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Remarks on Continuation of Real Analytic Solutions

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In [1], [2] and [3] we have given some results on continuation of real analytic solutions of linear partial differential equations with constant coefficients to convex sets K of various types. In this note we remark that the assumption of the convexity of K can be much weakened. First we prove the following general assertion. A same type result has been given by Komatsu [6] for other classes of functions.

THEOREM Let $p(D)$ be a $t \times s$ matrix of linear partial differential operators with constant coefficients. Let A_p be the sheaf of the real analytic solutions of $p(D)u = 0$. Let K be a compact subset of R^n. Then, for any open neighborhood U of K we have $A_p(U \setminus K)/A_p(U) = H^1_K(U, A_p)$. Hence, this space does not depend on U.

PROOF We have the following long exact sequence in the general cohomology theory :

$$0 \longrightarrow A_p(U) \longrightarrow A_p(U \setminus K)$$
$$\longrightarrow H^1_K(U, A_p) \longrightarrow H^1(U, A_p) \longrightarrow H^1(U \setminus K, A_p).$$

Thus it suffices to show that the restriction mapping $H^1(U, A_p) \longrightarrow H^1(U \setminus K, A_p)$ is injective. Since the cohomology groups $H^k(V, A)$ vanish for $k \geq 1$, for any open set $V \subset R^n$, we can calculate $H^1(U, A_p)$ and $H^1(U \setminus K, A_p)$ employing the resolution :

$$0 \longrightarrow A_p \longrightarrow A^s \xrightarrow{p} A^t \xrightarrow{p_1} A^{t_2} \longrightarrow \ldots .$$

Thus we have

(1) $$H^1(U, A_p) \cong A_{p_1}(U)/p(D)[A(U)]^s,$$

(2) $$H^1(U \setminus K, A_p) \cong A_{p_1}(U \setminus K)/p(D)[A(U \setminus K)]^s.$$

Take a representative $u(x) \in A_{p_1}(U)$ of an element of $H^1(U, A_p)$ which goes to zero cohomology class by the restriction. This obviously implies that $u|_{U \setminus K} = p(D)v$ for some $v \in [A(U \setminus K)]^s$. Now we consider v as a section of $\tilde{\mathcal{O}}^s$ on $U \setminus K$, where $\tilde{\mathcal{O}}$ denotes the sheaf of slowly increasing holomorphic functions on $D^n \times iR^n$; D^n is the directional compactification of R^n and $\tilde{\mathcal{O}}|_{R^n}$ agrees with A (see [4]). We have $H^1(V, \tilde{\mathcal{O}}) = 0$ for any open set $V \subset D^n$ ([4], Theorem 3.1.8). Thus we can find $f \in [\tilde{\mathcal{O}}(D^n \setminus K)]^s$ and $g \in [A(U)]^s$ such that $v = f - g$ on $U \setminus K$.

We have

$$p(D)f = p(D)v + p(D)g = u + p(D)g$$

on $U \setminus K$. Hence $p(D)f$ can be extended analytically to K. The extended element h obviously satisfies $p_1(D)h = 0$ and belongs to $[\tilde{\mathcal{O}}(D^n)]^s$. The latter implies especially that h is holomorphic on a complex strip around R^n with a fixed breadth. Thus by the existence theorem ([5], Theorem 1) we can find $w \in [A(R^n)]^s$ such that $p(D)w = h$. Thus we conclude that $w - g \in [A(U)]^s$. This implies that u represents the zero cohomology class also in $H^1(U, A_p)$. The injectivity is proved. Due to the excision theorem, $H^1_K(U, A_p)$ does not depend on U. q.e.d.

COROLLARY 1 Let K be a compact set in R^n such that $R^n \setminus K$ is connected. Let $p(D)$ be as above. Assume that $\mathrm{Hom}(\mathrm{Coker}\ p', P) = 0$ and that $\mathrm{Ext}^1(\mathrm{Coker}\ p', P)$ has no elliptic components, where p' denotes the transpose of p and P denotes the ring of polynomials of D. Then for any open neighborhood U of K we have $A_p(U \setminus K)/A_p(U) = 0$, namely, every real analytic solution of $p(D)u = 0$ in $U \setminus K$ can be uniquely continued to U.

PROOF We only have to prove $A_p(U \setminus K)/A_p(U) = 0$ for a convex neighborhood U of K. Let $ch(K)$ denote the convex hull of K. The restriction mapping

$$A_p(U \setminus K)/A_p(U) \longrightarrow A_p(U \setminus ch(K))/A_p(U)$$

is injective because of the assumption on K. Theorem 2.3 of [2] asserts that the second term vanishes. q.e.d.

We can make a similar generalization also for the results in [3]. We neglect to write it down explicitly.

COROLLARY 2 Let p be as above. Let p_1 be its compatibility system. Let U be an open set in R^n. Then for $f \in A_{p_1}(U)$ we can find a solution $u \in [A(U)]^s$ of $p(D)u = f$ if and only if for some compact set $K \subset U$ there exists $v \in [A(U \setminus K)]^s$ satisfying $p(D)v = f$. Namely, the solvability is determined only at the boundary and at infinity.

PROOF In the proof of THEOREM we have shown that the restriction mapping from (1) to (2) is injective. The above assertion is a mere paraphrase of this fact. q.e.d.

REFERENCES

[1] A. Kaneko, On continuation of regular solutions of partial differential equations to compact convex sets, J. Fac. Sci. Univ. Tokyo, Sec.IA, 17 (1970), 567-580.

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[4] T. Kawai, On the theory of Fourier hyperfunctions and its applications to partial differential equations with constant coefficients, J. Fac. Sci. Univ. Tokyo, Sec.IA. 17 (1970), 467-517.

[5] H. Komatsu, Resolutions by hyperfunctions of sheaves of solutions of differential equations with constant coefficients, Math. Annalen, 176 (1968), 77-86.

[6] H. Komatsu, Relative cohomology of sheaves of solutions of differential equations, Séminaire Lions-Schwartz, 1966/7. Reprinted in Lecture Notes in Mathematics 287, pp. 192-261, Springer, 1973.

APOLOGY and CORRECTION

In my note " Teisûkeisû-senkei-henbibun-hôteishiki no Kai no Senjô-tokui-shûgô ni tsuite " issued in Sûrikaiseki-kenkyûsho Kôkyûroku 226 (1975), pp.1-20, the condition (2.1) has been modified. Its original form

$$(2.1) \qquad |\mathrm{Im}\,\tau(\zeta')| \leq a|\mathrm{Re}\,\zeta_2|^q + b|\mathrm{Im}\,\zeta_2| + c(|\zeta_3|+\cdots+|\zeta_n|)$$

given at the lecture was not sufficient to prove "TEIRI 5". Nevertheless, at the beginning of §3 Hattari, we should have referred not the modified (2.1) but the above original one.