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Classical invariant theory

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This report describes the use of classical invariant theory in the theory of moduli of algebraic varieties and in elementary geometry. Naturally there is some overlap with the recent paper of Dieudonné [2] and the book Dieudonné and Carrell [3].

Invariant theory was first related to number theory going back to the Disquisitiones Arithmeticae of Gauß and the theory of quadratic forms in two variables. Let $f = ax^2 + 2bxy + cy^2$ be such a quadratic form; then if the variables x,y are changed by $x = \alpha x' + \beta y'$, $y = \gamma x' + \delta y'$ with $\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} \in SL(2,\mathbb{C})$, another quadratic form $f' = a'x'^2 + 2b'x'y' + c'y'^2$ is obtained. The discriminants of these forms are related by $b'^2 - a'c' = (b^2 - ac)(\alpha\delta - \beta\gamma)^2$ and therefore are the same. Gauß knew that the discriminant $\Delta = b^2 - ac$ is the main invariant of binary quadratic forms, i.e. that every polynomial F(a,b,c) in the coefficients a,b,c which is invariant when $SL(2,\mathbb{C})$ is applied according to the above rule, is a polynomial in Δ[1]. Starting from this observation, Gauß developped his arithmetic theory of positive definite quadratic forms with integral coefficients. He determined the number of possible representations of an integer by a given quadratic form and introduced the fundamental notion of equivalence of quadratic forms and the class number of the discriminant. We refer to [5] for this beautiful theory.

[1] Gauß knew this fact also for quadratic forms in three variables. Compare [6].

In the middle of the 19th. century, invariant theory branched away from number theory in various directions. Cayley [1], and later F. Klein in his Erlanger Programm [12], considered invariant theory to be the algebraic counterpart of the geometry of those days (elementary geometry in todays language) and used it to classify the elementary geometries. Sylvester, Hermite, and the German school with Aronhold, Clebsch, Gordan and others treated invariant theory as a purely algebraic theory. Their aim was to find explicit algorithms. In 1890, Hilbert solved the main problem of classical invariant theory by showing that there exists a finite basis for the invariants of the n-ary forms of degree r with respect to the action of SL(n,ℂ). After Hilbert's success, a big unsolved problem did not exist anymore, mathematicians lost interest in invariant theory until the 1930's. At this time, representation theory of the classical groups was being developped by Schur, Weyl and others, and a part of classical invariant theory was recognized to be a special case of this. But even then, no essential new contribution was made to invariant theory. In 1963, Mumford [14] revived the study of classical invariant theory and found the geometry behind the invariant theory of n-ary forms, in particular behind Hilberts papers [9] and [10], and showed that the theory of moduli of algebraic varieties is the geometric frame for classical invariant theory.

We first deal with the relationship of classical invariant theory to the theory of moduli of algebraic varieties, and then, in a second part, describe its connection to elemenary geometry.

(Unless otherwise stated, we take the complex number field to be our base field.)

In 1845, Cayley posed the problem of determination of all relative invariants of the n-ary forms of degree r. What does this mean?

Consider the general n-ary form of degree r, $f(x_1,\dots,x_n) = \sum A_\alpha x_1^{\alpha_1}\dots x_n^{\alpha_n}$ with coefficients A_α which are indeterminates over $\mathbb{C}$. Consider the action of $GL(n)$ on the tuple (A_α) given by the rule

(*) $(A_\alpha) \longrightarrow (\sigma(A_\alpha)) = (A'_\alpha)$ where A_α and A'_α are related via the corresponding polynomials and by having $\sigma \in GL(n)$ act on the variables $x_1,\dots,x_n$, i.e. by $f(\sigma(x_1),\dots,\sigma(x_n)) = \sum A'_\alpha x_1^{\alpha_1}\dots x_n^{\alpha_n}$.

A homogeneous polynomial $F(A_\alpha)$ in the indeterminates A_α is called a <u>relative invariant</u> if

$$F(\sigma(A_\alpha)) = \chi(\sigma)\cdot F(A_\alpha)$$

holds for all $\sigma \in GL(n,\mathbb{C})$, with $\chi(\sigma)$ a character of $GL(n)$. It is called an <u>absolute invariant</u> if $\chi(\sigma) = 1$, $\forall \sigma \in GL(n)$. Cayley's aim was to determine all these invariants explicitly by an algorithm.

