ell \leq m-1$, then

$$\sum_{km-|I|=p} C_3^k = \sum_{k=1}^{p} \binom{p-1}{k-1} C_3^k \leq (C_3 + 1)^p .$$

Consequently we obtain

$$(3.38) \qquad u_j(x) \ll A \sum_{p=p(j)}^{\infty} (C_3 + 1)^p\, C^{j+[(\sigma-1)p]}\, \theta^{(j-p)}(t) ,$$

where

$$p(j) = \begin{cases} 0 & \text{if } j \geqq 0 \text{ or } \sigma = 1 , \\ -[j/(\sigma - 1)] & \text{if } j < 0 \text{ and } \sigma > 1. \end{cases}$$

If $0 \leqq t < r/2$, we have the inequalities

$$(3.39) \qquad \theta^{(k)}(t) \leqq (2/r)^{k+1} k! , \qquad k \geqq 0 ,$$

$$(3.40) \qquad \theta^{(-k)}(t) \leqq 2r^{-1} t^k / k!, \qquad k > 0.$$

Let V be a neighborhood of $x_0 = 0$ on which $\rho|x_1|+|x_2|+ \cdots +|x_n| < r/2$. If $j \geqq 0$, then we have for $x \in V$

$$|u_j(x)| \leqq A\, C^j\Big(\sum_{p=0}^{j} C_4{}^p (2/r)^{j-p+1}(j-p)!$$

$$+ \sum_{p=j+1}^{\infty} C_4{}^p\, 2r^{-1} t^{p-j}/(p-j)!\Big)$$

$$\leqq A_1 C_5{}^j((j+1)! + e^{C_4 t}).$$

Since $j \leqq 2^j$, (3.32) holds on V for C sufficiently large.

If $\sigma > 1$ and $j < 0$, then we have for $x \in V$

$$|u_j(x)| \leqq A\, C^j \sum_{p=p(j)}^{\infty} C_6{}^p\, t^{p-j} / (p-j)!$$

$$\leqq A_1\, C^j\, C_6{}^{p(j)} e^{C_6 t}\, t^{p(j)-j}/(p(j)-j)!$$

where $p(j) = -[j/(\sigma-1)]$. Hence we obtain by Stirling's formula (3.33) for C sufficiently large.

4. Ultradifferentiable functions and ultradistributions.

Let M_p, $p = 0, 1, 2, \cdots$, be a sequence of positive numbers and let Ω be an open set in $\mathbb{R}^n$. An infinitely differentiable function $f(x)$ on Ω is said to be an ultradifferentiable function of class $\{M_p\}$ (resp. of class (M_p)) if for each compact set K in Ω there are positive constants C and h (resp. if for each compact set K in Ω and $h > 0$ there is a constant C) such that

(4.1) $$\sup_{x \in K} |\partial^\alpha f(x)| \le C h^{|\alpha|} M_{|\alpha|}, \quad |\alpha| = 0, 1, 2, \ldots .$$

$\mathcal{E}^{\{M_p\}}(\Omega)$ (resp. $\mathcal{E}^{(M_p)}(\Omega)$) denotes the space of all ultradifferentiable functions of class $\{M_p\}$ (resp. class (M_p)) on Ω.

We assume that the sequence M_p satisfies the following conditions:

(M.0) $$M_0 = 1;$$

(M.1) (logarithmic convexity)

$$M_p^2 \le M_{p-1} M_{p+1}, \quad p = 1, 2, \ldots ;$$

(M.2)' (stability under differentiation) There are constants A and H such that

(4.2) $$M_{p+1} \le A H^p M_p, \quad p = 0, 1, 2, \ldots. ;$$

(M.3)' (non-quasi-analyticity)

(4.3) $$\sum_{p=1}^{\infty} M_{p-1}/M_p < \infty .$$

We write $m_p = M_p / M_{p-1}$. By (M.1) and (M.3)' m_p is an increasing sequence of positive numbers satisfying

(4.4) $$\sum_{p=1}^{\infty} 1/m_p < \infty .$$

We consider also the following stronger conditions:

(M.2) There are constants A and H such that

$$(4.5) \qquad M_p \leqq AH^p \min_{0 \leqq q \leqq p} M_q M_{p-q}, \quad p = 0, 1, 2, \ldots .$$

(M.3) There is a constant A such that

$$(4.6) \qquad \sum_{q=p+1}^{\infty} \frac{M_{q-1}}{M_q} \leqq Ap \frac{M_p}{M_{p+1}}, \quad p = 1, 2, \ldots .$$

If $s > 1$, the Gevrey sequence

$$M_p = (p!)^s$$

satisfies conditions (M.0), (M.1), (M.2) and (M.3). In this case, we will write $\{s\}$ and (s) for $\{M_p\}$ and (M_p) respectively.

Lemma 4.1 (cf. Lemma 11.4 of [14]). We set for $\mathrm{Re}\, z < 0$,

$$(4.7) \qquad \Psi(z) = \frac{1}{2\pi i} \int_0^{\infty} (1 + \zeta)^{-2} \prod_{p=1}^{\infty} \left(1 + \frac{\zeta}{m_p}\right)^{-1} e^{z\zeta} d\zeta .$$

Then $\Psi(z)$ is a holomorphic function which can be continued analytically to the Riemann domain $\{z \,;\, -\pi/2 < \arg z < 5\pi/2\}$.

On the domain $\{z \,;\, 0 \leqq \arg z \leqq 2\pi\}$ we have the uniform estimates:

$$(4.8) \qquad \left| \frac{d^p}{dz^p} \Psi(z) \right| \leqq \frac{1}{4} M_p, \quad p = 0, 1, 2, \ldots .$$

In particular, the boundary value

$$(4.9) \qquad \psi(x) = \Psi(x + i0) - \Psi(x - i0)$$

is an ultradifferentiable function of class $\{M_p\}$ vanishing on the negative real axis.

We have, further,

$$(4.10) \qquad \psi(x) \geqq 0 \quad \text{and} \quad \int_0^{\infty} \psi(x)\,dx = 1.$$

When M_p satisfies (M.2) and (M.3), $\psi(x)$ is not an ultra-differentiable function of class (M_p) on any neighborhood of the origin.

Proof. Because of (4.4) $\prod(1 + \zeta/m_p)$ converges absolutely and represents an entire function in ζ. As we have shown in [14], we can find for each $\varepsilon > 0$ and $0 < \theta < \pi$ a constant C such that

$$\left|(1+\zeta)^{-1}\prod(1 + \zeta/m_p)^{-1}\right| \leqq Ce^{\varepsilon|\zeta|}, \quad |\arg \zeta| \leqq \theta .$$

Hence integral (4.7) converges absolutely for $\mathrm{Re}\, z < 0$. Rotating the path of integration into the ray from 0 to $\infty e^{i\alpha}$ for $-\pi < \alpha < \pi$, we obtain an analytic continuation to the Riemann domain $\{ z;\ -\pi/2 < \arg z < 5\pi/2 \}$.

If $\mathrm{Im}\, z > 0$, we can choose the positive imaginary axis as the path of integration. Since the integral may be differentiated under the integral sign, we have

$$\left|\frac{d^p}{dz^p}\Psi(z)\right| \leqq \frac{1}{2\pi}\int_0^\infty \left|(i\eta)^p\prod(1 + i\eta/m_p)^{-1}\right| |1 + i\eta|^{-2}d\eta$$

$$\leqq m_1 m_2 \cdots m_p \frac{1}{2\pi}\int_0^\infty (1+\eta^2)^{-1}d\eta$$

$$= M_p/4.$$

The proof is similar in the case where $\mathrm{Im}\, z < 0$. Since $d^p\Psi(z)/dz^p$ is continuous, we obtain (4.8) for the closed Riemann domain $\{z;\ 0 \leqq \arg z \leqq 2\pi\}$.

