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Remarks on the relation between the Lee-Yang circle theorem and the correlation inequalities

by

Keiichi R. Ito

Research Institute for Mathematical Sciences

and

Department of physics

Kyoto University , Kyoto, Japan.

Abstract

We investigate the relation between the Lee-Yang circle theorem and the correlation inequalities.These results are general and independent of models. General properties of the partition functions which belongs to the Lee-Yang class are given.

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1. Introduction

Recently several authors have investigated the Euclidean boson quantum field models (the so-called $P(\phi)_d$-models) as a classical statistical mechanics [1],[2]. In these articles we see that the Lee-Yang circle theorem and the correlation inequalities do play a central role in the studying. On the other hand, Griffiths et al conjectured that a set of correlation inequalities will determine the forms of the interactions [3],[4] .From the view points of these applications and the conjectures, it is an interesting problem to decide the partition functions which satisfy the Lee-Yang circle theorem or the desired correlation inequalities.

Adding to these problems, Newman recently proved that the Lee-Yang circle theorem leads to some correlation inequalities [5]. Therefore it is also an interesting problem to discuss the relation between the Lee-Yang circle theorem and the correlation inequalities. Finally since the properties of the partition functions which satisfy the Lee-Yang circle theorem seem to be open, we investigate the general properties of them.

We organize the paper as follows:
In section 2, we define classes of the partition functions $\mathcal{L}_0$, $\mathcal{L}$, $\mathcal{D}$, $\mathcal{A}$,and summarize the relevant correlation inequalities without proof. In section 3, we investigate the

Griffiths first (G-I) and the second (G-II) inequalities and discuss the relation between these inequalities and the Lee-Yang circle theorem. In sections 4 and 5, we investigate the Griffiths-Hurst-Sherman inequality (GHS-inequality) and the Lebowitz inequality. In section 6, general properties of the partition functions which belong to the Lee-Yang class are given.

2. Classes $\mathcal{L}, \mathcal{D}, \mathcal{J}$.

We summarize notations and definitions used in the following [6] :

D ; unit desk $= \{ z \in C ; |z| \leq 1 \}$

∂D ; boundary of $D = \{ z \in C ; |z| = 1 \}$

$\mathcal{L}_e^{(n)}$ or $\mathcal{L}_e$; polynomials of n-variables $z_1, \cdots, z_n$ which are linear with respect to each z_i, and satisfy $P(z_1^{-1}, \cdots, z_n^{-1}) = P(z_1, \cdots, z_n) \prod z_i^{-1}$ with $P(0, 0 \cdots, 0) = 1$.

For the sake of the brevity, we restrict ourselves to the case where all the coefficients are real, thus $P \in \mathcal{L}_e$ is typically given by

$$P = (1 + z_1 z_2 \cdots z_n) + \Sigma \beta_i^{(1)} (z_i + z_1 z_2 \cdots \hat{z}_i \cdots z_n) + \Sigma \beta_{ij}^{(2)} (z_i z_j + z_1 z_2 \cdots \hat{z}_i \cdots \hat{z}_j \cdots z_n) + \cdots \quad (2\text{-}1)$$

with $\beta_{i_1 i_2 \cdots i_\ell}^{(\ell)} \in (-\infty, \infty)$.

Here $\hat{z}_i$ (or $\hat{i}$) means that the variable z_i should be omitted.

$\mathcal{L}^{(n)}$ or $\mathcal{L}$; the Lee-Yang class $\subset \mathcal{L}_e$. We say that $P \in \mathcal{L}_e$ belongs to $\mathcal{L}$ provided that any root of $P=0$ satisfies $z_i(z_j;\ j \neq i) \in D^c$ provided $z_j \in D$ $(j \neq i)$ and $z_k \in D^o$ for some k $(k \neq i)$.

$\mathcal{J}$; Set of $P \in \mathcal{L}_e$ such that all the roots of $P(z,..,z)=0$ lies on ∂D. Obviously $\mathcal{J} \supset \mathcal{L}$.

These definitions are general and independent of models. In order to define class $\mathcal{D}$, we use the Ising model of spin 1/2 where there are only ferromagnetic pair interactions :

$$H_\Lambda = - \Sigma_{i<j}\ J_{ij}\ (s_i s_j - 1)/2 - \Sigma h_i (s_i + 1)/2 \qquad (2\text{-}2)$$

where $0 \leq J_{ij} \leq \infty$ and s_i $(i \in \Lambda)$ is a random variable at the lattice site $i \in \Lambda$ which takes the values ± 1 . Let P be the relevant partition function;

$$P = \Sigma_{\{s_i = \pm 1\}} \exp(-H_\Lambda) \qquad (2\text{-}3)$$

with $z_i = \exp\ (h_i)$.

