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Dimension formula for the Landau singularity.

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Abstract Dimension formula for the Landau singularity corresponding to a Feynman graph whose vertices are all external is obtained and the codimension 1 intersections of Landau singularities are determined.

§1, Holonomy structure of Landau singularities.

In the micro-local study of S-matrix and related quantities ([2], [3], [4], [5], [6]) the following problem is fundamental.

Problem determine the holonomy structure of Landau singularities.

We shall give an answer to this problem in the case when the corresponding Feynman graph has no internal vertices.

Let G be a Feynman graph with $n(G)$ vertices and $N(G)$ internal lines.[1)] We assume that G has no internal vertices throughout this paper. Let $b_i(G)$ $(i=0,1)$ be the i-th Betti number of G. As is well known $b_0(G)$ is the number of connected components of G and $b_1(G)$ is the number of independent loops. We also denote by $\chi(G)$ the Euler characteristic of G, i.e. $\chi(G)= n(G) - N(G) = b_0(G) - b_1(G)$, and by $m(G)$ the integer $n(G) - b_0(G) = N(G) - b_1(G)$.

The leading Landau singularity Λ_G is an analytic set in

1) See [5] as for definitions and notations which we do not mention.

the $(p, i\infty x)$ space defined by the following Landau equations:

$$p_j - \xi_j(k) = 0 \qquad (j=1,\ldots,n),$$
$$k_\ell^2 - m_\ell^2 = 0, \qquad \eta_\ell(x) = \alpha_\ell \cdot k_\ell \qquad (\ell=1,\ldots,N). \tag{1}$$

It can be expressed by a (multi-valued) holomorphic function of homogeneous degree 1

$$H(x) = \sum_{\ell=1}^{N} m_\ell \sqrt{(\eta_\ell(x))^2} \tag{2}$$

as follows ([3], [6]).

$$\Lambda_G = \{(p:i\infty x) \mid p_j = \frac{\partial}{\partial x_j} H(x)\}. \tag{3}$$

This expression implies the following theorem.

Theorem 1. Λ_G is a non empty, irreducible holonomic manifold.

We denote by L_G the projection of Λ_G to the p-space. L_G is an analytic set contained in the linear submanifold $P_0 = \{\sum_{j=1}^{n} p_j = 0\}$ corresponding to the over-all energy-momentum conservation. Usually, physicists say that there are no leading singularities if the codimension of L_G in P_0 is more than one ([3], [6]). But in micro-local analysis we consider that the singularities of certain quantities lie not simply on L_G but on Λ_G, the conormal bundle of L_G. Therefore we cannot say that there are no leading singularities if only L_G is not a hypersurface in P_0. For example, the singularity structure of $\frac{1}{x_1+i0} + \frac{1}{x_2+i0}$ and that of $\frac{1}{x_1+i0} \cdot \frac{1}{x_2+i0}$ are different. The former has no singulatities on the conormal bundle of the origin but the latter does have. The following formula tells us how to calculate the codimension of L_G. In what follows ν denotes the dimension of space-time.

Theorem 2. (Dimension formula)

$$\dim L_G = \min_{\tilde{G}=G_1\sqcup\cdots\sqcup G_k}(-k+\nu m(\tilde{G})) \tag{4}$$

We explain the notation $\tilde{G}$. Let I_0 be the set of indices of internal lines. To any partition $I_1\sqcup\cdots\sqcup I_k$ of I_0 corresponds the partition $\tilde{G} = G_1\sqcup\cdots\sqcup G_k$ of G. Here G_i is the subgraph of G composed of the internal lines in I_i. $m(\tilde{G})$ means the sum $\sum_{i=1}^{k} m(G_i)$. A partition will be called good if it attains the minimum in the formula.

It is easy to rewrite the above formula to the codimension formula. Here we consider codimension in P, not in P_0.

Theorem 2'. (Codimension formula)

$$\operatorname{codim} L_G = \nu\chi(G) + \max_{\tilde{G}=G_1\sqcup\cdots\sqcup G_k}(k+\nu b_1(\tilde{G})), \tag{5}$$

where $b_1(\tilde{G}) = \sum_{i=1}^{k} b_1(G_i)$.

This expression is often convenient because $\operatorname{codim} L_{\tau(G),\tau} = \operatorname{codim} L_{\tau(G)}$ holds for a contracted graph $\tau(G)$ ([6]).