By the continuity we have also

$$\psi(x) = \frac{1}{2\pi}\int_{-\infty}^\infty (1 + i\eta)^{-2}\prod(1 + i\eta/m_p)^{-1}e^{ix\eta}d\eta .$$

Since each factor

$$1 + i\eta/m_p = m_p \int_0^\infty e^{-m_p x} e^{-ix\eta} dx$$

is a positive definite function, the product $(1 + i\eta)^{-2}\prod(1+i\eta/m_p)^{-1}$ is also a positive function in η. Hence its Fourier transform $\psi(x)$ is non-negative and the integral $\int \psi(x)dx$ coincides with the value 1 of $(1 + i\eta)^{-2}\prod(1 + i\eta/m_p)^{-1}$ at $\eta = 0$.

When M_p satisfies (M.2) and (M.3)

$$P(d/dz) = (1 + d/dz)^2 \prod_{p=1}^{\infty} (1 + m_p^{-1} d/dz)$$

is an ultradifferential operator of class (M_p) (see Proposition 4.6 of [14]). On the other hand, we have clearly

$$P(d/dz)\Psi(z) = \frac{-1}{2\pi i}\frac{1}{z}$$

and hence

$$P(d/dx)\psi(x) = \delta(x).$$

Thus $\psi(x)$ can not be an ultradifferentiable function of class (M_p).

<u>Lemma</u> 4.2. <u>Let</u> $s > 1$ <u>and set for</u> z <u>with</u> $0 < \arg z < 2\pi$

$$(4.11) \qquad \Psi(z) = \frac{-1}{2\pi i}\int_0^\infty \frac{1}{z-x}\, d \exp(-x^{-\frac{1}{s-1}}).$$

<u>Then</u> $\Psi(z)$ <u>is a holomorphic function which can be continued analytically to the Riemann domain</u> $\{z \neq 0 ; -\infty < \arg z < \infty\}$. <u>If</u> $0 \leqq \theta < (s-1)\pi/2$, <u>we can find constants</u> B <u>and</u> h <u>such that we have the uniform estimates:</u>

$$(4.12) \qquad \left|\frac{d^p}{dz^p}\Psi(z)\right| \leqq Bh^p(p!)^s$$

<u>on the Riemann domain</u> $\{z; -\theta \leqq \arg z \leqq 2\pi + \theta\}$.

<u>The boundary value</u> $\psi(x)$ <u>defined by</u> (4.9) <u>is an ultradifferentiable function of class</u> $\{s\}$ <u>but not of class</u> (s) <u>and satisfies</u> (4.10).

Proof. First we consider the function

$$F(z) = \exp\left(-z^{-\frac{1}{s-1}}\right)$$

on the domain $\{z \in \mathbb{C}_j ;\ |\arg z| < (s-1)\pi/2\}$ and prove that on each subdomain $\{z \in \mathbb{C} ;\ |\arg z| < \theta\}$ with $0 < \theta < (s-1)\pi/2$ there are constants B and h such that

$$(4.13) \qquad \left|\frac{d^p}{dz^p} F(z)\right| \leqq Bh^p (p!)^s.$$

Choose a sufficiently small positive number k so that the disk with center at z in the subdomain and of radius $k|z|$ is included in the sector $\{z ;\ |\arg z| \leqq (s-1)\theta_0\}$ for a $\theta_0 < \pi/2$. Then we have by Cauchy's integral formula

$$\left|\frac{d^p}{dz^p} F(z)\right| \leqq p!\ (k|z|)^{-p} \sup_{|w-z|\leqq k|z|} |F(w)|$$

$$\leqq p!\ (k|z|)^{-p} \exp\left(-((1+k)|z|)^{\frac{1}{s-1}} \cos\theta_0\right).$$

Hence the estimate (4.13) follows from the inequality

$$\sup_{0<t<\infty} t^{-p} \exp\left(-Lt^{-\frac{1}{s-1}}\right) = \left(\frac{s-1}{Le}\right)^{(s-1)p} p^{(s-1)p}$$

$$\leqq \left(\frac{s-1}{L}\right)^{(s-1)p} (p!)^{s-1}.$$

We have further

$$\int_0^\infty t^{-p} \exp\left(-Lt^{-\frac{1}{s-1}}\right)dt = \frac{s-1}{L^{(s-1)(p-1)}} \Gamma((s-1)(p-1)).$$

Consequently the function

$$\psi(x) = \begin{cases} dF(x)/dx, & x > 0, \\ 0, & x \leqq 0, \end{cases}$$

satisfies

$$\| d^p \psi(x)/dx^p \|_{L^2(\mathbb{R})} \leq Ck^p (p!)^s, \quad p = 0, 1, 2, \ldots,$$

for some constants C and k.

If $y \neq 0$, we have

$$\frac{d^p \Psi}{dz^p}(x + iy) = \frac{-1}{2\pi i} \int_{-\infty}^{\infty} \frac{x - t - iy}{(x - t)^2 + y^2} \frac{d^p \psi(t)}{dt^p} dt$$

and therefore we obtain

$$\left(\int_{-\infty}^{\infty} | \Psi^{(p)}(x + iy) |^2 dx \right)^{1/2} \leq C k^p (p!)^s, \quad p = 0, 1, 2, \ldots \quad .$$

Now (4.12) for $0 < \arg z < 2\pi$ follows from Sobolev's inequality and (M.2)'.

The integrand of (4.11) is a holomorphic function on the Riemann domain $\{z \,;\, -\infty < \arg z < \infty\}$. Hence we can continue Ψ to the Riemann domain by deforming the path of integration. Let $\Psi^+(z)$ (resp. $\Psi^-(z)$) be the branch on the domain $\{z \,;\, -\pi/2 < \arg z < 2\pi\}$ (resp. $\{z \,;\, 0 < \arg z < 5\pi/2\}$). Then we have for $-\pi/2 < \arg z < \pi/2$

$$\Psi^+(z) - \Psi^-(z) = \frac{-1}{2\pi i} \oint_{|w-z|=\varepsilon} \frac{1}{z - w} F'(w)\, dw$$

$$= F'(z). \tag{4.14}$$

In view of (4.13) we have therefore the estimates (4.12) on the domain $\{z \,;\, -\theta \leq \arg z \leq 2\pi + \theta\}$ for every $0 < \theta < (s - 1)\pi/2$.

Since $\exp(-x^{-\frac{1}{s-1}})$ increases from 0 to 1 as x varies from 0 to ∞, it is clear that $\Psi(x + i0) - \Psi(x - i0) = \psi(x)$ satisfies (4.10).

Lastly to prove that $\psi(x)$ is not in $\mathcal{E}^{(s)}(-\varepsilon, \varepsilon)$, we assume to the contrary that for each $h > 0$ there is a constant C such that

$$|\psi^{(p)}(x)| \leqq Ch^p\, p!^{\,s}, \quad x \in (-\varepsilon, \varepsilon).$$

Since

$$F(x) = \exp\left(-x^{-\frac{1}{s-1}}\right)$$
$$= \int_0^x \frac{(x-y)^p}{p!}\, \psi^{(p)}(y)\,dy, \quad x > 0,$$

we have

$$|F(x)| \leqq C \inf_p \left(\frac{x^{p+1}}{(p+1)!}\, h^p (p!)^s \right)$$
$$\leqq \frac{C}{h} \inf_p \left[(xh)^{\frac{p+1}{s-1}} (p+1)! \right]^{s-1}, \quad 0 < x < \varepsilon .$$

Let $p + 1 = \left[(xh)^{-\frac{1}{s-1}} \right]$. Then we have

$$|F(x)| \leqq \frac{C}{h} (2\pi)^{\frac{s-1}{2}} \left[(xh)^{-\frac{1}{s-1}} \right]^{\frac{s-1}{2}} \exp\left(-(s-1) \left[(xh)^{-\frac{1}{s-1}} \right] \right)$$

for sufficiently small $x > 0$. This is impossible, however, if $h < (s-1)^{s-1}$.

