

On the disruption of Whitney's lemma for simply connected
4-manifolds (in piecewise-linear and homotopy versions)

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Whitney's lemma [7] states that intersection points of smooth n -submanifolds of a simply connected $2n$ -manifold can be eliminated if the intersection number of the two submanifolds is equal to zero and $2n \geq 6$. However, this lemma fails for $2n = 4$. This was first pointed out by Kervaire and Milnor [2] who found 2-dimensional homology classes ξ_1, ξ_2 of a simply connected 4-manifold such that (i) ξ_1 and ξ_2 are represented by smoothly embedded 2-spheres, (ii) the intersection number $\xi_1 \cdot \xi_2 = 0$ but (iii) there are no smoothly embedded disjoint 2-spheres which represent ξ_1, ξ_2 respectively. However, one can easily verify that their classes ξ_1, ξ_2 can be represented by disjoint piecewise-linearly (PL) embedded 2-spheres (with locally knotted points).

In this paper we shall give an example (Example 1) which shows that it is not always possible to represent two homology classes ξ_1, ξ_2 with $\xi_1 \cdot \xi_2 = 0$ by disjoint PL embedded 2-spheres. We shall also give an example (Example 2) in which one cannot represent a homology class ξ with $\xi \cdot [S_1] = 0$ ($\{S_1\}$ being a finite set of embedded 2-spheres) by a continuous map of a

2-sphere whose image is disjoint of these 2-spheres $\{S_i\}$.

§1. The PL case.

EXAMPLE 1. There exists a compact 1-connected 4-manifold W^4 (with boundary) which satisfies the following conditions :
 (i) There are two primitive homology classes $\xi_1, \xi_2 \in H(W^4; \mathbb{Z})$
with $\xi_1 \cdot \xi_2 = 0$, but (ii) one cannot represent ξ_1, ξ_2 by PL
embedded 2-spheres with disjoint images.

We start with the following link :

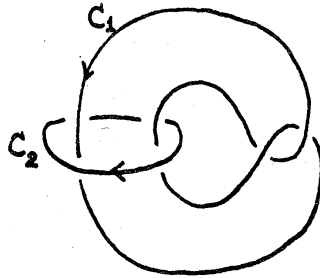


Fig. 1.

Since each of the components C_1, C_2 is a trivial knot, it has a trivial framing in S^3 : $C_1 \times D^2, C_2 \times D^2$. Attach 2-handles h_1, h_2 to D^4 along these trivially framed circles. Then we obtain the 1-connected 4-manifold W^4 with boundary. Clearly $H_2(W; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$ of which each summand is generated by the respective 2-handles. Let ξ_1, ξ_2 be the two generators.

LEMMA 1. Suppose that ξ_1 (or ξ_2) is represented by a PL embedded 2-sphere Σ^2 which has a singular point (i.e., a locally knotted point) of knot type k (Cf. Fox and Milnor [1]).

Then $\varphi(k)=0$, where $\varphi(k)$ denotes the Robertello invariant of the knot k . (See Robertello [3].)

LEMMA 2. Suppose that $\xi_1 + \xi_2$ is represented by a PL embedded 2-sphere Σ^2 with a singular point of knot type k . Then $\varphi(k)=1$.

These lemmas will be proved later. Since the linking number of our link is equal to zero, the intersection number $\xi_1 \cdot \xi_2 = 0$. Now we shall show that ξ_1, ξ_2 cannot be represented by disjoint PL embedded spheres. Otherwise, we would have two 2-spheres Σ_1, Σ_2 ($\subset W^4$) which represent ξ_1, ξ_2 respectively. By Lemma 1, the singularities k_1, k_2 of these 2-spheres have Robertello invariant zero. We take the connected sum of these two spheres and would ^{obtain} a PL embedded 2-sphere $\Sigma_1 \# \Sigma_2$ ($\subset W^4$) which represents $\xi_1 + \xi_2$ and whose singularity has Robertello invariant $\varphi(k_1) + \varphi(k_2) = 0$. This contradicts Lemma 2.

Proof of Lemma 1. We shall prove the lemma for ξ_1 . The proof for ξ_2 is the same. Suppose ξ_1 is represented by a PL embedded 2-sphere Σ^2 with a singularity k . Let D_1, D_2 be transverse ^{disks} of the attached 2-handles h_1, h_2 (i.e. cocores in the terminology of Rourke and Sanderson [4, p.74]). We may assume that Σ^2 intersects D_1, D_2 transversally with algebraic intersection numbers $1, 0$, respectively. Let U_1, U_2 be (sufficiently thin) tubular neighbourhoods of D_1, D_2 in W^4 . Then $V^4 = \overline{W^4 - (U_1 \cup U_2)}$ is PL-homeomorphic with a 4-disk, and on the boundary of V^4 we have a link $\ell = \Sigma^2 \cap (\partial U_1 \cup \partial U_2)$. observe

that one can obtain the link \mathcal{L} starting with the (trivial) knot C_1 (Fig.2) or with the link of Fig.3 by adding a finite number of $(0,L,K)$ -pairs in Tristram's sense ([6], Def.3.1), where L is the knot C_1 or the link of Fig.3 and K is any component of L . (This construction of \mathcal{L} will be referred to as the explicit construction.) Thus \mathcal{L} is a proper link in the sense of Robertello [3, p.546]. \mathcal{L} is clearly related (in Robertello's sense [3, p.547]) to the singularity knot k . Since \mathcal{L} is a proper link, the Robertello invariant of a knot which is related to \mathcal{L} depends only on \mathcal{L} . Therefore, we can compute $\varphi(k)$ by any knot which is related to \mathcal{L} ([3], Th.2). However, from the explicit construction of \mathcal{L} it is easily verified that \mathcal{L} is related to a trivial knot C_1 . This implies that $\varphi(k)=0$.

