数理解析研究所講究録
第 255 巻 1975 年 57-75

On the periodic problem for the equations of the KdV series

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1. In this paper we study the KdV equation and a family of the generalized KdV equations (the KdV series) under the periodic boundary condition. The KdV series is firstly derived by Lax by means of his principle for the unitary equivalence of a one parameter family of selfadjoint operators. We shall give another two derivations for the KdV series, which works in the periodic case. One is based upon a criterion for the existence of a t-independent monodromy matrix for the Hill's equation whose potential u depends smoothly on a parameter t. The other is based upon a characterization of the invariance of eigenvalues of the Schrödinger operator $L(t) = d^2/dx^2 + u(x,t)$ with t regarded as parameter t in terms of the Green's function of the associated heat equation :

$$\varphi_s = \varphi_{xx} + u(x,t)\varphi, \qquad s > 0.$$

We shall show that the trace of the Green's function is a generating function for the constants of motion (the conserved functionals) of any member of the KdV series, that is, the coefficients of the asymptotic expansion of the trace of the Gree's function as $s \searrow 0$ yield the constants of motion.

Among sufficiently differentiable solutions of the KdV equation in the class of functions which tend to zero rapidly as $|x| \longrightarrow \infty$, multi-soliton solutions plays an important role. Multi-soliton solutions correspond to the case when the associated Schrödinger operator L(t) has only a finite number of discrete spectrums. We shall present a family of special solutions which are periodic analogues of multi-soliton solutions and correspond to the case when the Hill's equation has only a finite number of instability intervals. In this connection we establish the necessary and sufficient condition in order that the Hill's equation has only a finite number of instability intervals.

In the final section, we discuss the extension to more general nonlinear evolution equations, including, in particular, the modified KdV equation, the nonlinear Schrödinger equation, the sine-Gordon equation and multi-dimentional analogues of them.

At the Conference on " Solitons" held in July 1975 at the Reseach Institute for Mathematical Science ,Kyoto University, I learned that Novikov and Dubrovin [7], Its and Matveev [3], and Lax [6] have done similar works on this subject.

2. Derivation of the KdV series, I. Let u(x,t) be a real smooth function which is periodic in x with period 1. We consider the Hill's equation with t regarded as a parameter :

(2.1) $$\psi_{xx} + (u(x,t) + \lambda)\psi = 0, \quad -\infty < x,t < +\infty .$$

Put
$$\Psi = \begin{pmatrix} \psi \\ \psi_x \end{pmatrix}, \quad A(x,t) = \begin{pmatrix} 0 & 1 \\ -u(x,t) & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

Then, the equation (2.1) is rewritten in the matrix form :

(2.2) $$\Psi_x = (A(x,t) + \lambda J)\Psi , \quad -\infty < x,t < +\infty .$$

Let $X(x,t;\lambda)$ be a fundamental matrix solution of (2.2) and $Q_X(t;\lambda)$ be a monodromy matrix defined by

(2.3) $$X(x+1,t;\lambda) = X(x,t;\lambda)Q_X(t;\lambda).$$

Theorem 1. The following three assertions are equivalent :

1). There exists a fundamental matrix solution $X(x,t;\lambda)$ of the equation (2.2) such that $Q_X(t;\lambda)$ does not depend on t.

2). There exists a 2×2 matrix function $H(x,t;\lambda)$ which is smooth, periodic in x with period 1 and satisfies

(2.4) $$A_t - H_x + \left[A + \lambda J,\ H\right] = 0,$$

where the square bracket indicates the commutator.

3). There exists a smooth function $p(x,t;\lambda)$ which is periodic in x with period 1 and satisfies

(2.5) $$p_{xxx} + 4(u+\lambda)p_x + 2u_x p = 2u_t.$$

Remark 1. Let $X(x,t;\lambda)$ be a fundamental matrix solution such that $d\,Q_X(t;\lambda)/dt = 0$. Then, X satisfies

(2.6) $$X_t(x,t;\lambda) = H(x,t;\lambda)\ X(x,t;\lambda).$$

Remark 2. Let $X_0(x,t;\lambda)$ be a fundamental matrix solution of (2.2) with $X_0(0,t;\lambda) = I$, the identity, for all t and λ. Then,

(2.7) $$X_{0t} = H(x,t;\lambda)X_0 - X_0H(0,t;\lambda)$$

and

(2.8) $$d\,Q_{X_0}/dt = \left[H(0,t;\lambda),\ Q_{X_0}\right].$$

Proof of Theorem 1 may be accomplished by utilizing Floquet's theorem (stated below).