The vector space $\mathcal{D}^{\{M_p\}}(\Omega)$ (resp. $\mathcal{D}^{(M_p)}(\Omega)$) of all ultradifferentiable functions of class $\{M_p\}$ (resp. of class (M_p)) and with compact supports has a natural locally convex topology. The elements in the dual space $\mathcal{D}^{\{M_p\}'}(\Omega)$ (resp. $\mathcal{D}^{(M_p)'}(\Omega)$) are called <u>ultradistributions of class</u> $\{M_p\}$ (resp. <u>of class</u> (M_p)). (For the theory of ultradistributions see Roumieu [24], Björck [0] and Komatsu [14].) The space $\mathcal{E}^{\{M_p\}}(\Omega)$ (resp. $\mathcal{E}^{(M_p)}(\Omega)$) has also a natural locally convex topology and the dual $\mathcal{E}^{\{M_p\}'}(\Omega)$ (resp. $\mathcal{E}^{(M_p)'}(\Omega)$) is identified with the space of all ultradistributions of class $\{M_p\}$ (resp. of class (M_p)) and with compact supports in Ω.

<u>Lemma 4.3</u>. <u>Suppose that</u> M_p <u>satisfies</u> (M, 0), (M, 1), (M, 2) <u>and</u> (M, 3). <u>Then</u>

$$\Psi(z) = \frac{-1}{2\pi i} \prod_{p=1}^{\infty} \left(1 + \frac{\partial/\partial z}{m_p}\right) \frac{1}{z} \tag{4.15}$$

<u>is a holomorphic function on the punctured plane</u> $\{z \in \mathbb{C};\ z \neq 0\}$ <u>and the boundary value</u>

$$\psi(x) = \Psi(x + i0) - \Psi(x - i0) \tag{4.16}$$

<u>in the sense of hyperfunction is an ultradistribution of class</u> (M_p) <u>which is not of class</u> $\{M_p\}$.

<u>Proof</u>. By Proposition 4.6 of [14]

$$Q(\partial) = \prod_{p=1}^{\infty} \left(1 + \frac{\partial/\partial z}{m_p}\right)$$

is an ultradifferential operator of class (M_p). Hence it follows from Lemma 11.3 of [14] that $\Psi(z)$ is holomorphic on $\{z \in \mathbb{C};\ z \neq 0\}$.

Since

$$\psi(x) = Q(\partial)\left(\frac{-1}{2\pi i}\frac{1}{x+i0} - \frac{-1}{2\pi i}\frac{1}{x-i0}\right)$$

$$= Q(\partial)\,\delta(x),$$

it is clearly an ultradistribution of class (M_p) with the support at the origin.

If it were an ultradistribution of class $\{M_p\}$, the convolution of $\psi(x)$ and the ultradifferentiable function $\phi(x)$ constructed in in Lemma 4.1 would be an ultradifferentiable function of class $\{M_p\}$ by Theorem 6.10 of [14]. However,

$$\begin{aligned}\psi(x) * \phi(x) &= Q(\partial)\,\delta(x) * \phi(x) \\ &= \delta(x) * Q(\partial)\,\phi(x) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{ix\eta}}{(1+i\eta)^2}\, d\eta\end{aligned}$$

is not differentiable at the origin.

5. Construction of null-solutions.

We are now able to prove our main theorems. We assume that the linear partial differential operator $P(x, \partial)$, the characteristic surface S and the sequence M_p of positive numbers satisfy the assumptions in the preceeding sections.

A solution $u(x)$ of

(5.1) $\quad P(x, \partial)u(x) = 0$

is said to be a null-solution on a neighborhood of x_0 in S if it is defined on a neighborhood of x_0, vanishes on one side of S and never vanishes on every neighborhood of x_0.

Theorem 5.1. Suppose that the characteristic surface S is regular and totally real. Then for each sequence M_p and point x_0 in S we can construct an ultradifferentiable null-solution $u(x)$ of class $\{M_p\}$ on a neighborhood of x_0. If M_p satisfies $(M, 2)$ and $(M, 3)$, then the null-solution $u(x)$ is not an ultradifferentiable function of class (M_p) on any neighborhood of x_0.

Proof. In this case we can choose the characteristic function $\varphi(x)$ as a local coordinate function. Since the classes $\{M_p\}$ and (M_p) are invariant under real analytic coordinate transformations by Théorème 13 of Roumieu [24], we may assume without loss of generality that $\varphi(x) = x_n$ and that $x_0 = 0$.

Employing $\psi(x)$ of Lemma 4.1, we define

$$(5.2) \qquad \Phi_j(s) = \begin{cases} \displaystyle\int_0^s \frac{(s-t)^{j-1}}{(j-1)!}\psi(t)dt, & j > 0, \\ \left(\dfrac{d}{ds}\right)^{-j}\psi(s), & j \leqq 0. \end{cases}$$

Clearly the sequence $\Phi_j(s)$ satisfies (2.9). Moreover, we have the estimates

$$(5.3) \qquad |\partial^\alpha(\Phi_j(\varphi(x)))| \leq \begin{cases} \dfrac{|\varphi(x)|^{j-|\alpha|}}{(j-|\alpha|)!}, & |\alpha| < j, \\ M_{|\alpha|-j}, & |\alpha| \geqq j. \end{cases}$$

In fact, if $\alpha = (\alpha_1, \ldots, \alpha_n)$ contains a non-zero component other than α_n, the left hand side vanishes. If it is of the form $\alpha = (0, \ldots, 0, \alpha_n)$, then $\partial^\alpha(\Phi_j(\varphi(x))) = \Phi_{j-|\alpha|}(\varphi(x))$ satisfies the estimates because of (4.8).

We write $f_j(x) = \Phi_j(\varphi(x))$. Then

$$(5.4) \qquad \begin{aligned} &\partial^\alpha\Big(\sum_{j=0}^{\infty} u_j(x)\, \Phi_j(\varphi(x))\Big) \\ &= \sum_{j=0}^{\infty} \sum_{\substack{0 \leqq \beta \leqq \alpha \\ |\beta|+k=j}} \binom{\alpha}{\beta} \partial^\beta u_k(x)\, \partial^{\alpha-\beta} f_k(x). \end{aligned}$$

Suppose that the polydisk of radius δ and with center at x is included in the neighborhood V of x_0 of Theorem 3.7. Then it follows from (3.32) and Cauchy's inequality that

$$(5.5) \qquad |\partial^\beta u_k(x)| \leqq C^{k+1}\, k!\, \delta^{-|\beta|} |\beta|! \quad .$$

If $\delta > 0$ is sufficiently small, those x form a neighborhood of x_0. We also note that

$$(5.6) \qquad \sum_{|\beta|=\ell} \binom{\alpha}{\beta} \leqq \binom{|\alpha|}{\ell} \quad .$$

In case $0 \leqq j \leqq |\alpha|$, we have by (5.3) and (5.5)

$$(5.7) \qquad \sum_{\substack{0 \leqq \beta \leqq \alpha \\ |\beta|+k=j}} \left|\binom{\alpha}{\beta} \partial^\beta u_k(x)\, \partial^{\alpha-\beta} f_k(x)\right|$$

$$\leqq \sum_{k=0}^{j} \binom{|\alpha|}{j-k} C^{k+1} k!\, \delta^{-j+k} (j-k)!\, M_{|\alpha|-j}$$

$$= C\, |\alpha|!\, \delta^{-j} M_{|\alpha|-j} \sum_{k=0}^{j} \frac{k!}{(|\alpha|-j+k)!} (C\delta)^k \quad .$$

By (7.11) of [14] we can find for each $H>0$ a constant A such that

(5.8) $$j! \leqq AH^j M_j \quad .$$

(M, 0) and (M, 1) imply

$$M_j M_{|\alpha|-j} \leqq M_{|\alpha|} \quad .$$

Hence if $\delta < C^{-1}$, the right hand side of (5.7) is bounded by

$$\frac{AC}{1-C\delta} \binom{|\alpha|}{j} \left(\frac{H}{\delta}\right)^j M_{|\alpha|} \quad .$$

Consequently

(5.9) $$\sum_{j=0}^{|\alpha|} \sum \left| \binom{\alpha}{\beta} \partial^\beta u_k(x)\, \partial^{\alpha-\beta} f_k(x) \right|$$

$$\leqq \frac{AC}{1-C\delta} \left(1+\frac{H}{\delta}\right)^{|\alpha|} M_{|\alpha|} \quad .$$

In case $j > |\alpha|$, we write $i = j - |\alpha|$.