In today's context of algebraic geometry, we may formulate Cayley's problem as follows: First check that if you take the group $SL(n,\mathbb{C})$ instead of $GL(n,\mathbb{C})$ and its action on (A_α) by (*), the relative invariants of $GL(n)$ coincide with the absolute invariants of $SL(n)$.

Next consider all n-ary forms $f = \sum a_\alpha x_1^{\alpha_1} \dots x_n^{\alpha_n}$ of degree r (with coefficients in $\mathbb{C}$) and parametrize them by the points of the affine space $A^{N+1}(\mathbb{C})$ via their coefficient (N+1)-tuples $(a_o,\dots,a_N)$. Then (*) induces an action of SL(n) on the affine space A^{N+1} and also on the polynomial ring $R = \mathbb{C}[A_o,\dots,A_N]$ of A^{N+1} which is homogeneous with respect to the natural grading of R. The absolute invariants of this operation form a graded subring S = S(n,r) of R consisting of all polynomials which are fixed by the action of SL(n). Cayley's problem was to determine the structure of S as a ring in today's mathematical terminology.

At this point, Mumford's interpretation of the invariant theory of forms can be explained. Mumford's intention is to classify the hypersurfaces of $\mathbb{P}^{n-1}(\mathbb{C})$ of degre r up to projective equivalence and to make the set of isomorphism classes of these hypersurfaces (isomorphism classes according to projective equivalence) into an algebraic variety in a natural way. As the hypersurfaces in question are parametrized by the points of $\mathbb{P}^N$ = Proj(R), and the equivalence under consideration is given by the action of PGL(N) or SL(N) on R via the rule (*), Mumford looks at the rational map $\varphi: \mathbb{P}^N = \mathrm{Proj}(R) \longrightarrow \mathrm{Proj}(S)$ and at those points of $\mathbb{P}^N$ where φ is defined. If N_o is the set of all points of $\mathbb{P}^N$ where all the non-constant invariants of R vanish, (The set of Nullforms, in Hilbert's terminology), then N_o is a closed subvariety of $\mathbb{P}^N$ and φ is defined exactly for the points in $\mathbb{P}^N - N_o$, the complement of N_o in $\mathbb{P}^N$. In Mumford's terminology, $\mathbb{P}^N - N_o$ is the set of semistable points of $\mathbb{P}^N$ with respect to the action of SL(n) on $\mathbb{P}^N$ and is denoted by $(P^N)^{ss}$. The map $\varphi: (\mathbb{P}^N)^{ss} \longrightarrow \mathrm{Proj}(S)$ is a categorical quotient of $(\mathbb{P}^N)^{ss}$ with respect to the action of SL(n) in the category of schemes satisfying certain additional properties. However Proj(S) and the map $\varphi: (P^N)^{ss} \longrightarrow \mathrm{Proj}(S)$ do not yet give the desired classification for hypersurfaces corresponding to the points $(P^N)^{ss}$. In general the action of SL(n) on $(\mathbb{P}^N)^{ss}$ is not with closed orbits and therefore φ does not separate orbits. We have to restrict ourselves to the subset of stable points $(\mathbb{P}^N)^s = \mathbb{P}^s$ of $(\mathbb{P}^N)^{ss}$, which are defined by the properties that their orbit is closed in $(\mathbb{P}^N)^{ss}$ and is of maximal dimension. The stable points form a Zariski open subset of $\mathbb{P}^N$; $\varphi(\mathbb{P}^s)$ is an open subset of Proj(S) which parametrizes the orbits of P^s by SL(n). In Mumford's sense, the map $\varphi: P^s \longrightarrow \varphi(P^s)$ is a geometric quotient of $\mathbb{P}^s$ by SL(n), and $\varphi(P^s)$ classifies the hypersurfaces of P^{n-1} which correspond to stable points of P^N up to projective equivalence. This interpretation of the invariant theory of n-ary forms of degree r leads Mumford to his theory of quotients by group actions in the category of schemes and to his proof of the existence of a quasi-projective scheme which is a coarse moduli space for curves of genus g and for polarized abelian varieties of a fixed dimension. In this sense, the invariant theory of forms becomes a part of the theory of quotients by group actions in the category of schemes and of moduli theory and provides us with the possibility to describing moduli spaces explicitly.

Let us return to the structure of the ring S. The problem can be attacked on two levels. The first and easier is to determine the graded parts of S. This is a linear problem and can be solved by representation theory. The method for the second level is to determine the ring structure of S by finding generators and relations for S. This is much more difficult and only in a few cases is the result known explicitly.