Therefore P is given by the coefficients

$$\beta^{(\ell)}_{i_1 i_2 \cdot\cdot i_\ell} = \Pi_{i=i_1}^{i_\ell}\ \Pi_{j \in \Lambda ; j \notin \{i_1 i_2 \cdot\cdot i_\ell\}}\ \gamma_{ij} \qquad (2\text{-}4)$$

with $\gamma_{ij} = \exp\ (-J_{ij})$.

Then obviously $0 \leq \gamma_{ij} \leq 1$, however we extend this as $-1 \leq \gamma_{ij} \leq 1$, and denote the resultant set by $\mathcal{D}$.

For $P \in \mathcal{L}_e$, we identify P with its coefficients $\{\beta^{\ell}_{i_1 i_2 \cdot\cdot i_\ell}\}$ $\in R^d$ $(d = 2^{n-1} - 1)$, and consider the sets of functions $\mathcal{L}_e$ and $\mathcal{L}$ as the set of the coefficients. Inthis sense ,we denote the convex hulls of $\mathcal{L}, \mathcal{D}, \mathcal{J}$ by $\hat{\mathcal{L}}, \hat{\mathcal{D}}$ and $\hat{\mathcal{J}}$ respectively, and the closures of $\mathcal{L}$ by $\bar{\mathcal{L}}$.

Finally in order to study the correlation inequalities, we sometimes restrict ourselves to the subsets where all the coefficients are real non negative . We denote these by $\mathcal{L}_e^+, \mathcal{L}^+$ and $\mathcal{D}^+$ respectively.

Now we define the so-called ursell functions:
for $P \in \mathcal{L}_e^+$, we define

$$\left.\begin{aligned} u^{(\ell)}(i_1,\cdots,i_\ell) &= \left(\Pi_{i=i_1}^{i_\ell} z_i\partial/\partial z_i\right)\log P \quad \ell\geq 2 \\ u^{(1)}(i) &= z_i\,\partial/\partial z_i \log P - 1/2 \end{aligned}\right\} \quad (2\text{-}5)$$

As is well known $\mathcal{D}\subset\bar{\mathcal{L}}$ ($\mathcal{D}^+\subset\bar{\mathcal{L}}^+$), and for $P\in\mathcal{D}^+$, we see [3],[4],[7] [8],[9] :

Griffiths first inequality ; $u^{(1)}(i)\geq 0$ for $z_j \geq 1, j\in\Lambda$,

Griffiths second inequality ; $u^{(2)}(i,j) \geq 0$ for $z_j \geq 1, j\in\Lambda$,

GHS-inequality ; $u^{(3)}(i_1,i_2,i_3)\leq 0$ for $z_j \geq 1$, $j\in\Lambda$,

Lebowithz inequality; $u^{(4)}(i_1,i_2,i_3,i_4)\leq 0$ for $z_j=1, j\in\Lambda$,

Sylvester inequality ; $u^{(6)}(i_1,\cdot\cdot,i_6) \geq 0$ for $z_j=1, j\in\Lambda$,

where $\Lambda = \{1,2,\cdots,n\}$.

Following inequalities are conjectured by Newman for $P\in\mathcal{D}^+$[5],[8]:

$$(-1)^{\ell-1}\, u^{(2\ell)}(i_1,\cdot\cdot,i_{2\ell})\geq 0 \text{ for } z_j=1, j\in\Lambda.$$

C_i ; The set of the partition functions $P\in\mathcal{L}_e^+$ which satisfy the expected inequality for the i'th ursell function.

3. $\mathcal{L}^+$ and $u^{(1)}, u^{(2)}$

Lemma 1. Let $P\in\mathcal{L}^+$, then $u^{(1)}(i) \geq 0$ provided $z_j\geq 1$, $j\in\Lambda$.

proof. Let P be given by

$$P=B(z_1,\cdots,z_{n-1}) + A(z_1,\cdots,z_{n-1})z_n$$

where A,B are linear functions of $z_1,\cdots,z_{n-1}$ with positive coefficients. $P\in\mathcal{L}$ implies

$$|B/A|\leq 1 \text{ provided } |z_i|\geq 1 \quad i=1,2,\cdots,n-1.$$

On the other hand,

$$u^{(1)}(n)=z_n(Az_n-B)/2P \;.$$ ▨

Lemma 2. For $P\in\mathcal{L}_e$, followings hold:

(i) If $u^{(1)}(i)\geq 0$ provided $z_j\geq 1$, $j\in\Lambda$, then

$$u^{(2)}(i_1,i_2)|_{z=1}\geq 0 \;.$$

(ii) If $u^{(2)}(i_1,i_2)\geq 0$ provided $z_j\geq 1, j\in\Lambda$, then

$$u^{(1)}(i_1)\geq 0 \text{ provided } z_j\geq 1, j\in\Lambda$$

proof. (i) Let all z except z_{i_2} be equal to 1. Since $P\in\mathcal{L}_e$, $u^{(1)}(i_1)=(z_{i_2}-1)f(z_{i_2})$ where the G-1 inequality ensures $f(z_{i_2})\geq 0$ provided $z_{i_2}\geq 1$. Thus $u^{(2)}(i_1,i_2)|_{z=1}=f(1)\geq 0$.
(ii) Since $P\in\mathcal{L}_e$, $u^{(1)}(i_1)|_{z=1}=0$. ▨

However, unfortunately $P\in\mathcal{L}^+$ does not necessarily imply the second Griffiths inequality with positive external fields, i.e., $P\in\mathcal{L}^+$ does not imply

$$u^{(2)}(i_1,i_2)\geq 0 \text{ with } z_j\geq 1 \quad ,j\in\Lambda.$$

An explicit counterexample is given in the next section.