Then we have

(5.10) $$\sum_{\substack{0\leqq\beta\leqq\alpha \\ |\beta|+k=j}} \left| \binom{\alpha}{\beta} \partial^\beta u_k(x)\, \partial^{\alpha-\beta} f_k(x) \right|$$

$$\leqq \sum_{\ell=0}^{|\alpha|} \binom{|\alpha|}{|\alpha|-\ell} C^{i+\ell+1} (i+\ell)!\, \delta^{-|\alpha|+\ell} (|\alpha|-\ell)!\, \frac{|\varphi(x)|^i}{i!}$$

$$\leqq C^{i+1} (C+\delta^{-1})^{|\alpha|} (|\alpha|+i)!\, \frac{|\varphi(x)|^i}{i!} \quad .$$

Hence if $|\varphi(x)| < r_0 < (2C)^{-1}$, we have

(5.11) $$\sum_{j=|\alpha|+1}^{\infty} \sum \left| \binom{\alpha}{\beta} \partial^\beta u_k(x)\, \partial^{\alpha-\beta} f_k(x) \right|$$

$$\leq C(C+\delta^{-1})^{|\alpha|}|\alpha|!\sum_{i=1}^{\infty}(C r_0)^i \, 2^{|\alpha|+i}$$

$$\leq \frac{C(2(C+\delta^{-1}))^{|\alpha|}|\alpha|!}{1-2Cr_0} .$$

Combined with (5.8) and (5.9), this proves that if r_0 is sufficiently small, then (5.4) is majorized on the domain $\Omega_0 = \{x \in \Omega ; |x| < r_0\}$ by a sequence whose sum does not exceed $C_1^{|\alpha|+1} M_{|\alpha|}$ for a constant C_1. Therefore

(5.12) $$u(x) = \sum_{j=0}^{\infty} u_j(x)\, \Phi_j(\varphi(x))$$

converges in the topology of $\mathcal{E}^{\{M_p\}}(\Omega_0)$ (see [14]). Then it is clear that $u(x)$ is a solution of (5.1) on Ω_0 and that $u(x)$ vanishes when $\varphi(x) < 0$.

Taking into account the initial conditions (3.35) of $u_j(x)$, we see that

$$u(0, x') = \Phi_0(\varphi(0, x')) = \psi(x_n)$$

It follows from Lemma 4.1 that the right hand side does not vanish when $\varphi(x) = x_n > 0$. In particular, $x_0 = 0$ belongs to the support of $u(x)$.

If M_p satisfies (M, 2) and (M, 3), $\psi(x_n)$ is not an ultradifferentiable function of class (M_p) on any neighborbood of 0. Hence $u(x)$ is not either.

Theorem 5.2. Suppose that S is a real analytic characteristic surface of irregularity $\sigma > 1$. Then for each $1 < s \leq \sigma/(\sigma - 1)$ and point x_0 in S we can construct a null-solution $u(x)$ on a neighborhood Ω_0 of x_0 which is an ultradifferentiable function of Gevrey class $\{s\}$ on Ω_0 but not of class (s) on any neighborhood of x_0.

Proof. Let $\varphi(x)$ be the holomorphic characteristic function onstructed in Proposition 1.5, where we choose a $\theta > 0$ smaller han $(s - 1)\pi/2$.

Then we define the sequence $\Phi_j(z)$ of holomorphic functions on he Riemann domain $\Sigma_\theta = \{z \neq 0;\ -\theta < \arg \Phi_j(z) < 2\pi + \theta\}$ by

$$(5.13) \qquad \Phi_j(z) = \begin{cases} \displaystyle\int_0^z \frac{(z - w)^{j-1}}{(j - 1)!} \Psi(w)\,dw, & j > 0 \\ \left(\frac{d}{dz}\right)^{-j} \Psi(z), & j \leq 0, \end{cases}$$

here $\Psi(z)$ is the function constructed in Lemma 4.2. We denote by $\Phi_j^+(z)$ (resp. $\Phi_j^-(z)$) the branch of $\Phi_j(z)$ continued from the ipper half plane (resp. the lower half plane).

If we choose a sufficiently small complex neighborhood V of x_0, the functions

$$(5.14) \qquad F_j(x) = \Phi_j(\varphi(x))$$

are defined and holomorphic on (a covering space of) $V \setminus S$. To distinguish two branches of $F_j(x)$ we will also use the notations

$$(5.14)' \qquad F_j^{\pm}(x) = \Phi_j^{\pm}(\varphi(x)).$$

Firstly we have to prove that the derivatives of F_j are estimated as follows: For each $r_0 > 0$ there are constants B and L such that Lr_0 is bounded as $r_0 \to 0$ and that if $z \in V \setminus S$ and $|\varphi(x)| < r_0$ then

$$(5.15) \qquad |\partial^\alpha F_j(x)| \leq \begin{cases} \dfrac{B\, L^{|\alpha|} r_0^{\,j}}{(j - |\alpha|)!}, & |\alpha| < j, \\ B\, L^{|\alpha| + |j|} (|\alpha| - j)!^{\,s}, & |\alpha| \geq j. \end{cases}$$

Since we have (4.12) on the Riemann domain Σ_θ, (5.15) holds for $|\alpha| = 0$ when L is greater than h.

To prove it for $|\alpha| > 0$, we follow Roumieu's arguments in [24].

Let x and y be two points in V. If V is sufficiently small, we can find two constants R and M independent of x and y such that

$$\varphi(x) - \varphi(y) \ll \frac{M\,t}{R - t} ,$$

where

$$t = (x_1 - y_1) + \cdots + (x_n - y_n) .$$

Similarly we have

$$\Phi_j(z) - \Phi_j(w) \ll \frac{N_1}{1!} s + \frac{N_2}{2!} s^2 + \cdots + \frac{N_q}{q!} s^q + \cdots ,$$

where

$$s \doteq z - w$$

and

$$(5.16) \qquad N_q \leqq \begin{cases} B\, r_0^{j-q}/(j-q)! , & q < j , \\ B\, h^{q-j}(q-j)!^s , & q \geqq j . \end{cases}$$

Hence we have

$$(5.17) \quad \Phi_j(\varphi(x)) - \Phi_j(\varphi(y)) \ll \frac{N_1}{1!}\left(\frac{M\,t}{R-t}\right) + \cdots + \frac{N_q}{q!}\left(\frac{M\,t}{R-t}\right)^q + \cdots$$

$$= \sum_{p=0}^{\infty} K_p\, t^p/p! ,$$

where

$$K_p = \sum_{q=1}^{p} \frac{N_q}{q!} \frac{M^q}{R^p} \frac{(p-1)!\; p!}{(q-1)!\;(p-q)!} .$$

In case $p = |\alpha| > j \geqq 0$, we have therefore

$$|\partial^\alpha F_j(x)| \leqq \sum_{q=1}^{j} \frac{M^q}{R^p} \frac{p!}{q!} \frac{(p-1)!}{(q-1)!} \frac{1}{(p-q)!} B\, r_0^{j-q} \frac{1}{(j-q)!}$$

$$+ \sum_{q=j+1}^{p} \frac{M^q}{R^p} \frac{p!}{q!} \frac{(p-1)!}{(q-1)!} \frac{1}{(p-q)!} B h^{q-j} (q-j)!^s$$

$$\leqq B\,\frac{(p-j)!^s}{R^p}\left\{\sum_{q=1}^{j}\frac{p!}{q!}\;\frac{(p-1)!}{(q-1)!}\;\frac{1}{(p-q)!^2}\left(\frac{M}{r_0}\right)^q r_0{}^j\frac{(p-q)!}{(j-q)!(p-j)!}\right.$$

$$\left.+\sum_{q=j+1}^{p}\frac{p!}{q!}\;\frac{(p-1)!}{(q-1)!}\;\frac{1}{(p-q)!^2}(M\,h)^q h^{-j}\,\frac{(p-q)!^s(q-j)!^s}{(p-j)!^s}\right\}$$

$$\leqq B\,\frac{(p-j)!^s}{R^p}\,2^p(\sqrt{K}+1)^{2p}\,K^j$$

when

$$K \geqq \max\left\{M/r_0,\;\; r_0,\; M\,h,\; h^{-1}\right\}.$$

The proofs in the other cases where $p = |\alpha| < j$ and $j \leqq 0$ are similar and easier.