Our codimension formula also gives a necessary and sufficient condition for L_G to have codimension $\nu+1$ in P (or equivalently 1 in P_0), in which case we call G is primary. In other words, a graph G is primary if and only if it is connected and has only the mobility of translation and similar enlargement.

Corollary 1. (Primary graph) *G is primary if and only if $1 + \nu b_1(G) \geq k+\nu b_1(\tilde{G})$ for any partition $\tilde{G}$.*

Since the union of two primary subgraphs with an internal line in common is itself primary, there exists a unique partition $\tilde{G}_{max} = G_1 \sqcup \cdots \sqcup G_k$ composed of all the maximal primary subgraphs of G. This maximal primary partition $\tilde{G}_{max}$ is a good partition. In fact, take any good partition $\tilde{G} = G'_1 \sqcup \cdots \sqcup G'_h$. Each G'_j is necessarily a primary graph, so that $\tilde{G}$ gives a partition of $\tilde{G}_{max}$. Because each G_i is primary, we have $\sum_{i=1}^{k} \dim L_{G_i} = -k+\nu m(\tilde{G}_{max}) \leq -h+\nu m(\tilde{G})$ and, together with the inequality $\dim L_G = -h+\nu m(\tilde{G}) \leq -k+\nu m(\tilde{G}_{max})$, we conclude that $\tilde{G}_{max}$ is good.

When the partition $\{1\} \sqcup \cdots \sqcup \{N\}$ is good, we call G is ample. An ample graph has codimension $N+\nu\chi(G)$ in P. In [3] and [6] the first author and M. Kashiwara mentioned a necessary and sufficient condition for the ampleness of G. But their condition must be corrected as follows:

<u>Corollary 2</u>. (<u>Ample graph</u>) <u>G is ample if and only if</u>

$$N \geq k + \nu b_1(\tilde{G}) \quad \underline{\text{for any partition } \tilde{G}.}$$

We give the proof of the formula in section 2. Here are some examples.

<u>Example 1</u>. (<u>1 loop graph</u>)([3]) $\operatorname{codim} L_G = \max(\nu+1, N)$.

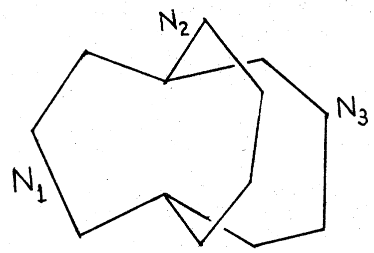
G is primary if and only if $\nu+1 \geq N$, and is ample if and only if $\nu + 1 \leq N$.

<u>Example 2</u>. (<u>2 loop graph</u>)

$$\operatorname{codim} L_G = \max(\nu+1,\ N_1+1,\ N_2+1,\ N_3+1,\ N_1+N_2+N_3-\nu),$$

where N_i is the number of internal lines composing each arm (See Fig.1). G is primary if and only if $\nu \geq N_1, N_2, N_3$, and $2\nu+1 \geq N_1+N_2+N_3$. G is ample if and only if $N_1+N_2+N_3 \geq 2\nu+1$ and $N_1+N_2, N_1+N_3, N_2+N_3 \geq \nu+1$.

Figure 1.



Now we state some applications of the formula.

Theorem 3.

Let G_0 be a full subgraph of G and $\Lambda_{\tau(G),\tau} = \Lambda_{G/G_0,\tau}$ be the Landau singularity obtained by contracting it ([6]). Then $\Lambda_{G/G_0,\tau}$ intersects Λ_G regularly ([3]) and

$$\dim \Lambda_{G/G_0,\tau} \cap \Lambda_G = \nu n(G) - (\operatorname{codim} L_{G_0} - \nu b_0(G_0)).$$

Proof.

Put $H(x) = H'(x) + H''(x)$, $H'(x) = \sum_{\ell'} m_{\ell'} \sqrt{(\eta_{\ell'}(x))^2}$, where the summation is taken over all the internal lines of G_0. In a neighborhood of $\Lambda_{G/G_0,\tau} \cap \Lambda_G$, $H''(x)$ is a homogeneous holomorphic function of homogeneous degree 1.