Finally for $P\in\mathcal{L}^+$, we can show the correlation inequalities which correspond to $\langle s_1s_2\cdots s_\ell\rangle\geq 0$ provided $h_i\geq 0, i\in\Lambda$. This is the G-1 inequality in usual sense.

Theorem 1. Let $P(z_1,\cdot\cdot,z_n)\in\mathcal{L}^+$ then

$$\Pi_{i\in S}(z_i\ \partial/\partial z_i)[P(z_1,\cdot\cdot,z_n)(\Pi_{i=1}^{n}z_i)^{-1/2}]\geq 0 \quad (3\text{-}1)$$

provided $z_i\geq 1$, $i\in\Lambda$ where $S\subset\Lambda$ denotes the set of indices.

proof. It is sufficient to consider the case that all the indices are different. Let P be given by

$$P=\Sigma_{\{i_1,i_2,\cdot\cdot,i_\ell\}\subset S}\ a_{i_1,i_2..i_\ell}z_{i_1}z_{i_2}..z_{i_\ell}$$

where $\{a_{i_1,i_2,\cdot\cdot,i_\ell}\}$ are linear functions of $z_j\in\Lambda\backslash S$ with positive cofficients. Then

$$\Pi_{i\in S}(z_i\ \partial/\partial z_i)[P\ \Pi_{i=1}^{n}z_i^{-1/2}]=[2^{|S|}\Pi_{i=1}^{n}z_i^{1/2}]^{-1}Q,$$

where

$$Q=2^{|S|}z^S P_S-2^{|S|-1}\Sigma_{i\in S}z^{S\backslash i}P_{S\backslash i}+2^{|S|-2}\Sigma_{i,j\in S}z^{S\backslash\{i,j\}}P_{S\backslash(i,j)}$$
$$+\cdots+(-1)^{|S|}P$$
$$=\Sigma_{\{i_1,i_2,\cdot\cdot,i_\ell\}=I\subset S}(-1)^{|S|-|I|}a_I z^I \quad (3\text{-}2)$$

with

$$a_I=a_{i_1,i_2,\cdot\cdot,i_\ell}\ ;\ z^I=\Pi_{i\in I}z_i;\ P_I=\Pi_{i\in I}\ \partial/\partial z_i\ P$$

Let $z_j\in\Lambda\backslash S$ be fixed and ≥ 1, thus we study the necessary and sufficient condition that ensures $Q\geq 0$ provided that $z_i\geq 1$, $i\in S$. Following Lemma 4, which will be proved later, this is

(1) $a_{1,2,..,\ell}\geq 0$

(2) $a_{1,2,..,\ell}-a_{1,2,.\hat{i}.\ell}\geq 0 \quad i\in S$,

(3) $a_{1,2,..,\ell}-a_{1,2,.\hat{i}.\ell}-a_{1,2,.\hat{j}.\ell}+a_{1,2,.\hat{i}.\hat{j}.\ell}\geq 0, i,j\in S,$

....

(ℓ+1) $a_{1,2,..\ell}-\Sigma_{i\in S}a_{1,2,.\hat{i}.\ell}+\Sigma_{i,j\in S}a_{1,2,.\hat{i}.\hat{j}.\ell}$
$+\cdots\cdot+(-1)^{\ell}a_{\hat{1},\hat{2},..\hat{\ell}}\geq 0$.

Here without loss of generality, we put $S=\{1,2,3,..,\ell\}\subset\Lambda$.

These conditions are equivalent to

$$(-1)^{|S\setminus I|} P_I(z_j=-1;j\in S\setminus I)$$
$$= (-1)^{|S\setminus I|} (\Pi_{i\in I}\partial/\partial z_i) P|_{z_j=-1;j\in S\setminus I} \geq 0$$

for any subset $I\subset S$. Since $P\in\mathcal{L}^+$, all the roots of $P_I(z_j=z;j\in S\setminus I)=0$ lie in the unit disk D. Now we investigate the sign of $P_I(-1)$.