Next we prove that

$$(5.18)\qquad \sum_{j=0}^{\infty} u_j(x)\;F_j(x)$$

and

$$(5.19)\qquad \sum_{j=-\infty}^{-1} u_j(x)\;F_j(x)$$

converge in $\mathcal{E}^{\{s\}}(\,V\setminus S\,)$.

The proof of convergence of (5.18) is similar to that of (5.12). In (5.7) $M_{|\alpha|-j}$ has to be replaced by $B\;L^{2k+|\alpha|-j}(|\alpha| - j\,)!^s$. Hence the right hand side of (5.9) becames

$$\frac{A\;B\;C\;L^{|\alpha|}}{1\,-\,C\,\delta\,L^2}\;\left(1+\frac{H}{L\,\delta}\right)^{|\alpha|}|\alpha|!^s$$

if δ is sufficiently small.

$|\varphi(x)|^i/i!$ in (5.10) has to be replaced by $B\;L^{\ell}r_0{}^{\ell+i}/i!$ Consequently if $|\varphi(x)| < r_0 < (\,2C\,)^{-1}$, (5.11) holds with the right hand side replaced by

$$\frac{B\,C\,(2\,(C\,L\,r_0 + \delta^{-1}))^{|\alpha|}\,|\alpha|!}{1 - 2\,C\,r_0}\ .$$

We now turn to the proof of the convergence of (5.19). We have

$$\partial^{\alpha}(\sum_{j=1}^{\infty} u_{-j}(x)\,F_{-j}(x))$$
$$=\sum_{j=1-|\alpha|}^{\infty}\ \sum_{\substack{0\leqq\beta\leqq\alpha\\ |\beta|-k=-j}} \binom{\alpha}{\beta}\,\partial^{\beta}u_{-k}(x)\,\partial^{\alpha-\beta}F_{-k}(x)\ .$$

It follows from Theorem 3.7 and (5.15) that if $|x| < r - \delta$, then

(5.20)
$$\sum_{\substack{0\leqq\beta\leqq\alpha\\ |\beta|-k=-j}} \left|\binom{\alpha}{\beta}\,\partial^{\beta}u_{-k}(x)\,\partial^{\alpha-\beta}F_{-k}(x)\right|$$
$$\leqq \sum_{\substack{\ell=0\\ \ell>-j}}^{|\alpha|} \binom{|\alpha|}{\ell} C^{j+\ell+1}\,r^{\tau(j+\ell)}(j+\ell)!^{-\tau}\,\delta^{-\ell}\,\ell!\,B L^{|\alpha|+j}(|\alpha|+j)!^{s},$$

where $\tau = \sigma/(\sigma - 1)$.

If $j \geqq 0$, we have $j!^{\tau}\ell!(j+\ell)!^{-\tau} \leqq 1$ and hence the right hand side is bounded by

$$B\,C^{j+1}L^{|\alpha|+j}\,r^{\tau j}(C\,r^{\tau}\delta^{-1} + 1)^{|\alpha|}(|\alpha| + j)!^{s} / j!^{\tau}\ .$$

Consequently

(5.21)
$$\sum_{j=0}^{\infty}\ \sum_{\substack{0\leqq\beta\leqq\alpha\\ |\beta|-k=-j}} \left|\binom{\alpha}{\beta}\,\partial^{\beta}u_{-k}(x)\,\partial^{\alpha-\beta}F_{-k}(x)\right|$$
$$\leqq B\,C\,L^{|\alpha|}(C\,r^{\tau}\delta^{-1} + 1)^{|\alpha|}|\alpha|!^{s}\sum_{j=0}^{\infty} C^{j}L^{j}\,r^{\tau j}\,\frac{(|\alpha|+j)!^{s}}{|\alpha|!^{s}\,j!^{s}}$$
$$\leqq \frac{B\,C(2^{s}L(C\,r^{\tau}\delta^{-1} + 1))^{|\alpha|}\,|\alpha|!^{s}}{1 - 2^{s}C\,L\,r^{\tau}}$$

converges if r is sufficiently small.

In case $1 - |\alpha| \leqq j < 0$, we have

$$(j + \ell)!^{-\tau}\,\ell!\,(|\alpha| + j)!^{s} \leqq 2^{s|\alpha|}|\alpha|!^{s}.$$

Hence

$$(5.22)\quad \sum_{j=1-|\alpha|}^{-1} \sum \left| \binom{\alpha}{\beta} \partial^{\beta} u_{-k}(x)\, \partial^{\alpha-\beta} F_{-k}(x) \right|$$

$$\leqq \frac{B\, C^2\, L\, r^{\tau}\, (C^{-1} r^{-\tau} (2^s\, C\, r^{\tau} \delta^{-1} + 1))^{|\alpha|}\, |\alpha|!^s}{1 - C\, L\, r^{\tau}}$$

converges if r is sufficiently small.

Now we define $u(x)$ on $\Omega_0 = \{x \in \mathbb{R}^n;\ |x| < r\}$ by

$$(5.23)\qquad u(x) = \sum_{j=-\infty}^{\infty} u_j(x)(F_j^+(x) - F_j^-(x)).$$

(5.21) and (5.22) prove that this converges in $\mathcal{E}^{\{s\}}(\Omega_0)$ and represents a solution of (5.1). Since two branches $F_j^{\pm}(x)$ coincide when $\mathrm{Re}\ \varphi(x) < 0$, $u(x)$ vanishes on one side of S. On the other hand,

$$u(0, x') = \Phi_0^+(\varphi(0, x')) - \Phi_0^-(\varphi(0, x'))$$

cannot vanish on any neighborhood of $x' = 0$. Therefore $u(x)$ is an ultradifferentiable null-solution of class $\{s\}$ on a neighborhood of $x_0 = 0$.

By (4.14) $u(0, x')$ is equal to $F'(\varphi(0, x'))$ on the other side of S, where

$$F'(z) = d \exp(- z^{-\frac{1}{s-1}}) / dz.$$

Thus we can prove that $u(0, x')$ is not an ultradifferentiable function of class (s) on any neighborhood of x_0 by the same estimate of $u(0, x')$ as in the proof of Lemma 4.2.

<u>Remark</u>. We had to shrink the domain to make (5.11), (5.21) and (5.22) converge. If we make use of

$$(5.24) \qquad \Phi_j(z) = \begin{cases} \varepsilon^{-j} \displaystyle\int_0^{\varepsilon z} \frac{(\varepsilon z - w)^{j-1}}{(j-1)!} \Psi(w)\, dw, & j > 0, \\ \varepsilon^{-j} \left(\dfrac{d}{d\varepsilon z}\right)^{-j} \Psi(\varepsilon z), & j \leqq 0, \end{cases}$$

instead of (5.13) for sufficiently small $\varepsilon > 0$, we have a wider domain of definition of $u(x)$.

Theorem 5.3. *If the characteristic surface* S *is regular, then for each sequence* M_p *satisfying* (M, 0), (M, 1), (M, 2) *and* (M, 3) *and for each* $x_0 \in S$ *we can construct an ultradistribution null-solution* $u(x)$ *of class* (M_p) *on a neighborhood of* x_0.