Consider the following transformation

$$p_j \longmapsto p_j - \frac{\partial H''}{\partial x_j}, \quad x_j \longmapsto x_j.$$

Due to the homogeneity of $H''(x)$, this is a well defined contact transformation ([7]) and reduces the problem to the case where $G = G_0$ and $\tau(G) = pt$. The theorem is then obvious.

Corollary 1. $\operatorname{codim} \Lambda_{G/G_0,\tau} \cap \Lambda_G$ in $\Lambda_G = 1$ if and only if G_0 is primary.

Thus the holonomy diagram for a Feynman integral is completely determined when G has no internal vertices.

<u>Theorem 4</u>. <u>Assume that</u> G_0 <u>is a primary subgraph of</u> G <u>and take the maximal primary subgraph</u> G_1 <u>containing</u> G_0.

$$\mathrm{codim}L_{G/G_0} - \mathrm{codim}L_G = -1,\ 0,\ 1,\ 2,\ \ldots$$

<u>Moreover we have</u>

$$\mathrm{codim}L_{G/G_0} = \mathrm{codim}L_G - 1 \quad \underline{\text{if and only if}} \quad G_0 = G_1. \tag{6}$$

$$\mathrm{codim}L_{G/G_0} = \mathrm{codim}L_G \quad \underline{\text{if and only if}} \quad G_1/G_0 \quad \underline{\text{is primary}}. \tag{7}$$

<u>Proof</u>.

Let the maximal primary partition of G, G/G_0 and G_1/G_0 be $\tilde{G}_{max} = G_1 \sqcup \ldots \sqcup G_k$, $(\widetilde{G/G_0})_{max} = H_1 \sqcup \ldots \sqcup H_h$ and $(\widetilde{G_1/G_0})_{max} = K_1 \sqcup \ldots \sqcup K_m$, respectively.

Now that $K_1 \sqcup \ldots \sqcup K_m \sqcup G_2 \sqcup \ldots \sqcup G_k$ gives a partition of G/G_0,

$$\begin{aligned} \mathrm{codim}L_{G/G_0} - \mathrm{codim}L_G &\geq \nu\chi(G/G_0) + (k+m-1) + \nu\sum_{j=1}^{m} b_1(K_j) + \nu\sum_{j=2}^{k} b_1(G_i) \\ &\quad - (\nu\chi(G) + k + \nu\sum_{i=1}^{k} b_1(G_i)) \qquad (8) \\ &= (\nu\chi(G_1/G_0) + m + \nu\sum_{j=1}^{m} b_1(K_j)) - (\nu\chi(G_1) + 1 + \nu b_1(G_1)) \\ &= \mathrm{codim}L_{G_1/G_0} - \mathrm{codim}L_{G_1} \\ &\geq -1, \end{aligned}$$

where we have used $\chi(G/G_0) - \chi(G) = 1 - \chi(G_0) = \chi(G_1/G_0) - \chi(G_1)$.

In particular we see that the equality holds only if $G_0 = G_1$.

Conversely assume $G_0 = G_1$. Considered as a subgraph of G/G_1, each G_i $(i=2,\dots,k)$ is still primary but may not be a maximal one, so we may suppose $H_1,\dots,H_h$ have the form $H_1 = G_2 \cup \dots \cup G_{i_1}, \dots, H_h = G_{i_{h-1}+1} \cup \dots \cup G_{i_h}$ $(i_1+\dots+i_h=k-1)$ as subsets of internal lines. If we put $\bar{H}_s = G_1 \cup G_{i_{s-1}+1} \cup \dots \cup G_{i_s}$ $(s=1,\dots,h)$, we have $\bar{H}_s/G_1 = H_s$, and $G_1 \sqcup G_{i_{s-1}+1} \sqcup \dots \sqcup G_{i_s}$ gives the maximal primary, hence a good partition of $\bar{H}_s$. Since $\mathrm{codim} L_{\bar{H}_s} \geq \nu+2$, $\mathrm{codim} L_{H_s} = \nu+1$, and $\mathrm{codim} L_{\bar{H}_s/G_1} - \mathrm{codim} L_{\bar{H}_s} \geq -1$ by (8), $\mathrm{codim} L_{\bar{H}_s}$ must be $\nu+2$ and we have