$$P_I(z)= a_S z^{|S\setminus I|}+\cdots+a_I = \Sigma_{J;I\subset J\subset S}\, a_J\, z^{|J\setminus I|}$$
$$= a_S \Pi_{i=1}^{r}[(z-\omega_i)(z-\bar{\omega}_i)]\, \Pi_{j=1}^{|S\setminus I|-2r}(z-\zeta_j)$$

where $\{\omega_i,\bar{\omega}_i \; |\omega|\leq 1\}$ are the complex roots, and $\{\zeta_j;|\zeta|\leq 1\}$ are the real roots of $P_I=0$. Since $a_S>0$, $(-1-\omega_i)(-1-\bar{\omega}_i)>0$ and $\mathrm{sgn}\, \Pi_{j=1}^{|S\setminus I|-2r}(-1-\zeta_j) = \mathrm{sgn}\,(-1)^{|S\setminus I|}$, we see

$$(-1)^{|S\setminus I|} P_I(-1) \geq 0.$$

This completes the proof. ▨

Finally we would like to point out that if $P\in\mathcal{L}$, $u^{(1)}(i)$ or $<s_1 s_2 \cdots s_\ell> \propto \Pi_{i\in S}\, z_i\partial/\partial z_i\, [P\, \Pi z_j^{-1/2}]$ also satisfy the definition of $\mathcal{L}$ except the eveness condition $P(z_1^{-1},\ldots,z_n^{-1})=P(z_1,\ldots,z_n^{-1})\,\Pi z_i^{-1}$. This is obvious because if $P(z_1,\ldots,z_n)\in\mathcal{L}$, $P(\exp(i\theta_1)z_1,\ldots,\exp(i\theta_n)z_n)$ with $\theta_i\in R$ again satisfies the definition of $\mathcal{L}$ except the eveness condition, and these correlation functions are essentially given by (3-2). However, this is not true for the higher order ursell functions. In fact if it were true, the higher order ursell functions would have definite signs in $\{z_i \geq 1; i\in\Lambda\}$.

4. $\mathcal{L}^+$ and $u^{(3)}$, $u^{(4)}$

In the cases of n= 1,2, $\bar{\mathcal{L}} = \mathcal{D}$ $(\bar{\mathcal{L}}^+ = \mathcal{D}^+)$, and $C_i = \mathcal{D}^+ = \mathcal{L}^+$, (i=1,2,3,4) . In the case of n= 3, we will easily see that $\bar{\mathcal{L}} = \hat{\mathcal{D}}$ $(\bar{\mathcal{L}}^+ = \hat{\mathcal{D}}^+)$, and $P \in \mathcal{L}^+$ does not imply the desired inequality .

Lemma 3. Let $P \in \mathcal{L}_e$, and be given by

$$P = 1 + z_1 z_2 z_3 + \Sigma_{i=1}^{3} \beta_i (z_i + z_1 \hat{z}_i z_3) . \tag{4-1}$$

Then $P \in \mathcal{L}$ if and only if

$$|1 \pm \beta_i| > |\beta_j \pm \beta_k| . \tag{4-2}$$

proof. It is necessary and sufficient that

$$(z_3)^{-1} = -[\beta_3 + \beta_1 z_1 + \beta_2 z_2 + z_1 z_2]/[1 + \beta_1 z_1 + \beta_2 z_2 + \beta_3 z_1 z_2] \in D^o$$

provided $z_1, z_2 \in D$ and some of them $\in D^o$. Remark that the Shilov boundary of the polydisk $D \otimes D \otimes \ldots \otimes D$ is $\partial D \otimes \ldots \otimes \partial D$. Since $z_3 \in \partial D$ provided $(z_1, z_2) \in \partial D \times \partial D$, the problem reduces to obtain a condition which is equivalent to

$$P(z_1, z_2, z_3 = 0) = 1 + \beta_1 z_1 + \beta_2 z_2 + \beta_3 z_1 z_2 \neq 0$$

provided $(z_1, z_2) \in D \otimes D$.

Therefore

$$|z_2|^{-1} = |(\beta_2 + \beta_3 z_1)/(1 + \beta_1 z_1)| < 1 \text{ provided } z_1 \in D.$$

This completes the proof. ▨

Now we investigate the correlation inequalies for $P \in \mathcal{L}$ given in the previous lemma 3. When all the arguments are different with each other, we have

$$u^{(3)}(1,2,3) \, z_1 z_2 z_3 \Pi \partial / \partial z_i \log P = z_1 z_2 z_3 \, f_3 / P^3$$

where

$$f_3 = P^2 P_{1,2,3} - P\Sigma\, P_i P_{j,k} + P_1 P_2 P_3$$

$$= s_o(1-z_1z_2z_3) + s_1(z_1 - z_2z_3) + s_2(z_2 - z_3z_1) + s_3(z_3 - z_1z_2)$$

with

$$\left.\begin{aligned} s_0 &= 1-\beta_1^2 - \beta_2^2 - \beta_3^2 + 2\beta_1\beta_2\beta_3 \,, \\ s_1 &= \beta_1(1-\beta_1^2 + \beta_2^2 + \beta_3^2) - 2\beta_2\beta_3 \,, \\ s_2 &= \beta_2(1 + \beta_1^2 - \beta_2^2 + \beta_3^2) - 2\beta_3\beta_1 , \\ s_3 &= \beta_3(1 + \beta_1^2 + \beta_2^2 - \beta_3^2) - 2\beta_1\beta_2 \,. \end{aligned}\right\} \qquad (4\text{-}3)$$