We now show that the equation (2.5) can be reduced to nonlinear partial differential equations for unknown $u(x,t)$ (which are (generalized) KdV equations) if we choose $p(x,t;\lambda)$ appropriately.

To obtain the KdV series, we must require that the equation (2.5) with suitable choice of $p(x,t;\lambda)$ does not depend explicitly on λ. Under this requirement, we shall give a method which leads us to obtain the KdV series recursively.

For an arbitrary positive integer N, put

$$p^N(x,t;\lambda) = \sum_{k=0}^{N} p_{N-k}(x,t)\,\lambda^k, \tag{2.9}$$

where $p_{N-k}(x,t)$ are smooth functions to be specified below. Substituting (2.9) into (2.5), we get

$$p_{N-k_{xxx}} + 4up_{N-k_x} + 2u_x p_{N-k} = -4p_{N-k+1_x}, \quad k = 1,2,\cdots,N, \tag{2.10}$$

$$p_{N_{xxx}} + 4up_{N_x} + 2u_x p_N = 2u_t, \tag{2.11}$$

$$p_{0_x} = 0, \tag{2.12}$$

which is a recursion formula for the p_k, $k = 0,1,\cdots,N$.

Introduce the operator

$$\mathcal{L}_k = -\frac{1}{4}\frac{\partial^2}{\partial x^2} - u(x,t) + \frac{1}{2}\int^x dx\cdot u_x(x,t) + C_k, \tag{2.13}$$

where C_k denote arbitrary constants. Then, from (2.10) and (2.11), we have

$$p_k = \prod_{n=1}^{k} \mathcal{L}_n\cdot p_0, \quad p_0 \text{ constant}, \tag{2.14}$$

and

$$u_t + 2\,p_{N+1_x} = 0, \quad N = 1,2,\cdots, \tag{2.15}$$

where p_k is a polynomial of u, u_x, $\cdots$, $\partial^{2(k-1)}u/\partial x^{2(k-1)}$ whose maximal degree is N.

Let us take $N = 1$ and $p^1(x,t;\lambda) = -2u(x,t) + 4\lambda$. Then, we have the KdV equation :

(2.16) $$u_t + 6uu_x + u_{xxx} = 0.$$

Theorem 2. The potential $u(x,t)$ of the Hill's equation (2.1) evolves according to the (generalized) KdV equation (2.15), then there exists a fundamental matrix solution $X(x,t;\lambda)$ of (2.3) which satisfies (2.6) and the corresponding monodromy matrix $Q_X(t;\lambda)$ does not depend on t. Furthermore, for every λ, the characteristic multipliers $\rho_{\pm}(\lambda)$ are constants of motion of (2.15).

3. The Hill's equation. Let $\varphi(x;\lambda)$ and $\theta(x;\lambda)$ be the solutions of the Hill's equation :

(3.1) $$\psi_{xx} + (u(x) + \lambda)\psi = 0,$$

satisfying the condition

(3.2) $$\varphi(0;\lambda) = 0,\ \varphi_x(0;\lambda) = 1\ ;\ \theta(0;\lambda) = 1,\ \theta_x(0;\lambda) = 0.$$

Put

$$X_0(x;\lambda) = \begin{pmatrix} \varphi & \theta \\ \varphi_x & \theta_x \end{pmatrix}.$$

In this case, the monodromy matrix is

(3.3) $$Q_{X_0}(\lambda) = X_0(1;\lambda).$$

The characteristic multipliers $\rho_{\pm}(\lambda)$ (which are eigenvalues of $Q_{X_0}(\lambda)$) are the roots of the equation

(3.4) $$\rho^2 - 2\Delta(\lambda)\rho + 1 = 0,$$

where

(3.5) $$2\Delta(\lambda) = \varphi_x(1;\lambda) + \theta(1;\lambda).$$

Hence, we have

(3.6) $$\rho_{\pm}(\lambda) = \Delta(\lambda) \mp \sqrt{\Delta^2(\lambda) - 1}.$$

Floquet's theorem. If $\rho_+(\lambda) \neq \rho_-(\lambda)$, the equation (3.1) has two linearly independent solutions

(3.7) $$\psi_+(x;\lambda) = \rho_+^x \mathcal{X}_+(x;\lambda),\quad \psi_-(x;\lambda) = \rho_-^x \mathcal{X}_-(x;\lambda),$$

where $\chi_+(x;\lambda)$ and $\chi_-(x;\lambda)$ are smooth periodic functions with period 1. If $\rho_+(\lambda) = \rho_-(\lambda)$, the equation (3.1) has a nontrivial periodic solution with period 1 (when $\rho_+ = \rho_- = 1$) or period 2 (when $\rho_+ = \rho_- = -1$). Let $\psi_p(x;\lambda)$ denote such a periodic solution. Then, another solution $\psi^*(x,t)$, linearly independent of $\psi_p(x;\lambda)$, satisfies