$$
\begin{aligned}
0 &= 1 + \mathrm{codim} L_{H_s} - \mathrm{codim} L_{\bar{H}_s} \\
&= 1+(\nu\chi(H_s)+1+\nu b_1(H_s)) - (\nu\chi(\bar{H}_s)+(i_s+1)+\nu b_1(G_1)+\nu b_1(G_{i_{s-1}+1})+\dots \\
&\qquad +\nu b_1(G_{i_s})) \\
&= (1+\nu b_1(H_s))-(1+\nu b_1(G_{i_{s-1}+1}))-\dots-(1+\nu b_1(G_{i_s})).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathrm{codim} L_{G/G_1}-\mathrm{codim} L_G &= \nu\chi(G/G_1)+h+\nu\sum_{s=1}^{h} b_1(H_s)-(\nu\chi(G)+k+\nu\sum_{i=1}^{k} b_1(G_i)) \\
&= \nu(\chi(G/G_1) - \chi(G) - b_1(G_1)) - 1 \\
&\quad +\sum\{(1+\nu b_1(H_s))-(1+\nu b_1(G_{i_{s-1}+1}))-\dots-(1+b_1(G_{i_s}))\} \\
&= -1, \qquad (9)
\end{aligned}
$$

because $b_0(G_1) = 1$. This proves (6).

From (6) and (8) follows the "only if" part of (7). Let us prove the "if" part. Assume G_1/G_0 is primary. We may suppose $H_1 = G_1/G_0 \cup G_2 \cup \ldots \cup G_{i_1}$, $H_2 = G_2 \cup \ldots \cup G_{i_2}, \ldots, H_h = G_{i_{h-1}+1} \cup \ldots \cup G_{i_h}$ $(i_1+\ldots+i_h=k)$. Actually $i_1 = 1$, because setting $\bar{H}_1 = G_1 \cup G_2 \cup \ldots \cup G_{i_1}$, we have $\bar{H}_1/G_0 = H_1$ and $\mathrm{codim} L_{\bar{H}_1/G_0} - \mathrm{codim} L_{\bar{H}_1} \geq 0$ so that $\mathrm{codim} L_{\bar{H}_1} = \nu+1$. In the same way as in (9), we get

$$\mathrm{codim} L_{G/G_0} - \mathrm{codim} L_G = \sum_{s=2}^{h} (\mathrm{codim} L_{H_s} - \mathrm{codim} L_{G_0 \cup G_{i_{s-1}+1} \cup \ldots \cup G_{i_s}}) = 0,$$

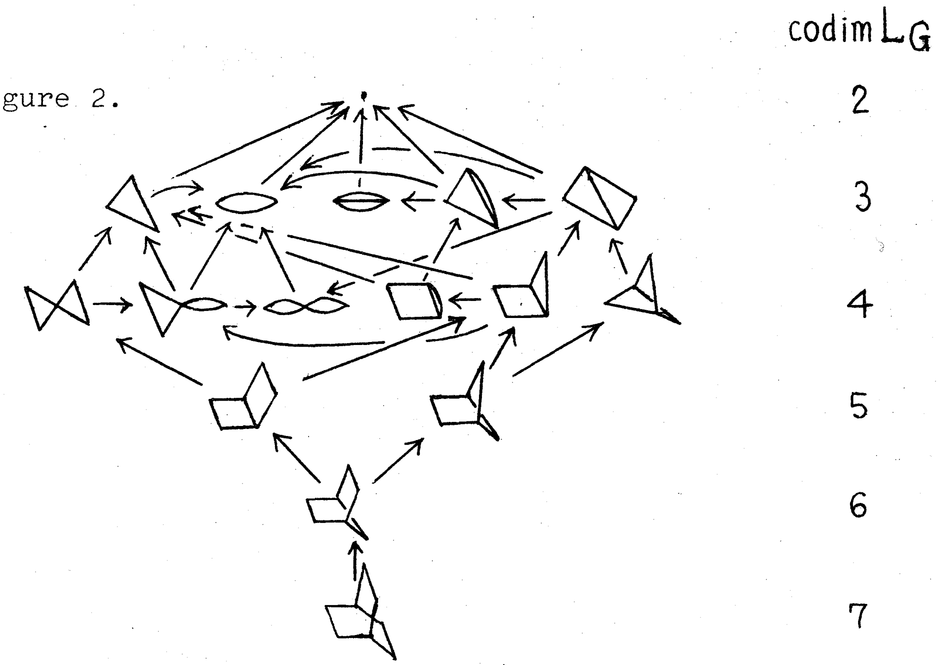
and the proof is now complete.