Lemma 4. Let

$$f = a_o z_1 z_2 \ldots z_n + \Sigma a_i z_1 z_2 \ldots \hat{z}_i \ldots z_n + \Sigma a_{ij}\, z_1 z_2 \ldots \hat{z}_i \ldots \hat{z}_j \ldots z_n$$
$$+ \ldots + a_{1,2,\ldots n} \,.$$

Then the necessary and sufficient condition that $f \geq 0$ provided $z_i \geq 1$ $(i \in \Lambda)$ is

$$a_o \geq 0$$
$$a_o + a_i \geq 0 \quad i \in \Lambda$$
$$a_o + a_i + a_j + a_{i,j} \geq 0 \quad i,j \in \Lambda$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_o + \Sigma a_i + \Sigma a_{i,j} + \ldots + a_{1,2,\ldots n} \geq 0$$

proof. Remark that f is a linear function with respect to each variable. Therefore the necessary and sufficient condition that $f \geq 0$ provided $z_i \geq 1$ is

$$f(z_1, z_2, \ldots, z_{n-1}, 1) \geq 0$$
$$\partial/\partial z_n\, f(z_1, \ldots, z_n) = \partial/\partial z_n f(z_1, \ldots, z_n)\,|_{z_n=1} \geq 0$$

provided $z_i \geq 1$, $i \in \Lambda$. This discussion leads to the following condition:

for any $I \subset \Lambda$, $\left(\prod_{i \in I} \partial/\partial z_i\right) f|_{z=1} \geq 0$. ▨

Theorem 2. For $P \in \mathcal{L}^+$, we see:

(i) $u^{(3)}(1,2,3) \leq 0$ provided $z_1, z_2, z_3 \geq 1$ and $z_4 = z_5 = \ldots = z_n = 1$.

(ii) $u^{(2)}(1,2)$ is not necessarily positive provided $z_i \geq 1$, however, $u^{(2)}(1,2) \geq 0$ provided $z_1, z_2 \geq 1$ and $z_3 = z_4 = \ldots = z_n = 1$.

(iii) For $P \in \mathcal{L}_e^+$, let $\tilde{P} = P|_{z_4 = \cdots = z_n = 1}$, then for $\tilde{P}$ we see that if $u^{(1)}(i) \geq 0$ provided $z_j \geq 1$ $(j=1,2,3)$, $\tilde{P} \in \bar{\mathcal{L}}^+$.

proof. (i) It is sufficient to consider P given by (4-1). Thus following lemma 4 and (4-3), we must prove

$$s_0 + s_i + s_j - s_k \geq 0, \qquad (i,j,k) = (1,2,3)$$

$$s_0 + s_i \geq 0, \qquad (i = 1,2,3)$$

$$s_0 \geq 0$$

provided $(\beta_1, \beta_2, \beta_3) \in \mathcal{L}^+$. It is a straightforward but tedious calculation.

(ii) We present an example. For P given by (4-1),

$$u^{(2)}(1,2) = z_1 z_2 [(\beta_3 - \beta_1\beta_2)(1 + z_3^2) + (1 - \beta_1^2 - \beta_2^2 + \beta_3^2) z_3] / P^2.$$

Thus obviously $u^{(2)} \geq 0$ provided $P \in \mathcal{L}^+$ and $z_3 = 1$ (this includes general cases), however, consider the point $(1/3, 1/3, 0) \in \bar{\mathcal{L}}^+$. At the point $\beta_3 - \beta_1\beta_2 = -1/9$, then if z_3 is large enough, we see $u^{(2)} < 0$.

(iii) Following lemma 4, the necessary sufficient condition that $u^{(1)}(i) \geq 0$ provided $z_j \geq 1$ $(j=1,2,3)$ is

$$1 + \beta_i \geq \beta_j + \beta_k \qquad (i,j,k) = (1,2,3)$$

Finally we investigate the fourth ursell function [7],[8]:

$$u^{(4)}(1,2,3,4) = \prod_{i=1}^{4} (z_i \partial / \partial z_i) \log P.$$

As is well known, even if $P \in \mathcal{D}^+$, $u^{(4)}$ does not necessarily negative when $z_i \geq 1$, but negative provided $z_i=1$ (the so-called Lebowitz Inequality). Contrary to the case of $u^{(2)}$, $u^{(4)}$ is not necessarily negative even if $P \in \mathcal{L}^+$ and $z_i=1$ ($i \in \Lambda$). In fact let $P(z_1,z_2,z_3,z_4) = P\,|_{z_5=\cdots=z_n=1}$

$$= \text{const.}[(1+z_1..z_4) + \Sigma\ \beta_i(z_i+z_1..\hat{z}_i..z_4) + \Sigma\ \beta_{ij}(z_iz_j+z_1\hat{z}_i\hat{z}_jz_4)\ ] \qquad (4\text{-}4)$$