(3.8) $$\psi^*(x+1;\lambda) = \rho_+ \psi^*(x;\lambda) + \alpha \psi_p(x;\lambda),$$

where α is a constant. In (3.8), $\alpha = 0$ if and only if

(3.9) $$\varphi_x(1;\lambda) + \theta(1;\lambda) = \pm 2, \quad \varphi(1.\lambda) = \theta_x(1;\lambda) = 0.$$

The solution ψ_+ and ψ_- can be written in the form

(3.10) $$\psi_+(x;\lambda) = \theta(x;\lambda) + m_+(\lambda)\varphi(x;\lambda),$$

and

(3.11) $$\psi_-(x;\lambda) = \theta(x;\lambda) + m_-(\lambda)\varphi(x;\lambda),$$

where

(3.12) $$m_\pm(\lambda) = \varphi(1;\lambda)^{-1}(\varphi_x(1;\lambda) - \theta(1;\lambda))/2 \mp \varphi(1;\lambda)^{-1}\sqrt{\Delta^2(\lambda) - 1}.$$

Their Wronskian is

$$W(\psi_+,\psi_-) = 2\,\varphi(1.\lambda)^{-1}\sqrt{\Delta^2(\lambda) - 1}.$$

It is known that $\Delta(\lambda)$ is an entire analytic function of oder 1/2, and accordingly, the functions $\Delta(\lambda) - 1$ and $\Delta(\lambda) + 1$ have infinitely many zeros on the real axis with no finite limit point. The zeros of $\Delta(\lambda) - 1$ are denoted by λ_k ($k = 0,1,\cdots$, the enumeration of increasing λ_k). They furnish the eigenvalues of the selfadjoint boundary value problem for (3.1) with the boundary condition

(3.13) $$\psi(x+1;\lambda) = \psi(x;\lambda).$$

Similarly the zeros of $\Delta(\lambda) + 1$ are the eigenvalues of the problem for (3.1) with

(3.14) $$\psi(x+1;\lambda) = -\psi(x;\lambda).$$

We denote these by $\hat{\lambda}_k$ (k = 1,2,$\cdots$). These two sequences of zeros are distributed on the real axis in the following order :

$$-\infty < \lambda_0 < \hat{\lambda}_1 \leqq \hat{\lambda}_2 < \lambda_1 \leqq \lambda_2 < \hat{\lambda}_3 \leqq \hat{\lambda}_4 \quad \cdots .$$

The intervals of the type $(\lambda_{2k}, \hat{\lambda}_{2k+1})$ and $(\hat{\lambda}_{2k}, \lambda_{2k-1})$ are referred to as the stability intervals. The intervals of the type $(- \quad , \lambda_0]$, $[\lambda_{2k-1}, \lambda_{2k}]$, $[\hat{\lambda}_{2k-1}, \hat{\lambda}_{2k}]$ are referred to as the instability intervals. In general, it is possible that some instability interval degenerates into a point. When this occurs, two linearly independent solutions of the Hill's equation (3.1) are of period 1 or 2. We shall call this an instance of coexistence (Magnus and Winkler [4]). In this connection we have

Lemma 1. The coexistence of periodic solutions of (3.1) with period 1 or 2 occurs if and only if the equation $\Delta(\lambda) - 1 = 0$ or $\Delta(\lambda) + 1 = 0$ has a double root, which is equivalent to the equalities (3.9) holds.

In the next section, we shall study the coexistence problem in detail.

The spectrum of $L = d^2/dx^2 + u(x)$ consists of the closure of the stability intervals. We see that if $\lambda \in \mathbb{C} \setminus \bigcup \overline{\{\text{stability intervals}\}}$, the resolvent kernel for L at λ is

$$G(x,y;\lambda) = \frac{\varphi(1;\lambda)}{2\sqrt{\Delta^2(\lambda) - 1}} \times \begin{cases} \psi_-(x;\lambda)\psi_+(y;\lambda) & x \leqq y \\ \psi_+(x;\lambda)\psi_-(y;\lambda) & x > y. \end{cases}$$