Proof. Let

$$(5.25) \qquad Q(\partial/\partial z) = \prod_{p=1}^{\infty} \left(1 - \frac{i\partial/\partial z}{m_p} \right)$$

and define the sequence $\Phi_j(z)$ of holomorphic functions on the Riemann domain $\{ z ; -\theta < \arg z < 2\pi + \theta \}$ for a $\theta > 0$ by

$$(5.26) \qquad \Phi_0(z) = \frac{-1}{2\pi i} Q\left(\frac{\partial}{\partial z}\right) \frac{1}{z}$$

$$(5.27) \qquad \Phi_j(z) = \frac{-1}{2\pi i} Q\left(\frac{\partial}{\partial z}\right) \left(\frac{z^{j-1}}{(j-1)!} \log z - \frac{z^{j-1}}{(j-1)!}\left(1+\frac{1}{2}+\cdots+\frac{1}{j-1}\right)\right), \quad j > 0.$$

It is easy to see that the sequence $\Phi_j(z)$ satisfies (2.9). We will prove that

$$(5.28) \qquad U(x) = \sum_{j=0}^{\infty} u_j(x)\, \Phi_j(\varphi(x))$$

converges on (the covering space of) $V \setminus S$ and that the boundary value

$$(5.29) \qquad u(x) = \sum_{j=0}^{\infty} u_j(x) \left(\Phi_j(\varphi(x) + i0) - \Phi_j(\varphi(x) - i0) \right)$$

is a desired null-solution.

Since $Q(\partial)$ is an ultradifferential operator of class (M_p), it follows from Lemma 11.3 of [14] that there are constants B' and L' such that

$$|Q(\partial) F(z)| \leqq B' \exp M^*(L'/t) \sup_{|w-z|=t} |F(w)|$$

for every holomorphic function F on a neighborhood of the disk $|w - z| \leqq t$, where

$$M^*(\rho) = \sup \log \frac{\rho^p p!}{M_p}.$$

In particular, we can find constants B and L such that

(5.30) $|\Phi_j(z)| \leqq B^{j+1} |z|^{j-2} (j - 2)_+!^{-1} \exp M^*(L/|z|)$

for $|z| < 1$. Consequently if $|\varphi(x)| < r_0 < (B\,C)^{-1}$, we have

$$\sum_{j=2}^{\infty} |u_j(x)\, \Phi_j(\varphi(x))|$$

$$\leqq \sum_{j=2}^{\infty} C^{j+1} j!\, B^{j+1} r_0^{j-2} (j - 2)!^{-1} \exp M^*(L / |\varphi(x)|)$$

$$= 2B^2 C^2 (1 - B\,C\,r_0)^{-3} \exp M^*(L / |\varphi(x)|).$$

Making L a little larger, we can prove that

$$\sum_{j=0}^{\infty} |u_j(x)\, \Phi_j(\varphi(x))| \leqq C_1 \exp M^*(L / |\varphi(x)|) \tag{5.31}$$

for a constant C_1. Hence by Theorem 11.5 of [14] the boundary value (5.29) exists in the topology of $\mathcal{E}^{(M_p)'}(\Omega_0)$, where $\Omega_0 = \{x \in \Omega\,;\, |x - x_0| < r\}$.

Since

$$U(0, x') = \Phi_0(\varphi(0, x')), \tag{5.32}$$

$U(x)$ is not holomorphic at x_0. Therefore it follows from the edge of the wedge theorem that x_0 is in the support of $u(x)$.

Remark. In view of (5.32) we can easily prove that there exists an $L > 0$ such that the estimate

$$\sup_{x \in K} |U(x + iy)| \leq C \exp M^*(L/|y|)$$

does not hold for any compact neighborhood K of x_0 in Ω and constant C. If Theorem 11.8 of [14] is true in the n-dimensional case (and if $M_p \subset M_p^* p!$), this implies that the null-solution $u(x)$ is not an ultradistribution of class $\{M_p\}$ on any neighborhood of x_0.

Theorem 5.4. If the characteristic surface S is of irregularity $\sigma > 1$, then for each $1 < s \leq \sigma/(\sigma - 1)$ and point $x_0 \in S$ there is an ultradistribution null-solution $u(x)$ of class (s) on a neighborhood of x_0.

Proof. We employ the sequence $\Phi_j(z)$ defined by (5.26), (5.27) and

$$\Phi_j(z) = \left(\frac{d}{dz}\right)^{-j} \Phi_0(z), \quad j < 0. \tag{5.33}$$

Since $M^*(\rho)$ is equivalent to $(s - 1)\rho^{\frac{1}{s-1}}$ in this case, $\exp \rho^{\frac{1}{s-1}}$ may be used in place of $\exp M^*(\rho)$.

In view of (5.30) and Cauchy's inequality we can find constants B and L such that

$$|\Phi_j(z)| \leq \begin{cases} B^{j+1}|z|^j\, j!^{-1} \exp (L/|z|)^{\frac{1}{s-1}}, & j \geq 0, \\ B^{-j+1}|z|^{-j}\, (-j)! \exp (L/|z|)^{\frac{1}{s-1}}, & j < 0 \end{cases} \tag{5.34}$$

for $|z| < 1$.

We consider the series

$$U(x) = \sum_{j=-\infty}^{\infty} u_j(x)\, \Phi_j(\varphi(x)). \tag{5.35}$$

The sum over the non-negative j has already been estimated. To estimate the sum over the negative j, we first note that for each $a > 0$ and $\varepsilon > 0$ there is a constant A such that

$$\sum_{j=0}^{\infty} t^j / j!^a \leqq A \exp((a+\varepsilon)t^{1/a}), \quad 0 < t < \infty . \tag{5.36}$$

Applying this to the case where $a = (\sigma - 1)^{-1}$, we have

$$\begin{aligned}
&\sum_{j=1}^{\infty} |u_{-j}(x)\, \Phi_{-j}(\varphi(x))| \\
&\leqq \sum_{j=1}^{\infty} C^{j+1}(|x|^j/j!)^{\sigma/(\sigma-1)} B^{j+1} |\varphi(x)|^{-j} j! \exp(L/|\varphi(x)|)^{\frac{1}{s-1}} \\
&\leqq A\ B\ C \exp\left\{((\sigma-1)^{-1}+\varepsilon)(B\ C)^{\sigma-1}|x|^{\sigma}/|\varphi(x)|^{\sigma-1} + (L/|\varphi(x)|)^{\frac{1}{s-1}}\right\} \\
&\leqq C' \exp(L'/|\varphi(x)|)^{\frac{1}{s-1}}
\end{aligned}$$

for constants C' and L' when $|x| < r$.

The rest of the proof is the same as that of the previous theorem. We could also use the sequence

$$\Phi_j(z) = \begin{cases} \displaystyle\int_{-i0}^{z} \frac{(z-w)^{j-1}}{(j-1)!} \exp(-iw)^{-\frac{1}{s-1}} dw, & j > 0, \\[2ex] \left(\dfrac{d}{dz}\right)^{-j} \Phi_0(z), & j \leqq 0. \end{cases}$$

<u>Remark</u>. When S is irregular, we have a gap between the ultradifferentiable null-solution of class $\{\sigma/(\sigma-1)\}$ and the ultradistribution null-solution of class $(\sigma/(\sigma-1))$. We remark that this is unavoidable in general. For example, consider the differential operator

$$P(\partial) = \partial_1^{2m} + (-1)^{\ell-m}(\partial_2^2 + \cdots + \partial_n^2)^{\ell}$$

with $\ell < m$. The irregularity of the characteristic function $\varphi(x) = x_n$ is $m/(m - \ell)$. On the other hand, it is easily proved that there is a constant $A > 0$ such that if $\xi + i\eta \in \mathbb{R}^n + i\,\mathbb{R}^n$ satisfies

$$P(i\,(\xi + i\eta)) = 0,$$

then

$$|\eta| \geq A\,|\xi|^{\ell/m}.$$

Hence it follows that every ultradistribution solution $u(x)$ of class $\{\sigma/(\sigma - 1)\}$ of

$$P(\partial)u(x) = 0$$

on an open set in $\mathbb{R}^n$ is an ultradifferentiable function of class $\{\sigma/(\sigma - 1)\}$ (cf. Björck [0] and Chou [2]). Therefore there are actually no null-solutions with regularity or singularity in the gap.