We first treat the linear problem, but briefly, since it is well explained in the literature (cf [17], [2], [3]).

Consider more generally the following situation: Let E be a vector space over $\mathbb{C}$ of dimension m. Let $GL(n,\mathbb{C})$ operate on E linearly and let $f : E \longrightarrow \mathbb{C}$ be a relative homogeneous polynomial invariant of degree r (i.e. $f(x)$ is a homogeneous polynomial and $f(\sigma(\underline{x})) = \chi(\sigma) f(\underline{x})$ holds for all $\sigma \in GL(n,\mathbb{C})$.).[2] We show that to determine all such f's it suffices to determine the 1-dimensional invariant subspaces of an action of GL(n) on a certain other linear space. Consider for this purpose $f(\lambda_1\underline{x}_1 + \ldots + \lambda_r\underline{x}_r) = \sum_{\alpha=(\alpha_1,\ldots,\alpha_r)} c_\alpha \lambda^\alpha f_\alpha(\underline{x}_1,\ldots,\underline{x}_r)$ with vectors $\underline{x}_i \in E$ and indeterminates λ_i. Then $f_{(1,\ldots 1)}(\underline{x}_1,\ldots,\underline{x}_r)$ is a relative multilinear invariant of the vectors $\underline{x}_i$ provided f is a relative homogeneous invariant and vice versa. Moreover $f_{(1,\ldots,1)}(\underline{x},\ldots,\underline{x}) = r!\, f(\underline{x})$, and hence $f_{(1,\ldots,1)}(\underline{x}_1,\ldots,\underline{x}_r)$ determines $f(\underline{x})$.

[2] For applications, it is later necessary to consider more general rational relative invariants $f: E \longrightarrow \mathbb{C}$, i.e. rational functions $f(\underline{x})$ for which $f(\sigma\underline{x}) = \chi(\sigma)\cdot f(\underline{x})$ holds for all $\sigma \in GL(n)$. An easy argument (cf. [3], p. 7) shows, however, that the rational relative invariants are known, provided the homogeneous relative invariants can be determined.

We may therefore consider the multilinear invariant $u_f = f_{(1,\dots,1)} : E^r \longrightarrow \mathbb{C}$ instead of f. Next let $\tilde{u}_f : E^{\otimes r} \longrightarrow \mathbb{C}$ be the linear map determined by u_f. Then $\tilde{u}_f$ is a linear relative invariant of $E^{\otimes r}$ as u_f is a relative invariant and conversely. Finally consider the natural isomorphism $\mathrm{Hom}(E^{\otimes r}, \mathbb{C}) \simeq E^{*\otimes r} \otimes \mathbb{C}$ with the action of GL(n) on $E^{*\otimes r} \otimes \mathbb{C}$ induced by the given action of GL(n) on E and by the character $\chi(\sigma)$ of f on $\mathbb{C}$. Then $\tilde{u}_f$ determines a 1-dimensional subspace of $E^{*\otimes r} \otimes \mathbb{C}$ which is G-invariant. Conversely every such subspace leads to a linear relative invariant of $E^{\otimes r}$ with $\chi(\sigma)$ as character. So, we attempt to determine 1-dimensional invariant subspaces of the representation as described above of GL(n) on $E^{*\otimes r} \otimes \mathbb{C}$. Therefore we now consider the representation theory of GL(n).

Let $\underline{X} \longrightarrow \underline{F}(\underline{X})$ be a homogeneous representation of $GL(n,\mathbb{C})$ in $GL(N,\mathbb{C})$ of degree f, i.e., $\underline{F}$ defines a group homomorphism and the matrix elements $F_{hk}(\underline{X})$ of the matrix $\underline{F}(\underline{X}) = (F_{hk}(\underline{X})) \in GL(N)$ are homogeneous polynomials of degree f in the coefficients x_{ij} of the matrix $\underline{X} = (x_{ij}) \in GL(n,\mathbb{C})$.[3] Consider $\underline{F}$ as a map from GL(n) to $\mathrm{End}(\mathbb{C}^N)$. Then $\underline{F}$ factors over $\mathrm{End}((\mathbb{C}^n)^{\otimes f})$ as follows. There exists a commutative diagram

$$\begin{array}{ccc} GL(n) & \xrightarrow{\underline{F}} & \mathrm{End}(\mathbb{C}^N) \\ & \searrow^{\varphi} \quad \nearrow_{\underline{G}} & \\ & \mathrm{End}((\mathbb{C}^n)^{\otimes f}) & \end{array}$$

where φ is the tensor representation $\varphi : GL(n) \longrightarrow GL((\mathbb{C}^n)^{\otimes f})$ and $\underline{G}$ is a ring homomorphism from the subring A_f of $\mathrm{End}((\mathbb{C}^n)^{\otimes f})$,