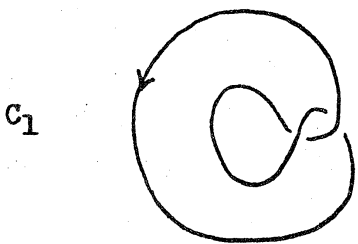


Fig.2

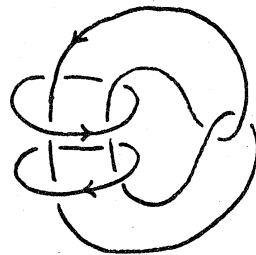


Fig.3

Q.E.D.

Proof of Lemma 2. Let Σ^2 be a PL embedded 2-sphere ($\subset W^4$) which represents $\xi_1 + \xi_2$. Then Σ^2 intersects D_1, D_2 with algebraic intersection numbers 1,1.

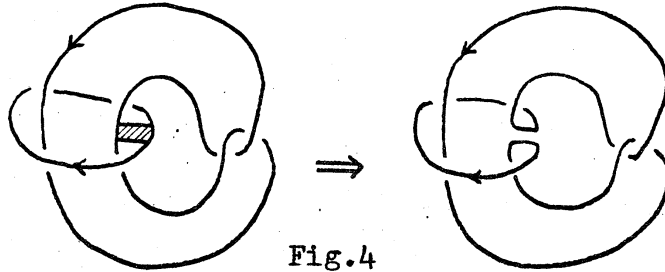


Fig.4

Thus, by the same reasoning as the previous proof, the link $\mathcal{L} = \Sigma^2 \cap (\partial U_1 \cup \partial U_2)$ ($\subset \partial V$) is proper and is related to the link of Fig.1. The link of Fig. 1 is related to a trefoil 3_1 (See Fig.4). Since $\varphi(3_1)=1$, we know that the singularity k of Σ^2 , which is also related to \mathcal{L} , has Robertello invariant 1.

Q.E.D.

PROBLEM 1. Find a closed example with the same property.

PROBLEM 2. Determine whether ξ_1, ξ_2 are represented by topologically embedded 2-spheres with disjoint images.

§2. The homotopy case.

EXAMPLE 2. There exists a closed 1-connected 4-manifold M^4 with the following properties : (i) There are smoothly embedded 16 2-spheres S_1, \dots, S_{16} with disjoint images, (ii) there is a continuous map $f: S^2 \rightarrow M^4$ of a 2-sphere to the manifold with $(f_*[S^2]) \cdot [S_i^2] = 0$ for $i=1, \dots, 16$, but (iii) f cannot be homotopic to any map $g: S^2 \rightarrow M^4$

with $g(S^2) \cap (\bigcup_{i=1}^{16} S_i^2) = \emptyset$.

The manifold M^4 is, in fact, a Kummer manifold (Cf. Spanier[5]).

Let us recall the construction. We take a 4-dimensional torus

$T^4 = S^1 \times S^1 \times S^1 \times S^1$ and consider the involution σ defined by $\sigma(z_1, z_2, z_3, z_4) = (\bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4)$, where we are considering $S^1 = \{z \in \mathbb{C}; |z|=1\}$. Then σ has

16 fixed points P_1, \dots, P_{16} . The quotient T^4/σ has thus 16 singular points each of which locally looks like a cone over a 3-dimensional

(real) projective space. Blow up these singularities, in other

words, delete small regular neighbourhoods of the singular points

and glue copies of the total space E of a λ^2 -disk bundle over S^2 with

Euler class -2 . Then we obtain a closed smooth 4-manifold M^4 which

contains 16 smoothly embedded 2-spheres (as exceptional curves or

zero-sections of E 's). Denote these spheres by S_1^2, \dots, S_{16}^2 . Note

that $[S_i^2] \cdot [S_j^2] = -2 \delta_{ij}$ (Kronecker's delta). It is known that the second

Betti number $b_2(M^4) = 22$ (cf. [5]). Thus we have a non-zero homology

class $\xi \in H_2(M^4; \mathbb{Z})$ such that $\xi \cdot [S_i^2] = 0$ ($\forall i=1, \dots, 16$). Since M^4 is

1-connected ([5]), $H_2(M^4; \mathbb{Z}) = \pi_2(M^4)$. Hence ξ is represented by a

continuous map $f: S^2 \rightarrow M^4$. Suppose $f \simeq g$ with $g(S^2) \cap (\bigcup_{i=1}^{16} S_i^2) = \emptyset$. Then,

since $M^4 - \bigcup_{i=1}^{16} S_i^2 = T^4/\sigma$ (the 16 points), the map g would be lifted to

$\tilde{g}: S^2 \rightarrow T^4$ (the 16 points). However, $\pi_2(T^4 - 16 \text{ points}) = \{0\}$. This

implies that $g=0$, which contradicts $\xi \neq 0$.

Q.E.D.

PROBLEM 3. Find a similar example with a smaller number of spheres.

References

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