6. Solutions with small singularity spectra.

Null-solutions are solutions with smallest possible supports. In this section we are concerned with solutions with smallest singularity spectra or singular supports in the sense of Sato-Kawai-Kashiwara [25], p.284.

It is easy to see that the null-solutions $u(x)$ constructed in § 5 are real analytic outside the characteristic surface S. More precisely it is shown that the singularity spectrum

(6.1) $$S\,S\,u(x) = \left\{ (x, \pm d\varphi\infty) \in S^*\Omega_0;\ \varphi(x) = 0 \right\},$$

which is a submanifold of $S^*\Omega_0$ of dimension $n-1$. When a characteristic element $(x_0, \xi_0\infty)$ is simple and the principal part $p(a, \partial)$ satisfies certain conditions, Zerner [28], Hörmander [11], [12] and Kawai [13] have proved that there is a solution $u(x)$ whose singularity spectrum is a zero or one or two dimensional submanifold passing through $(x_0, \xi_0\infty)$. We extend their results to the case where the multiplicity d is greater than one.

A curve $b : (x(t), \xi(t)\infty)$ in $S^*\Omega$ is said to be a (real) bicharacteristic strip of the operator $P(x, \partial)$ if a representative $(x(t), \xi(t)) \in T^*\Omega$ is a solution of Hamilton's canonical equations :

(6.2) $$\frac{dx_1}{\frac{\partial K}{\partial \xi_1}} = \cdots = \frac{dx_n}{\frac{\partial K}{\partial \xi_n}} = \frac{-d\xi_1}{\frac{\partial K}{\partial x_1}} = \cdots = \frac{-d\xi_n}{\frac{\partial K}{\partial x_n}}$$

and

(6.3) $$K(x(t), \xi(t)) = 0$$

of an irreducible factor $K(x, \xi)$ of the principal symbol $p(x, \xi)$.

We always assume that bicharacteristic strips b are non-singular or that every characteristic element $(x(t), \xi(t)\infty)$ on b is non-singular. Then, the representative $(x(t), \xi(t))$ is also a solution of (6.2) with $K(x, \xi)$ replaced by a factor $\xi_1 - \lambda(x, \xi')$ of $K(x, \xi)$ under a suitable coordinate system. Hence it follows that the curve $(x(t), \xi(t))$ in $T^*\Omega$ is real analytic and that the multiplicity d and the irregularity σ of the characteristic elements $(x(t), \xi(t)\infty)$ are constant on the bicharacteristic strip.

Theorem 6.1. Let b be a non-singular real bicharacteristic strip of irregularity σ and let $1 < s \leqq \sigma/(\sigma - 1)$. Then for each $(x_0, \xi_0\infty)$ in b there is a solution $u(x)$ of (5.1) on a neighborhood of x_0 whose singularity spectrum is included in b and contains $(x_0, \xi_0\infty)$ and which is ultradifferentiable of class $\{s\}$ but not of class (s) (resp. an ultradistribution of class (s)).

Proof. We may assume without loss of generality that $x_0 = 0$ and that the representative $(x(t), \xi(t))$ satisfies (6.2) and (6.3) with a simple holomorphic factor $K(x, \xi) = \xi_1 - \lambda(x, \xi')$ of $p(x, \xi)$. Then we can choose x_1 for the parameter t. Since $\lambda(x, \xi')$ is homogeneous in ξ', ξ'_0 cannot be zero.

Now we solve the first order equation

$$(6.4) \quad \frac{\partial \varphi(x)}{\partial x_1} - \lambda(x, \frac{\partial \varphi(x)}{\partial x'}) = 0$$

under the initial condition

$$(6.5) \quad \varphi(0, x') = \langle x', \xi'_0 \rangle + i\sum_{j=2}^{n} x_j^2 .$$

By the Cauchy-Kowalevsky theorem there is a unique holomorphic solution $\varphi(x)$ on a complex neighbourhood V of $x_0 = 0$. The equation can also be solved by integrating the bicharacteristic equation

(6.2) under the initial conditions derived from (6.5).

Since $\varphi(0) = 0$ and $\partial\varphi/\partial x' = \xi_0'$ at the origin, $\varphi(x)$ vanishes on the bicharacteristic curve $\pi b : x(t)$ and $\operatorname{grad}\varphi(x(t)) = \xi(t)$ is real on πb. By a simple calculation we have also on πb

$$\frac{\partial^2\varphi}{\partial x_1^2} = \frac{d\xi_1}{dt} - \sum_{k=2}^{n} \frac{dx_k}{dt}\frac{d\xi_k}{dt} + \sum_{k,\ell=2}^{n} \frac{dx_k}{dt}\frac{dx_\ell}{dt}\frac{\partial^2\varphi}{\partial x_k \partial x_\ell}$$

$$\frac{\partial^2\varphi}{\partial x_1 \partial x_j} = \frac{d\xi_j}{dt} - \sum_{k=2}^{n} \frac{dx_k}{dt}\frac{\partial^2\varphi}{\partial x_j \partial x_k}, \quad j = 2,3,\cdots,n$$

and hence

$$(6.7) \quad \sum_{i,j=1}^{n} \frac{\partial^2 \operatorname{Im}\varphi}{\partial x_i \partial x_j} t_i t_j = \sum_{k,\ell=2}^{n} \frac{\partial^2 \operatorname{Im}\varphi}{\partial x_k \partial x_\ell}\left(t_k - \frac{dx_k}{dt}t_1\right)\left(t_\ell - \frac{dx_\ell}{dt}t_1\right).$$

Thus the Hessian of $\operatorname{Im}\varphi$ has at least one zero eigenvalue at every point on the bicharacteristic curve. On the other hand, since it has $n - 1$ positive eigenvalues at the origin because of (6.5) and (6.7), the same is true on a neighborhood of the origin. Hence it follows that there is a real neighborhood Ω_0 of $x_0 = 0$ in V such that $\varphi(x)$ restricted to Ω_0 vanishes only on πb and $\operatorname{Im}\varphi(x) > 0$ on $\Omega_0 \setminus \pi b$.

Next we constract a holomorphic solution

$$(6.8) \quad U(x) = \sum_{j=-\infty}^{\infty} u_j(x)\Phi_j(\varphi(x))$$

of (5.1) as in the proofs of Theorem 5.2 and 5.4. $U(x)$ is defined on a covering space of $V\setminus S$, where S is the zeros of $\varphi(x)$. Then the boundary value

$$(6.9) \quad u(x) = \sum_{j=-\infty}^{\infty} u_j(x)\Phi_j(\varphi(x) + i0)$$

of $U(x)$ on Ω_0 gives a desired solution.

In fact, $u(x)$ is clearly real analytic outside πb and at each point $x(t)$ in πb it is the boundary value of the holomorphic $U(z)$ defined at least on the domain $\mathrm{Im}\,\varphi(z) > 0$. Since $\mathrm{grad}\,\varphi(x) = \xi(t)$, the domain contains every subsector of the half space $\langle \mathrm{Im}\, z, \xi(t)\rangle > 0$. Thus we have

$$SSu(x) \subset b.$$

On the other hand, since

$$u(0, x') = \Phi_0(\varphi(o, x') + i0)$$

is not real analytic at the origin, the singularity spectrum contains $(x_0, \xi_0 \infty)$.

The regularity or the singularity of $u(x)$ are proved in the same way as Theorems 5.2 and 5.4.