Thus

$$u^{(4)}(1,2,3,4)=z_1z_2z_3z_4[P^3\ P_{1,2,3,4} - P^2\Sigma P_{ij}P_{kl} - P^2\Sigma P_iP_{jkl} + 2P\Sigma P_{ij}P_kP_l - 6P_1P_2P_3P_4]\ /P^4 \qquad (4\text{-}5)$$

and

$$u^{(4)}(1,2,3,4)\,|_{z=1} = \text{positive const.}[\{(\Sigma\ \beta_i)^2/2 - 2\Sigma\ \beta_i\beta_j\ -1/2\} + \{(\ \Sigma\ \ \beta_{ij})^2/2 - \Sigma\ \ \beta_{ij}^2 + \Sigma\beta_{ij}\ \}] \qquad (4\text{-}6)$$

A point ($\beta_i=0$, $\beta_{ij}=1/3) \in R^7$ is $\in \mathcal{L}^+$, but at the point $[\ ]=2/3>0$. Thus we see that $P \in \mathcal{L}^+$ implies neither $u^{(4)} \leq 0$ with zero external fields nor $u^{(3)} \leq 0$ with positive external fields (see the next section).

5. Some remarks on the correlation inequalities.

Now we see that the partition functions which belong to $\mathcal{L}^+$ do not necessarily satisfy the correlation inequalities expected from the results seen in $P \in \mathcal{D}^+$. The reason is obvious, in fact $P \in \mathcal{L}^+$ is a property which is derived from the behavior of P on $D\otimes D\otimes..\otimes D \subset C^n$, and on the other hand correlation inequalities crucially depend on the behavior on $[1,\infty)^n \subset R^n$. Our examples suggest

$$\mathcal{D}^+ \subset \bigcap_i \begin{matrix} C_i \\ \cup \\ C_i \end{matrix} \subset \bar{\mathcal{L}}^+ \qquad (5\text{-}1)$$

However, Newman showed [5]

$$(-1)^{\ell-1}(z\,\partial/\partial z)^{2\ell}\log P(z,z,..,z)_{|z=1} \geq 0 \qquad (5\text{-}2)$$

provided $P \in \mathcal{D}$.

From our standing point of view, these are special ursell functions.

Our analysis implies that $\mathcal{L}^+$-class is too wide to satisfy all the correlation inequalities . Finally we show that the even'th correlation inequalities with zero external fields follow from the odd'th correlation inequalities with positive external fields (see also the note-added in [9]).

Theorem 3. For $P\in\mathcal{L}_e$, if $u^{(3)}(i,j,k) \leq 0$ with positive external fields holds, $u^{(4)}(i,j,k,l)_{z=1} \leq 0$.

proof.

$$u^{(4)}(i,j,k,l) = z_l\,\partial/\partial z_l\; u^{(3)}(i,j,k)_{|z=1}$$

Since we can put all z except z_l equal to 1 in $u^{(3)}$ and $P\in\mathcal{L}_e$, then we have

$$u^{(3)}(i,j,k) = (1-z_l)f(z_l) \qquad (5\text{-}3)$$

where the GHS inequality ensures $f(z_l) \geq 0$ provided $z_l \geq 1$. This compltes the proof. ▨

Remark. As is well known, the higher order ursell functions do not satisfy the expected inequality for $z_i \geq 1$

even for $P\in\mathcal{D}^+$. However, if they satisfy the conjectured inequalities (including odd'th ursell functions) for $1\leq z_i < 1+\varepsilon$ with $\varepsilon>0$, $i\in\Lambda$, we see that $(-1)^{\ell-1}u^{(2\ell)}|_{z=1}\geq 0$ can be derived from $(-1)^{\ell-1}u^{(2\ell-1)}\geq 0$ with $1\leq z_i<1+\varepsilon$. If this is true, $u^{(2\ell)}$ and $u^{(2\ell-1)}$ should be considered as a pair. See also the dicussions in lemma 2, and by the same discussions, we see that the converse is true.

Corollary 1. For $P\in\mathcal{L}^+$, $u^{(4)}(i,j,k,l)|_{z=1}\leq 0$ provided that at least two of (i,j,k,l) are equal.

proof. (i) Two arguments are equal.

Following Theorem 2 and Theorem 3, it is obvious.

(ii) Three arguments are equal.

Without loss of generality, let $(i,j,k,l)=(1,2,2,2)$. Thus $u^{(4)}|_{z=1}=(PP_{1,2}-P_1P_2)(P^2+6P_2^2-6PP_2)\,/\,P^4|_{z=1}$. This is negative since $P\in\mathcal{L}^+$. Finally if all the arguments are equal, the problem reduces to $P=\text{const.}(1+z)$. ▨

6. Structure of $\mathcal{L}$ and $\mathcal{D}$.

Before studying the topological structure of $\mathcal{L}$, we would like to point out that a product can be defined on $\mathcal{L}$ [6],[10],[11] .We call this product the Asano product.