4. Coexistence. We shall characterize the potential u(x) for which the Hill's equation has only a finite number of instability intervals. Hochstadt [2] showed that (1) if the Hill's equation (3.1) has only one instability interval $(-\infty, \lambda_0]$, the the potential u(x) = constant ; (2) if the Hill's equation has only two instability intervals if and only if the potential u(x) is a Weierstrass elliptic function ; (3) if the Hill's equation has only a finite num-

ber of instability intervals, then $u(x)$ is infinitely differentiable. We generalize Hochstadt's results :

Theorem 3. The Hill's equation (3.1) has only n+1 instability intervals if and only if the potential $u(x)$ is a solution of the ordinary differential equations of the form :

$$\sum_{k=0}^{n+1} C_{n+1-k} P_k(u, u_x, \cdots, d^{2(k-1)}u/dx^{2(k-1)}) = 0 \tag{4.1}$$

with periodic boundary condition

$$u(x+1) = u(x), \qquad -\infty < x < +\infty , \tag{4.2}$$

where $P_0 = 1$, each P_k $(k = 1,2,\cdots,n+1)$ is uniquely determined by the formula (4.15), (4.18) (stated below) and is a polynomial of u and their derivatives of order $2(k-1)$, whose maximal degree is k, and $C_0 = 1$, C_k $(k = 1,2,\cdots, n+1)$ are arbitrary real constants.

Proof of Theorem 1 will be partitined into several lemmas.

Consider the following equation with a parameter σ :

$$\psi_{xx} + (u(x+\sigma) + \lambda)\psi = 0. \tag{4.3}$$

Lemma 2. Let $\mu_i(\sigma)$ denote the eigenvalues of the equation (4.3) when subjected to the boundary conditions

$$\psi(0) = \psi(1) = 0. \tag{4.4}$$

Then, we have

$$\hat{\lambda}_1 \leqq \mu_1(\sigma) \leqq \hat{\lambda}_2, \quad \lambda_1 \leqq \mu_2(\sigma) \leqq \lambda_2, \cdots\cdots, \tag{4.5}$$

and

$$\hat{\lambda}_{2n-1} = \min_\sigma \mu_{2n-1}(\sigma), \quad \hat{\lambda}_{2n} = \max_\sigma \mu_{2n-1}(\sigma), \tag{4.6}$$

$$\lambda_{2n-1} = \min_\sigma \mu_{2n}(\sigma), \quad \lambda_{2n} = \max_\sigma \mu_{2n}(\sigma) \tag{4.7}$$

Note that $\mu_i(\sigma)$ $(i = 1,2,\cdots\cdot)$ are periodic functions of σ of period 1.

When we consider the case in which the Hill's equation (3.1) has only n+1 instability intervals, without loss of generality, we can always assume that the first n+1 instability intervals, that is, $(-\infty, \lambda_0]$, $[\hat{\lambda}_1, \hat{\lambda}_2], \cdots, [\hat{\lambda}_n, \hat{\lambda}_{n+1}]$ (n : odd) or $[\lambda_{n-1}, \lambda_n]$ (n : even) do not vanish, if necessary, changing the order of the enumeration of $\hat{\lambda}_k$ and λ_k.

Lemma 3. The Hill's equation (3.1) has only $n+1$ instability intervals if and only if

$$(4.8) \qquad \Phi(1;\lambda,\sigma) = \varphi(1;\lambda) \prod_{i=1}^{n} \frac{\lambda - \mu_i(\sigma)}{\lambda - \mu_i(0)}$$

$$\text{for all } \lambda \in \mathbb{C} \setminus \{\mu_1(0), \cdots, \mu_n(0)\},$$

where $\Phi(1;\lambda,\sigma)$ is the solution of (4.3) with the initial conditions

$$\Phi(0;\lambda,\sigma) = 0, \quad \Phi_x(0;\lambda,\sigma) = 1.$$

Here $\mu_i(\sigma)$ $(i = 1,2,\cdots,n)$ are C^∞ periodic functions of σ of period 1.

Lemma 2 and the necessity part of Lemma 3 were proved by Hochstadt [2].

Lemma 4. The condition (4.8) is equivalent to

$$(4.9) \qquad \psi_+(x;\lambda)\psi_-(x;\lambda) = \chi_+(x;\lambda)\chi_-(x;\lambda)$$

$$= \prod_{i=1}^{n} \frac{\lambda - \mu_i(x)}{\lambda - \mu_i(0)}.$$

Lemma 5. Let ψ_1 and ψ_2 be any two solutions of the Hill's equation (3.1). Let $\eta = \psi_1\psi_2$ be the product of these solutions. Then η satisfies the equation

$$(4.10) \qquad \eta_{xxx} + 4(u+\lambda)\eta_x + 2u_x\eta = 0, \quad \text{for all } \lambda.$$

Equation (4.10) has at least one nontrivial periodic solution with period 1.