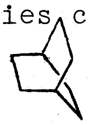
<u>Example 3</u>. (Holonomy diagram for 2 loop graph)

For the sake of simplicity we assume $\nu=2$ and $N_1=N_2=N_3=3$. Each arrow $G \to G'$ indicates the fact that the latter is obtained by contracting a primary subgroph of the former.

Here we have

Figure 2.



used a reduced expression. Each figure represents several Landau singularities corresponding to different contractions. for Example,  represents 9 Landau singularities corresponding to contractions of 9 different internal lines of  .

§2. Mobility of a graph and a mini-max theorem.

Fix a value of $x = (x_1, \ldots, x_n)$. x determines a realization of G in ν-dimensional space. We consider the mobility of this realization x to another x' not changing the direction of the internal lines. Obviously there is the mobility of ν-dimensional translation, and if G does not reduces to one point there is the mobility of 1 dimensional similar enlargement. In general, more mobility exists. For example the mobility of n-point simple loop is $\max(\nu+1, n)$ if the realization in ν dimensional space is generic.

Proposition 1.

$\mathrm{codim} L_G$ = the mobility of a generic realization of G in ν dimensional space.

Proof. Because Λ_G is νn-dimensional, the codimension of L_G in νn-dimensional p-space is equal to the maximal dimension of a fiber $(\pi|_{\Lambda_G})^{-1}(p) = \{(p; i\infty x) \in \Lambda_G\}$. In other words it is equal to the dimension of a generic fiber. As is shown in (3) Λ_G is locally bi-holomorphic to x space, therefor $\mathrm{codim} L_G$ is equal to the dimension of a generic fiber of the mapping $x \longmapsto p = \mathrm{grad}_x H(x)$. Note that if we choose a generic x the corresponding p is also generic. Therefore choosing a generic x we have

$$\mathrm{codim} L_G = \mathrm{corank\ Hess}\ H(x).$$

Corank of Hessian is equal to the degeneracy of the corresponding quadratic form with respect to y

$$Q(x,y) = \sum \frac{\partial^2 H(x)}{\partial x_j^{(\mu)} \partial x_{j'}^{(\mu')}} y_j^{(\mu)} y_{j'}^{(\mu')}$$

where the summation is taken over $1 \leqq j,\ j' \leqq n$ and $1 \leqq \mu$, $\mu' \leqq \nu$. A little caluculation shows that

$$Q(x,y) = \sum_{\ell=1}^{N} \frac{m_\ell}{\sqrt{\eta_\ell(x)^2}^{\,3}} \{\eta_\ell(x)^2 \eta_\ell(y)^2 - \langle \eta_\ell(x), \eta_\ell(y) \rangle^2\}.$$

We can calculate the degeneracy of $Q(x,y)$ considering it in a positive definite Euclidean vector space. On account of the Schwarz' inequality it is equal to dim $\{y | \eta_\ell(x) // \eta_\ell(y)\}$, that is, the dimension of the mobility of G when it is realized in a generic position.

<u>Remark 1</u>. The above proposition implies that if we fix the masses m_ℓ's and the external momenta p_j's, then the internal momenta k_ℓ's are uniquely determined ([1]).

We can calculate the mobility of the graph in a generic position using a mini-max theorem in linear algebra.

We denote by C(respectively by D) the incident matrix ([j:ℓ]) ($j=1,\ldots,n$, and $\ell=1,\ldots,N$) (respectively the circuit matrix ($[C_i:\ell]$) ($i=1,\ldots,b_1$, and $\ell=1,\ldots,N$)) of G ([6]). By the definition $\eta_\ell(x) = ({}^tC \otimes I_\nu \cdot x)_\ell$ where I_ν is $\nu \times \nu$ identity matrix. Note that a set of ν-vectors h_ℓ ($\ell = 1,\ldots,N$) can be written as $h_\ell = \eta_\ell(x)$ for some x if and only if $\sum_{\ell=1}^{N} d_\ell \otimes h_\ell = 0$, where d_ℓ's are column vectors of D.