Theorem 4. Let $\{\alpha^{(\ell)}_{i_1,..i_\ell}\}\in\mathcal{L}^{(n)}$, $\{\beta^{(\ell)}_{i_1,..i_\ell}\}\in\mathcal{L}^{(n)}$ then $\{\alpha^{(\ell)}_{i_1,..i_\ell}\beta^{(\ell)}_{i_1,..,i_\ell}\}\in\mathcal{L}^{(n)}$

This is a very well known theorem, and we do not repeat the proof. Details are shown in [6],[10],[11]. Therefore $\mathcal{L}$ has a semi-group structure by this product. Remark that $\mathcal{D}$ is also closed under the product. We denote this product by $\{\alpha\beta\}$ or $A[P_\alpha P_\beta]$.

The Lee-Yang class $\mathcal{L}$ is a much complicated set in the space of the d-coefficients ($d=2^{n-1}-1$). Let P be given by (2-1), then we identify P with $\{\beta^{(1)}_{1},\ldots,\beta^{(2)}_{1,2}\ldots\ldots\}\in R^d$.

Lemma 5. If $P\in\mathcal{L}$, then $-1\leq \beta^{(\ell)}_{i,.}\leq 1$.
proof. If $P\in\mathcal{L}$, then $A[P^N]=A[PA[\ldots.A[PP]]..]\in\mathcal{L}$. $A[P^N]$ is given by $\{\beta^{(\ell)N}_{i\ ..}\}$. Since all the coefficients of $\mathcal{L}$ must be bounded, $|\beta^{(\ell)}_{i,.}|\leq 1$. This completes the proof. ▨

Theorem 4.

(i) $\mathcal{L}$ is an open ,arcwise-connected set.

(ii) $\mathcal{L}$ is homeomorphic to d-dimensional opendisk $D^{(d)}$.

proof. (i) The openess of $\mathcal{L}$ follows from the definition. Let $P_t\in\mathcal{D}$ be given by putting all γ_{ij} $t\in[0,1]$. THerefore $P_1\in\bar{\mathcal{L}}$ and $P_t\in\mathcal{L}$ for $t\in[0,1)$. P_t is a continuous line connecting $\Pi^n(1+z_i)\in\bar{\mathcal{L}}$ and $1+\Pi^n z_i\in\mathcal{L}$,and lies in $\mathcal{L}$. For any $P\in\mathcal{L}$, $\tilde{P}_t=A[P_tP]\in\mathcal{L}$ is continuous with respect to $t\in[0,1]$, and $\tilde{P}_1=P\in\mathcal{L},\tilde{P}_0=1+\Pi z_i\in\mathcal{L}$. Thus $\mathcal{L}$ is arcwise connected.

(ii) From the above discussions ,we see ,by operating $A[P_t\cdot\cdot]$, that any sub set of $\mathcal{L}$ can be continuously contracted to the origin. This completes the proof. ▨

Remark. Even if the coefficients are complex, these statements can be extended by suitable redefinitions [6].

The main theorem in this section is:

Theorem 5.

(i) $\mathcal{D} \subset \bar{\mathcal{L}} \subset \hat{\mathcal{D}}$,

(ii) $\mathcal{D}^+ \subset \bar{\mathcal{L}}^+ \subset \hat{\mathcal{D}}^+$,

(iii) $\hat{\mathcal{L}} = \hat{\mathcal{D}}$; $\hat{\mathcal{D}}^+ = \hat{\mathcal{L}}^+$.

proof . (i) $\mathcal{D} \subset \bar{\mathcal{L}}$, $\mathcal{D}^+ \subset \bar{\mathcal{L}}^+$ are well known. Consider the following (d+1) functions :

$$\Pi^n (1 \pm z_i)$$

where the number of (-) sign is even, and which ensures that these functions belong to $\mathcal{L}_e$. We denote these functions by P_i (i=1,..,d+1), and remark that $P_i \in \mathcal{D}$ ($P_i \in \bar{\mathcal{L}}$) and these are all linearly independent.Then $P = \Sigma\, \alpha_i P_i$ with $\alpha_i \geq 0$, $\Sigma\, \alpha_i = 1$ becomes a d-dimensional convex cell in the d-dimensional space of the coefficients. Thus,denoting this convex cell by $\hat{\mathcal{D}}'$,we show $\partial \hat{\mathcal{D}}' \notin \mathcal{L}$ .If once it is proved,(i) follows from theorem 4 and the fact $\mathcal{D} \subset \bar{\mathcal{L}}$. Each of $\partial \hat{\mathcal{D}}'$ is a (d-1) dimensional convex cell.We rewrite P as

$$P = P_{|z_n=0} + z_n\, \partial/\partial z_n\, P$$
$$= B(z_1, \ldots, z_{n-1}) + A(z_1, \ldots, z_{n-1})\, z_n$$

Since $P \in \mathcal{L}$ is equivalent to

$$|A/B| < 1$$

provided $z_i \in D$ and some $z_j \in D^o$, and $P \in \mathcal{L}_e$ implies $|A/B| = 1$ provided all $z_j \in \partial D$, it is necessary and sufficient that $B \neq 0$ provided $(z_1, z_2, \ldots, z_{n-1}) \in \partial D^{n-1}$.