Lemma 6. If λ is not a double root of $\Delta^2(\lambda) - 1 = 0$, then any periodic solution of (4.10) is a constant multiple of $\psi_+(x;\lambda)\psi_-(x;\lambda)$ $(=\chi_+(x;\lambda)\chi_-(x;\lambda))$.

Outline of the proof of Theorem 3. Suppose that precisely $n+1$ instability intervals do not vanish. Then, from Lemma 3-5, we see that the function

$$\eta(x,\lambda) = \prod_{i=1}^{n} (\lambda - \mu_i(0))\psi_+(x;\lambda)\psi_-(x;\lambda)$$

is a solution of (4.10) and is a polynomial of λ of degree n. It is written in the form

$$\eta(x,\lambda) = \sum_{i=0}^{n} p_{n-i}(x)\,\lambda^i, \tag{4.11}$$

where $p_0(x) = 1$ and

$$p_k(x) = \Sigma\, \mu_{i_1}(x)\mu_{i_2}(x)\cdots \mu_{i_k}(x), \tag{4.12}$$

in which the right hand side of (4.12) represents the fundamental symmetric functions of $\mu_{i_1}(x)$, $\mu_{i_2}(x),\cdots \mu_{i_k}(x)$ whose degree is k. Substituting (4.11) into (4.10), we get

$$p_{n-i_{xxx}} + 4\, up_{n-i_x} + 2\, u_x p_{n-i} = -4\, p_{n-i+1_x}, \quad (i = 1,2,\cdots,n) \tag{4.13}$$

and

$$p_{n_{xxx}} + 4\, up_{n_x} + 2\, u_x p_n = 0. \tag{4.14}$$

Introducing the operator

$$\mathcal{L}_k = -\frac{1}{4}\frac{d^2}{dx^2} - u(x) + \frac{1}{2}\int^x dx\cdot u_x + C_k,$$

where C_k denote arbitrary constants, from (4.13) and (4.14) we have

$$p_k = \prod_{n=1}^{k} \mathcal{L}_n \cdot 1 \tag{4.15}$$

and

$$p_{n+1_x} = 0, \tag{4.16}$$

where we define

(4.17) $$p_{n+1} = \prod_{i=1}^{n+1} \mathcal{L}_i \cdot 1.$$

Put

(4.18) $$P_k = p_k \Big|_{C_1=C_2=\cdots=C_k = 0.}$$

Then, P_k is uniquely determied by u and its derivatives and (4.16) gives (4.1).

Proof of the sufficiency is accomplished by utilizing Lemma 3,4 and 6.

Theorem 4. Let $u(x)$ be a solution of (4.1). Then

(4.19) $$u(x,t) = u(x- C_n t/2)$$

is a solution of the generalized KdV equation

(4.20) $$u_t + \sum_{k=2}^{n+1} C_{n+1-k} P_{k_x} = 0.$$

Proof is easy.

5. Conservation laws of the KdV equation. Suppose that u varies according to the KdV equation (2.16). Let

$$X = \begin{pmatrix} \psi_1 & \psi_2 \\ \psi_{1_x} & \psi_{2_x} \end{pmatrix}$$

be a fundamental matrix solution of (2.2) such that $d\, Q_X(t;\lambda)/dt = 0$. Then $X(x,t;\lambda)$ satisfies (2.6), Here,

(5.1) $$H = \begin{pmatrix} u_x & -2u + 4\lambda \\ u_{xx} + 2(u+\lambda)(u-2\lambda) & - u_x \end{pmatrix}.$$

Hence we have

(5.2) $$\psi_{i_t} = u_x\psi_i - 2(u-2\lambda)\psi_{i_x}, \quad i = 1,2.$$

Let $\eta(x,t;\lambda)$ be any linear combination of the products ψ_1^2, $\psi_1\psi_2$ and ψ_2^2. Then, from (5.2) we have

$$\eta_t = 2u_x\eta - 2(u-2\lambda)\eta_x. \tag{5.3}$$

On the other hand, from Lemma 5, we get

$$\eta_{xxx} + 4(u+\lambda)\eta_x + 2u_x\eta = 0. \tag{5.4}$$

Combining (5.3) and (5.4), we obtain

$$\eta_t + 6\,u\,\eta_x + \eta_{xxx} = 0. \tag{5.5}$$

In virtue of the Floquet's theorem, we see that at least one linear combination of the products ψ_1^2, $\psi_1\psi_2$, ψ_2^2 is periodic with period 1. For such a periodic , we have