We have for a generic x

$$\dim \{y \mid \eta_\ell(x) \mathbin{//} \eta_\ell(y)\}$$

$$= \dim \{y \mid ({}^tC \otimes I_\nu)y = \begin{pmatrix} k_1 \\ \vdots \\ k_N \end{pmatrix} \text{ for some } k_\ell = \beta_\ell \cdot \eta_\ell(x)\}$$

$$= \dim \text{kernel } {}^tC \otimes I_\nu + \dim \{\beta \mid \sum_{\ell=1}^{N} d_\ell \otimes \beta_\ell \eta_\ell(x) = 0\}$$

$$= \nu b_0(G) + n - \max \text{rank } (d_1 \otimes h_1, \ldots, d_N \otimes h_N) \quad ,$$

where we take the maximum over $h_1, \ldots, h_N$ (ν - vectors) such that $d_1 \otimes h_1 + \ldots + d_N \otimes h_N = 0$.

We can state generally the following mini-max theorem.

Theorem 5. (Mini-max theorem)

Let V be a finite dimensional vector space over a field K. For subspaces $V_1, \ldots, V_N$, we have

$$\max_{\substack{x_1 \in V_1, \ldots, x_N \in V_N \\ x_1 + \ldots + x_N = 0}} \dim (Kx_1 + \ldots + Kx_N) = \min_{I_0 = I_1 \sqcup \ldots \sqcup I_k} (N - k + \dim V_{I_0 - I_1} \cap \ldots \cap V_{I_0 - I_k})$$

where $I_1 \sqcup \ldots \sqcup I_k$ is a partition of the set $I_0 = \{1, \ldots, N\}$ *and* V_I *means the sum* $\sum_{i \in I} V_i$.

Proof of theorem 2.

We take $\mathbb{C}^{\nu b_1}$ as V and $\{d_\ell \otimes y \mid y \in \mathbb{C}^\nu\}$ as V_ℓ. Then we have

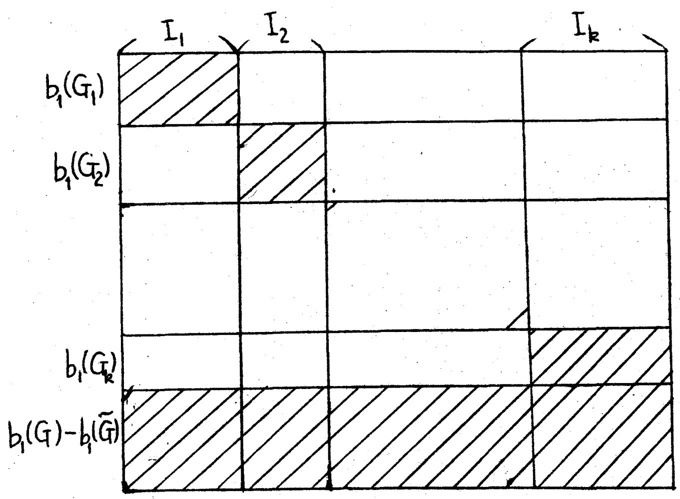
$$\text{codim} L_G = \nu b_0(G) + N(G) - \min (N(G) - k + \dim V_{I_0 - I_1} \cap \ldots \cap V_{I_0 - I_k})$$

$$= \nu\chi(G) + \max\,(\nu b_1(G) + k - \dim V_{I_0-I_1} \cap \ldots \cap V_{I_0-I_k}).$$

To prove $\dim V_{I_0-I_1} \cap \ldots \cap V_{I_0-I_k} = \nu(b_1(G) - b_1(\tilde{G}))$ we can assume $\nu = 1$ without loss of generality. The circuit matrix can be modified so that it has the form of Fig. 3, where blocks not shaded are zero matrices. It clarifies the identity

$$\dim V_{I_0-I_1} \cap \ldots \cap V_{I_0-I_k} = b_1(G) - b_1(\tilde{G}).$$

Fig. 3



Proof of the mini-max theorem. For notational convenience we write $V^{I_1,\ldots,I_k} = V_{I_0-I_1} \cap \ldots \cap V_{I_0-I_k}$, if $I_1,\ldots,I_k$ are subsets of I_0.