B is given by by $\Sigma\alpha_i P_i \;|z_n=0$, and consider the point $(z_1,z_2,\ldots,z_{n-1})=(\pm1,\pm1,\ldots,\pm1)$. There are $2^{n-1}=$ [illegible] points. For the given point, the function which does not vanish at the point is one of the following two possible functions:

$$\Pi_{i=1}^{n-1}(1\pm z_i)(1+z_n)$$

$$\Pi_{i=1}^{n-1}(1\pm z_i)(1-z_n)$$

and only one of these functions belongs to $\mathcal{L}_e$.Thus d-functions of $\{P_i\}$ vanish at the point. Therefore $\mathcal{L} \not\ni \partial\hat{\mathcal{D}}'$,and (i) follows.

(ii) Each hypersurface of $\partial\hat{\mathcal{L}}$ is given by

$$P = \Sigma_{i=1}^{d}\alpha_i P_i \qquad \alpha_i \geq 0, \quad \Sigma_{i=1}^{d}\alpha_i=1.$$

Since we restrict ourselves to $\mathcal{L}^+$, one of P_i (i=1,2,..,d) is $\Pi_{i=1}^{n}(1+z_i)$.These hypersurfaces intersect the positive part of the coordinate-axis at the points

$$(1+\Pi_{i\in I}z_i)(1+\Pi_{j\in I^c}z_j)\in\mathcal{D}^+$$

$$I\neq\phi,\Lambda.$$

The convex hull of these points together with $(1+\Pi_i z_i)$ and $\Pi_i(1+z_i)$ includes $\mathcal{L}^+$.

(iii) This is obvious from the above discussions. ▨

Remark

(i) Set $\mathcal{L}$ is much complicated, and $\partial\bar{\mathcal{L}}$ is constructed by algebraic manifolds. $\mathcal{L}$ seems to be a "concave set ".To confirm the conjecture,consider the function $P=(1-\lambda)(1+\Pi z_i)+\lambda\Pi(1+z_i)$ with $0\leq\lambda\leq 1, n\geq 3$. $P\in\mathcal{L}^+$ if and only if $0\leq\lambda<[1+(2\cos\pi/(n-1))^{n-1}]^{-1}$.

(ii) One may define the vertices of $\bar{\mathcal{L}}$., $\mathcal{D}$. In order to define the vertices, however, we must use the terminologies of algebraic geometry. Since $\bar{\mathcal{L}}$ and $\mathcal{D}$ are semi-analytic sets, we can define the vertices as the zero dimensional singularities of $\partial\bar{\mathcal{L}}$, $\partial\mathcal{D}$. This is usually done through the stratification of the singularity. We conjecture

$$\text{(i) } \mathrm{Ver}\,\bar{\mathcal{L}} = \mathrm{Ver}\,\mathcal{D} \quad ,$$

$$\text{(ii) } \mathrm{Ver}\,\bar{\mathcal{L}}^{+} = \mathrm{Ver}\,\mathcal{D}^{+}.$$

These can be confirmed for n=1,2,3,4.

Finally we comment on the some interesting properties of $\mathcal{J}$. Let $P(z,\ldots,z)=(1+z^n)+a_1(z+z^{n-1})+a_2(z^2+z^{n-2})+\ldots\in\mathcal{J}$. Recently Millard et al have obtained a generalization of the Ruelle's lemma [6],[12]:

Theorem 6. Let A and B be closed circular regions not containing the origin. If $f=\sum_{i=0}^{n} b_n z^i$ vanishes only in $A\subset C$, and $g=\sum_{i=0}^{n} c_i z^i$ vanishes only in $B\subset C$, then $A[fg]\equiv \sum_{i=0}^{n} {}_nC_i^{-1} b_i c_i z^i$ vanishes only in $AB=\{z\in C; z=-z_1z_2, z_1\in A, z_2\in B\}$.

Therefore, using the same techniques in Theorem 4, we have:

Theorem 7.

(i) $\mathcal{J}$ is closed, arcwise-connected set and all of the homotopies of $\mathcal{J}$ vanish.

(ii) Let $\mathcal{F}_S=\{z^n+a_{n-1}z^{n-1}+a_{n-2}z^{n-2}+\ldots+a_1z+1; (a_{n-1},\ldots,a_1)\in R^{n-1}\}$ be functions whose roots are all in an open region $S\subset C$ which is invariant under the rotation around the origin. Then $\mathcal{F}_S$ is homeomorphic to (n-1)-dimensional open disk D^{n-1}.

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