$$\frac{d}{dt}\int_0^1 \eta(x,t;\lambda)dx = 0 \tag{5.6}$$

Hence, $\int_0^1 \eta(x,t;\lambda)dx$ is a constant of motion of the KdV equation. Since $\eta(x,t;\lambda)$ is a solution of (5.4), we have an asymptotic expansion

$$\eta(x,t;\lambda) \sim \sum_{k=0}^{\infty} p_k(x,t)\,\lambda^{-k}, \quad \text{as } |\lambda| \longrightarrow \infty, \tag{5.7}$$

where $p_k(x,t)$ are determined by a recursion formula :

$$p_{k_{xxx}} + 4up_{k_x} + 2u_xp_k = -4p_{k+1_x},\ p_{0_x} = 0, \tag{5.8}$$

which is the same form as (2.10) or (4.13). From (5.5), (5.7) and (5.8), we obtain an infinite number of conservation laws :

$$p_{k_t} - (12\,p_{k+1} - 2p_{k_{xx}} - 6\,up_k)_x = 0. \tag{5.9}$$

Thus, we see that p_k are conserved densities of the KdV equation.

Remark 3. In much the same way, we obtain the conservation laws for the generalized KdV equation.

6. Special periodic solutions of the KdV equation.

We shall discuss an analogue of Lax's work for multi-soliton solutions of the KdV equation in the periodic case.

Theorem 5. If at some time t_0 the solution $u(x,t_0)$ of the KdV equation (2.16) satisfies the ordinary differential equation (4.1) with (4.2), then for all t, u(x,t) satisfies (4.1) with (4.2). In other words, at some t_0, the Hill's equation with the potential $u(x,t_0)$ has only n+1 instability intervals, then for all t, the Hill's equation with the potential u(x,t) has also only n+1 instability intervals while u(x,t) varies according to the KdV equation (2.16).

Remark 4. Detailed study has been done by Lax [6] on these special periodic solutions.

Remark 5. Explicit forms of the solutions of (4.1) with (4.2) may be found by solving the inverse problem for the Hill's equation. (see Novikov and Dubrovin [7], Its and Matveev [3]).

6. The trace of Green's function and constants of motion of the KdV equation.

In section 3, we see that the resolvent kernel for the Schrödinger operator $L(t) = d^2/dx^2 + u(x,t)$ on the whole real axis is given by

$$G(x,y;\lambda,t) = \frac{\varphi(1;\lambda,t)}{2\sqrt{\Delta^2(\lambda,t)-1}} \begin{cases} \psi_-(x;\lambda,t)\psi_+(x;\lambda,t) & x \leqq y \\ \psi_+(x;\lambda,t)\psi_-(x;\lambda,t) & x > y. \end{cases}$$

The trace kernel $G(x,x;\lambda,t)$ satisfies (5.4) and hence, is a generating function for the conserved densities for the KdV equation.

We now consider the periodic eigenvalue problem with a parameter t :

$$(6.1)\qquad \begin{cases} \psi_{xx} + u(x,t)\psi = -\lambda\psi \\ \psi(x+1) = \psi(x). \end{cases}$$

It is known that there exist a complete system of orthonormal eigenfunctions $\{\psi_j(x,t)\}$ and eigenvalues $\{\lambda_j(t)\}$ which are the zeros of $\Delta(\lambda,t) - 1$. If λ lies in the resolvent set of $L(t) = d^2/dx^2 + u(x,t)$ with periodic boundary condition, we have

$$(6.2)\quad R_\lambda(t)\varphi = (L(t)-\lambda I)^{-1}\varphi = \int_0^1 G_p(x,y;\lambda,t)\varphi(y)dy, \quad \forall\varphi \in L^2(0,1),$$

where

$$(6.3)\qquad G_p(x,y;\lambda,t) = -\sum_{j=0}^{\infty} \frac{1}{\lambda + \lambda_j} \psi_j(x,t)\psi_j(y,t).$$

The resolvent $R_\lambda(t)$ lies in the trace class and its trace is given by

$$(6.4)\qquad \mathrm{tr}\, R_\lambda(t) = \int_0^1 G_p(x,x;\lambda,t)dx = d \ln D(\lambda)/d\lambda,$$

where

$$(6.5)\qquad D(\lambda) = \prod_{j=0}^{\infty} (1 + \lambda/\lambda_j)$$

is the Fredholm's denominator. Since $\lambda_j(t)$ does not depend on t if u(x,t) evolves according to the KdV equation, $\mathrm{tr}\, R_\lambda(t)$ is a constant of motion of the KdV equation.