Remarks. If the bicharacteristic strip b is regular or the irregularity $\sigma = 1$, then we have of course analogues of Theorems 5.1 and 5.3. If we start, in this case, with

$$(6.10)\quad \Psi(z) = \frac{z^{k+1}}{(k+1)!}\left(\log z - 1 - \frac{1}{2} - \cdots - \frac{1}{k+1}\right) \quad \text{or} \quad z^{-k+1},$$

we obtain a proof of the existence of an exactly k times continuously differentiable solution $u(x)$ or a distribution solution $u(x)$ exactly of order k whose singularity spectrum coincides with b. This fact has been proved by Hörmander [12], [12'] and Kawai [13] under the assumption that the principal part $p(x, \partial)$ has real coefficients.

If $p(x, \partial)$ has real coefficients and if $(x_0, \xi_0 \infty)$ is a real non-singular characteristic element, then we can find a real bicharacteristic strip b passing through $(x_0, \xi_0\infty)$. By Theorem 6.1 there is therefore a solution u(x) of (5.1) whose singularity spectrum contains $(x_0, \xi_0\infty)$. This generalizes in a certain sense Theorem 4.1 of [16].

When $p(x, \partial)$ has complex coefficients, however, a real non-singular characteristic element $(x_0, \xi_0\infty)$ is not necessarily contained in the singularity spectrum of a solution u(x) of (5.1) (see Mizohata [22] and Kawai [13']). We will give here two sufficient conditions.

When $\mu(x, \xi)$ and $\nu(x, \xi)$ are functions on $T^*\Omega$ homogeneous in ξ, we define the <u>Poisson bracket</u> $\{\mu, \nu\}$ by

(6.11) $$\{\mu,\nu\}(x, \xi) = \frac{1}{i} \sum_{j=1}^{n} \left(\frac{\partial\mu}{\partial\xi_j} \frac{\partial\nu}{\partial x_j} - \frac{\partial\mu}{\partial x_j} \frac{\partial\nu}{\partial\xi_j} \right)$$

according to Sato-Kawai-Kashiwara [25] p.480. When $\mu(x, \xi)$ is a complex valued function, we write

(6.12) $$C(x, \xi) = -\{\mu, \bar{\mu}\}(x, \xi)$$

according to Hörmander [11], [12], where $\bar{\mu}(x, \xi)$ is the complex conjugate of $\mu(x, \xi)$. The following theorem is due to Hörmander [12], [12'] when the characteristic element is simple.

<u>Theorem 6.2</u>. <u>Let</u> $(x_0, \xi_0\infty)$ <u>be a real non-singular characteristic element of</u> $P(x, \partial)$ <u>and let</u> $\mu(x, \xi)$ <u>be either the irreducible factor</u> $K(x, \xi)$ <u>of</u> $p(x, \xi)$ <u>in</u> (1.6) <u>or the holomorphic factor</u> $\xi_1 - \lambda(x, \xi')$ <u>in</u> (1.4). <u>If</u>

(6.13) $$C(x_0, \xi_0) < 0$$

<u>and</u> σ <u>is the irregularity, then for each</u> $1 < s \leqq \sigma/(\sigma - 1)$ <u>there</u>

<u>is a solution</u> $u(x)$ <u>of (5.1) on a neighborhood of</u> x_0 <u>whose singularity spectrum coincides with the point</u> $(x_0, \xi_0\infty)$ <u>and which is ultradifferentiable of class</u> $\{s\}$ <u>but not of class</u> (s) (<u>resp. an ultradistribution of class</u> (s)).

<u>Proof</u>. By Lemma 6.1.3 of Hörmander [11] there is a holomorphic characteristic function $\varphi(x)$ defined on a neighborhood of $x_0 = 0$ such that

$$(6.14) \quad \varphi(x) = \langle x, \xi_0 \rangle + \frac{1}{2} \sum_{i,j=1}^{n} \alpha_{ij} x_i x_j + O(|x|^3), \quad x \longrightarrow 0,$$

with a symmetric matrix α_{ij} whose imaginary part is positive definite. Then the rest of the proof is the same as for Theorem 6.1.

<u>Remark</u>. The assumption (6.13) is used only to prove the existence of a holomorphic characteristic function $\varphi(x)$ on a neighborhood of x_0 such that $\operatorname{grad} \varphi(x_0) = \xi_0$ and $\operatorname{Im} \varphi(x) > 0$ on a real neighborhood of x_0 except at x_0 where $\varphi(x_0) = 0$. Such a characteristic function $\varphi(x)$ exists for the generalized Levi- Mizohata equation

$$(6.15) \quad \left(\frac{\partial}{\partial x_1} - i x_1^{2k+1} \frac{\partial}{\partial x_n}\right)^d u(x) = 0, \ k = 0, 1, 2, \cdots,$$

at $(0, dx_n\infty)$. However, (6.13) holds only when $k = 0$. An invariant characterization of such operators has been investigated by L. Nirenberg and F. Trèves but we will not go into the details.

Lastly we generalize a result of Zerner [28] and Hörmander [12], [12'] and constract a solution with a two dimensional singularity spectrum.

Let $(x_0, \xi_0\infty)$ be a real non-singular characteristic element and let $\mu(x, \xi)$ be as in Theorem 6.2. We consider the case where the real and imaginary parts of $\mathrm{grad}_{\xi}\, \mu(x, \xi)$ are linearly independent and

(6.16) $C(x, \xi) \equiv 0.$

We write the real and imaginary parts of $\mu(x, \xi)$, $\mu_1(x, \xi)$ and $\mu_2(x, \xi)$. Then (6.16) is equivalent to

(9.17) $\{\mu_1, \mu_2\} \equiv 0.$

Let H_k be the corresponding Hamilton fields defined by

(6.18) $H_k f(x, \xi) = i\{\mu_k, f\}.$

The vector fields H_k are tangential to the manifold M defined by $\mu(x, \xi) = 0$ in a neighborhood of (x_0, ξ_0) and form an involutive system by the Jacobi identity. Hence there is a unique two dimensional integral manifold $(x(t_1, t_2), \xi(t_1, t_2))$ passing through each point (x, ξ) in T*M. The two dimensional submanifold $(x(t_1, t_2), \xi(t_1, t_2)\infty)$ of $S^*\Omega$ is again called a <u>bicharacteristic strip</u>.

We choose a local coordinate system so that the linear submanifold $\{x_1 = x_2 = 0\}$ is transversal to the projection πb of the bicharacteristic strip b passing through $(x_0, \xi_0\infty)$. Then we can integrate the equation

(6.19) $\mu(x, \mathrm{grad}\, \varphi(x)) = 0$

with the initial condition

(6.20) $\varphi(0, 0, x'') = \langle x'', \xi''_0 \rangle + i\sum_{j=3}^{n} x_j^2$

by the Hamilton-Jacobi method (see Hörmander [12']). Here $x'' = (x_3, \cdots, x_n)$.

It is proved in the same way as in the proof of Theorem 6.1 that the solution $\varphi(x)$ is holomorphic on a complex neighborhood V of x_0, vanishes only on πb in $V \cap \mathbb{R}^n$ and has positive imaginary part on $V \cap \mathbb{R}^n \setminus \pi b$ and that grad $\varphi(x_0) = \xi_0$. Hence we obtain the following theorem.

Theorem 6.3. Let $(x_0, \xi_0 \infty)$ be a real non-singular characteristic element of irregularity σ which annihilates the simple factor $\mu(x, \xi)$ of $p(x, \xi)$. If the real and imaginary parts of $\mathrm{grad}_\xi \mu(x, \xi)$ are linearly independent and if (6.16) holds on a neighborhood of (x_0, ξ_0), then for each $1 < s \leq \sigma/(\sigma - 1)$ there is a solution $u(x)$ of (5.1) on a neighborhoos of x_0 whose singularity spectrum is included in the two dimensional bicharacteristic strip b passing through $(x_0, \xi_0 \infty)$ and which is ultradifferentiable of class $\{s\}$ but not of class (s) (resp. an ultradistribution of class (s)).

Footnote

1) After the paper was completed the author was informed from Prof. J. Persson that he proved in [29] the existence of an infinitely differentiable null-solution for every totally real characteristic surface of constant multiplicity.

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