i) Proof of the inequality $\leqq$. Choose $x_i \in V_i$ $(i \in I_0)$. For any partition $I_0 = I_1 \sqcup \ldots \sqcup I_k$ there is a canonical injection

$$(Kx_1+\ldots+Kx_N+V^{I_1,\ldots,I_k}) / V^{I_1,\ldots,I_k} \hookrightarrow$$

$$(Kx_1+\ldots+Kx_N+V^{I_1})/V^{I_1} \oplus \ldots \oplus (Kx_1+\ldots+Kx_N+V^{I_k}) / V^{I_k},$$

which yields the inequality

$$\dim(Kx_1+\dots+Kx_N) \leqq \dim(Kx_1+\dots+Kx_N+V^{I_1,\dots I_k})$$

$$\leqq \sum_{j=1}^{k} \dim(Kx_1+\dots+Kx_N+V^{I_j})/V^{I_j} + \dim V^{I_1,\dots, I_k}.$$

Let us assume that $\sum_{i\in I_0} x_i = 0$. Rewriting this equality in the form

$$\sum_{i\in I_j} x_i = - \sum_{i\in I_0-I_j} x_i \qquad (j=1,\dots,k),$$

we have inequalities $\dim(Kx_1+\dots+Kx_N+V^{I_j})/V^{I_j} \leqq \#(I_j)-1$, which together with the above inequality yield the desired one.

ii) Proof of the enequality $\geqq$. We shall make use of an induction on N. The case $N=0,1$ is trivial.

Case 1. The minimum on the right hand side is attained by a partition $I_1\sqcup\dots\sqcup I_k$ where $2 \leqq k \leqq N-1$.

We claim that for generic $x_i\in V_i$ such that $x_1+\dots+x_N = 0$,

$$\dim(Kx_1+\dots+Kx_N+V^{I_j})/V^{I_j} = \#(I_j) - 1 \qquad (j=1,\dots,N). \quad (10)$$

In fact, applying the induction hypothesis to the case of vector spaces

$$\bar{V}_i = (V_i + V^{I_j})/V^{I_j} \subset \bar{V} = V/V^{I_j} \qquad (i\in I_j),$$

we have a partition $I_{j1}\sqcup\dots\sqcup I_{jk_j} = I_j$ such that

$$\dim(\sum_{i\in I_j} Kx_i+V^{I_j})/V^{I_j} = \#(I_j) - k_j +\dim \bar{V}^{I_{j1},\dots,I_{jk_j}} \qquad (11)$$

holds. $I_1 \sqcup \dots \sqcup I_{j1} \sqcup \dots \sqcup I_{jk_j} \sqcup \dots \sqcup I_k = I_0$ gives another partition of I_0. In our case

$$-(k_j+k-1)+\dim V^{I_1,\dots,I_{j1},\dots,I_{jk_j},\dots,I_k} \geqq -k+\dim V^{I_1,\dots,I_k} \tag{12}$$

holds. Note that the following diagram exists:

$$\begin{array}{ccccc} V^{I_1,\dots,I_{j1},\dots,I_{jk_j},\dots,I_k} = & V & \supset & V\cap U & = V^{I_1,\dots,I_k} \\ & \cap & & \cap & \\ V^{I_{j1},\dots,I_{jk_j}} = & V+U & \supset & U & = V^{I_j}, \end{array}$$

Using the isomorphism theorem for vector spaces, we have

$$\begin{aligned} \dim \overline{V}^{I_{j1},\dots,I_{jk_j}} &= \dim V^{I_{j1},\dots,I_{jk_j}}/V^{I_j} \\ &= \dim V^{I_1,\dots,I_{j1},\dots,I_{jk_j},\dots,I_k} - \dim V^{I_1,\dots,I_k} . \end{aligned} \tag{13}$$

(10) follows from (11), (12) and (13).

Now applying the induction hypothesis to the case of vector spaces $V_{I_1},\dots,V_{I_k} \subset V$, we have a partition $J_1 \sqcup \dots \sqcup J_\ell = J_0$ of $J_0 = \{1,\dots,k\}$ such that for generic $y_j \in V_{I_j}$ satisfying $y_1+\dots+y_k = 0$,

$$\dim(Ky_1+\dots+Ky_k) = k - \ell + \dim V^{J_1,\dots,J_\ell}$$

holds. Denote by ρ this dimension. Noting that $(\bigcup_{j\in J_1} I_j) \sqcup \dots \sqcup (\bigcup_{j\in J_\ell} I_j) = I_0$ gives another partition of I_0, we have in our case $\rho \geqq \dim V^{I_1,\dots,I_k}$.