Theorem 6. We have an asymptotic expansion

$$(6.6)\qquad R_\lambda(t) = \sum_{k=0}^{\infty} (-1)^k \lambda^{-k-1/2} \int_0^1 \tilde{P}_k(x,t)dx$$

and hence,

$$\int_0^1 \tilde{P}_k(x,t)dx$$

are constants of motion of the KdV equation.

We have the asymptotic behavior of eigenvalues for large n :

(6.7) $$\lambda_{2n}, \lambda_{2n-1} \sim (2\pi n)^2 + c_0 + c_1(2\pi n)^{-2} + c_2(2\pi n)^{-4} + \cdots,$$

From Theorem 6, we obtain

Theorem 7. We have

(6.8) $$\sum_{r=1}^{m} \binom{m-1/2}{r} \quad \Sigma' c_{\alpha_1} \cdots c_{\alpha_r} = \int_0^1 P_m(x,t)dx,$$

where summation Σ' is taken over $\alpha_1 + \cdots + \alpha_r = m-r$.

Remark 6. Lax's formalism says that if a one parameter family of selfadjoint operators $L(t)$ are mutually unitary equivalent, the spectral structure of $L(t)$ do not depend on t. If this takes place, the resolvents $R_\lambda(t)$ are also unitary equivalent and, furthermore, if $R_\lambda(t)$ has the trace, then $\mathrm{tr}R_\lambda(t)$ does not depend on t.

Remark 7. Let $u(x,t)$ be a smooth function which decays rapidly as $|x| \longrightarrow \infty$. Consider the Schrödinger operator $L(t) = d^2/dx^2 + u(x,t)$ on the whole real axis. Although, in this case the resolvent $R_\lambda(t)$ does not lie in the trace class, $R_\lambda(t) - R_\lambda^0$ $(R_\lambda^0 = (d^2/dx^2 - \lambda)^{-1})$ has the trace for $\arg \lambda = 0$ and $\lambda = \lambda_j$ (eigenvalues) and

(6.9) $$\mathrm{tr}(R_\lambda(t) - R_\lambda^0) = - d \ln a(\sqrt{\lambda})/d\lambda,$$

where

$$a(\sqrt{\lambda}) = 1 - \frac{1}{2i\sqrt{\lambda}} \int_{-\infty}^{\infty} e^{i\sqrt{\lambda}\, s} u(s,t)\varphi(s,\sqrt{\lambda},t)ds,$$

from which we can derive an infinite number of conserved densities of the KdV equation (see Zakharov and Faddeev [7]).

Theorem 8. The eigenvalues of (6.1) are independent of t if and only if for all t and $\lambda \in \mathbb{C} \setminus \{\text{eigenvalues}\}$

(6.10) $$\int_0^1 u_t(x,t)G_p(x,x;\lambda,t)dx = 0.$$

7. Derivation of the KdV series, II. We consider the following theta function

(7.1) $$G(x,y;s,t) = \sum_{j=0}^{\infty} e^{-\lambda_j(t)s} \psi_j(x,t)\psi_j(y,t),$$

which is a Green function of the heat equation of the form :

(7.2) $$\begin{cases} G_s = G_{xx} + u(x,t)G \\ \lim_{s\searrow 0} G(x,y;s,t) = \delta(x-y), \\ G(x+1,y;s,t) = G(x,y;s,t). \end{cases}$$

Theorem 9. We have the asymptotic expansion as $s\searrow 0$,

(7.3) $$G(x,x;s,t) \sim \sum_{i=0}^{\infty} s^{i-(1/2)} B_i(x,t),$$

where B_i are uniquely determined by $u(x,t)$ and its derivatives.

Theorem (Menikoff). The eigenvalues of (6.1) are constants as t varies if and only if $u(x,t)$ satisfies

(7.4) $$\int_0^1 u_t(x,t)G(x,x;s,t)dx = 0 \quad \text{for all } s > 0 \text{ and } t.$$

Using the asymptotic expansion (7.3), we get

(7.5) $$\int_0^1 B_{i_x}(x,t)G(x,x;s,t)dx = 0, \quad i = 0,1,2,\cdots.$$

Since B_i is a constant multiple of P_i, from (7.4) and (7.5), we have again the KdV series :

(7.6) $$u_t = \sum_{\text{finite}} C_i B_{i_x}.$$

Hence, if $u(x,t)$ varies according to the (generalized) KdV equation, then the eigenvalues of (6.1) are constants (which we denote by $\{\lambda_j\}$) and we get

(7.7) $$\sum_{j=0}^{\infty} e^{-\lambda_j s}(\psi_j(x,t))^2 \sim \sum_{i=0}^{\infty} s^{i-(1/2)} B_i(x,t).$$

8. Extensions. We first consider a linear periodic system with t as a parameter :

(8.1) $$\varphi_x = (A(x,t) + \lambda J)\varphi, \quad -\infty < x,t < +\infty ,$$

where $\varphi = \varphi(x,t;\lambda)$ is a complex n-column vector, $A(x,t)$ a complex $n \times n$ matrix function, λ a parameter and J is a complex $n \times n$ constant matrix. We assume that $A(x,t)$ is infinitely differentiable and periodic with period 1. Then, we have

Theorem 10. There exists a fundamental matrix solution $X(x,t;\lambda)$ of the equation (8.1) such that the monodromy matrix $Q_X(t;\lambda)$ does not depend on t if and only if there exists a n n matrix function $H(x,t;\lambda)$ which is smooth, periodic in x with period 1 and satisfies

(8.2) $$A_t - H_x + [A + \lambda J, H] = 0.$$

From (8.2), we obtain an infinite number of nonlinear evolution equations of the form

(8.3) $$A_t = [H_{k_x} - A, H_k], \quad k = 1,2,$$

where H_k are determined by the following recursion formula :

(8.4) $$[J, H_0] = 0,$$
$$- H_{k-1_x} + [A, H_{k-1}] + [J, H_k] = 0.$$

The Modified Korteweg-de Vries equation, the nonlinear Schrödinger equation and so on, are the special cases of the equation (8.3). They are naturally derived when one considers the case in which $A(x,t) + \lambda J$ lies in the Lie algebra of $SL(2;C)$.

We next consider multidimensional analogues of the KdV series. Let M be an oriented C compact Riemannian manifold without boundary, dim M = n. The volume element of M willbe denoted by dvol. Let E be a complex vector bundle over M provided with an Hermitian inner product (,). $C^{\infty}(E)$ denoted the smooth sections of E. $L^2(E)$ is the completion of $C^{\infty}(E)$ in the norm

$$\|\psi\| = \left(\int_M (\psi,\psi)_x \, dvol\right)^{1/2}.$$

Let $L(t) : L^2(E) \longrightarrow L^2(E)$ be a positive self-adjoint elliptic operator of order $2m$ such that $L(t) = L_0 + U(x,t)$, where $U(x,t) \in C^{\infty}(E\times R^1)$ and L_0 has the symbol

$$\sigma L_0 = \sum_{i=1}^{2m} a_i(x,\xi) : T_x^*M \longrightarrow \mathrm{Hom}(E_x,E_x).$$

Let $\{\varphi_j(x,t)\}$ and $\{\lambda_j(t)\}$ be the complete system of orthnormal eigensections and eigenvalues, respectively, of $L(t)$.

Theorem 11. The eigenvalues $\{\lambda_j(t)\}$ of $L(t)$ are constants as t varies if and only if

$$(8.5) \qquad \int_M \mathrm{trace}(U_t(x,t)G(x,x,s,t))\,dvol = 0,$$

where $G(x,y,s,t) \in \mathrm{Hom}(E_x,E_y)$ is the kernel function of $e^{-sL(t)}$ and of the form

$$(8.6) \qquad G(x,y,s,t) = \sum_{j=0}^{\infty} e^{-\lambda_j(t)s} \psi_j(x,t)\psi_j^*(y,t),$$

where * denotes the conjugate transpose.

We have the asymptotic expansion as $s \searrow 0$:

$$(8.7) \qquad \mathrm{trace}\, G(x,x,s,t) \sim \sum_{k=0}^{\infty} s^{-(n/2m) + (k/2m)} F_k(x,L_0)$$

from which it follows that

$$(8.8) \qquad \sum_{j=0}^{\infty} e^{-\lambda_j(t)s} \sim \sum_{k=0}^{\infty} s^{-(n/2m)+(k/2m)} \int_M F_k(x,L_0)\,dvol.$$

We have

$$(8.9) \qquad \int_M \mathrm{trace}(\nabla_x \cdot F_k(x,L_0)G(x,x,s,t))\,dvol = 0,$$

where ∇_x denotes a volume preserving infinitesimal transformation. Hence we obtain multidimentional analogues of the KdV series :

$$U_t = \sum_{\text{finite}} C_k \nabla_x \cdot F_k(x, L_0),$$

with some additional conditions.

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