For generic $x_i \in V_i$ $(i=1,\dots,k)$ such that $x_1+\dots+x_k = 0$,

$y_j = \sum_{i \in I_j} x_i$ $(j=1,\ldots,k)$ are generic $y_j \in V_{I_j}$ satisfying $y_1+\ldots+y_\ell = 0$. Therefore we can assume without loss of generality that $y_1,\ldots,y_\ell$ are linearly independent, and also we can assume that $\#(I_j)-1$ elements of $\{x_i \mid i \in I_j\}$ are linearly independent modulo V^{I_j}. Then we have the linear independence of $s-k+\rho$ elements $\{x_i \mid i \in I_1\},\ldots, \{x_i \mid i \in I_\rho\}$, $\#(I_{\rho+1})-1$ elements of $\{x_i \mid i \in I_{\rho+1}\},\ldots,\#(I_k)-1$ elements of $\{x_i \mid i \in I_k\}$, which yields the inequality

$$\dim(Kx_1+\ldots+Kx_N) \geqq N-k+\rho \geqq N-k+\dim V^{I_1,\ldots,I_k} .$$

Case 2. The minimum is attained only by the partition $\{1\}\sqcup\ldots\sqcup\{N\}$.

Applying the induction hypothesis to the case of vector spaces $V_{\{1,2\}},V_3,\ldots,V_N$, we have a partition $J_1,\ldots,J_k$ of $\{\{1,2\},3,\ldots,N\}$ so that for generic $y_{12}\in V_{\{1,2\}},y_3\in V_3,\ldots,y_N\in V_N$,

$$\dim(ky_{12}+Ky_3+\ldots+Ky_N) = N-1-k+\dim V^{J_1,\ldots,J_k}$$

holds. Considering $J_1,\ldots, J_k$ as a partition of $\{1,2,\ldots,N\}$, we have in our case 2,

$$N - 1 - k + \dim V^{J_1,\ldots,J_k} >-1 + \dim V^{\{1\},\ldots,\{N\}} .$$

Case 3. The minimum is attained by the trivial partition $I_0 (k=1)$.

We are to prove that $N-1$ of $x_i \in V_i$ satisfying $x_1+\ldots+x_N = 0$ are linearly independent. As in case 2 the induction hypothesis applied to the case $V_{\{1,2\}}, V_3,\ldots,V_N$ assures that

$$\dim(Kx_1+\ldots+Kx_N) \geqq N - 2.$$

We shall show that it leads to a contradiction assuming that there are only $N-2$ independent elements. We can choose $x_i \in V_i$ so that $x_1+\ldots+x_N = 0$ and any $N-2$ of them are linearly independent. Next we can take $y_i \in V_i$ so that $y_1+\ldots+y_N = 0$ and y_N is independent of the x_i's. In fact, because $\dim V^{\{1\},\ldots,\{N\}} \geqq N-1$ in our case 3, we can take $y = y_2+\ldots+y_N \in V^{\{1\},\ldots,\{N\}}$ which is independent of the x_i's. Here we may assume without loss of generality that y_N is independent of the x_i's. Now that y can be written as $y = z_1+\ldots+z_{N-1}$, we have

$$-z_1+(y_2-z_2) +\ldots+ (y_{N-1}-z_{N-1}) + y_N = 0,$$

which is a desired expression.

We have $x_{N-1} = \alpha_1 x_1+\ldots+\alpha_{N-2}x_{N-2}$ where $\alpha_1 \neq 0,\ldots, \alpha_{N-2} \neq 0$. This follows from the fact that $N-2$ of x_i's are linearly independent. For $\lambda \neq 0$

$$(\alpha_1 x_1+\lambda y_1)+\ldots+(\alpha_{N-2}x_{N-2}+\lambda y_{N-2})+(-x_{N-1}+\lambda y_{N-1})+\lambda y_N = 0,$$

hence

$$N-2 \geqq \mathrm{rank}(\alpha_1 x_1+\lambda y_1,\ldots,\alpha_{N-2}x_{N-2}+\lambda y_{N-2},\lambda y_N)$$

$$= \mathrm{rank}(\alpha_1 x_1+\lambda y_1,\ldots,\alpha_{N-2}x_{N-2}+\lambda y_{N-2},y_N)$$

$$\geqq \mathrm{rank}(\alpha_1 x_1,\ldots,\alpha_{N-2}x_{N-2},y_N)$$

$$= \mathrm{rank}(x_1,\ldots,x_{N-2},y_N),$$

which is a contradiction.

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