3) In general rational representations $\underline{X} \longrightarrow \underline{F}(\underline{X})$ of GL(n) in GL(N) have to be considered. However, cf. [3], the homogeneous representation determine the rational representations.

generated by $\varphi(\underline{X})$ with $\underline{X} \in GL(n)$, to $End(\mathbb{C}^N)$. To prove this fact, we recall that if $e_1,\dots,e_n$ is a basis for $E = \mathbb{C}^n$, the set $e_{\underline{\alpha}} = e_{\alpha_1} \dots e_{\alpha_f}$ of vectors in $E^{\otimes f}$ forms a basis of $E^{\otimes f}$ where $\underline{\alpha} = (\alpha_1,\dots,\alpha_f)$ ranges over all multi-indices $\underline{\alpha} = (\alpha_1,\dots,\alpha_f)$, $1 \leq \alpha_i \leq n$. The matrizes of $End(E^{\otimes f})$ can then be written as $(t_{\underline{\alpha}\underline{\beta}})$ where $\underline{\alpha}, \underline{\beta}$ is a pair of multiindices and the tensor representation $GL(n) \longrightarrow GL(E^{\otimes f})$ is described by

$$\underline{X} = (x_{ij}) \longrightarrow (x_{\underline{\alpha}\underline{\beta}}) = \underline{X}^{\otimes f}$$

where $x_{\underline{\alpha}\underline{\beta}} = x_{\alpha_1\beta_1}\dots x_{\alpha_f\beta_f}$ is the monomial in the coefficients x_{ij} of $\underline{X}$ determined by $\underline{\alpha}$ and $\underline{\beta}$. Obviously $x_{\underline{\alpha}\underline{\beta}} = x_{\underline{\alpha}'\underline{\beta}'}$ if there exists a permutation $\pi \in \mathfrak{S}_f$ with $\pi\underline{\alpha} = \underline{\alpha}'$, $\pi\underline{\beta} = \underline{\beta}'$.

Next, the homogeneous polynomials $F_{hk}(\underline{X})$ can be uniquely written as

$$(*) \quad F_{hk}(\underline{X}) = \sum a_{hk\underline{\alpha}\underline{\beta}} \, x_{\underline{\alpha}\underline{\beta}}$$

if we impose the condition $a_{hk\,\pi\underline{\alpha}\,\pi\underline{\beta}} = a_{hk\underline{\alpha}\underline{\beta}}$ on the coefficients for all $\pi \in \mathfrak{S}_f$. Using the expression $(*)$, we define a map

$$\underline{G} : End((\mathbb{C}^n)^{\otimes f}) \longrightarrow End(\mathbb{C}^N)$$
$$\underline{T} = (t_{\underline{\alpha}\underline{\beta}}) \longrightarrow (G_{hk}(\underline{T}))$$

where $G_{hk}(\underline{T}) = \sum a_{hk\underline{\alpha}\underline{\beta}} \, t_{\underline{\alpha}\underline{\beta}}$. One may check that $\underline{G}(\underline{X}^{\otimes f}) = \underline{F}(\underline{X})$, and that $\underline{G}$ is a $\mathbb{C}$-algebra homomorphism of A_f into $End(\mathbb{C}^N)$.
A simple but very important observation is that the ring A_f is closely related to the natural representation of the permutation group $\mathfrak{S}_f$ of f elements on the linear space $(\mathbb{C}^n)^{\otimes f}$.
Clearly $\pi \in \mathfrak{S}_f$ acts linearly on $(\mathbb{C}^n)^{\otimes f}$ by the rule $\pi e_{\underline{\alpha}} = e_{\pi\underline{\alpha}}$, with $\pi\underline{\alpha} = (\alpha_{\pi^{-1}(1)},\dots,\alpha_{\pi^{-1}(f)})$. This induces an action of the group ring $\mathbb{C}[\mathfrak{S}_f]$ on $E^{\otimes f}$ and a $\mathbb{C}$-algebra homomorphism

$\varrho : \mathbb{C}[\mathfrak{S}_f] \longrightarrow \mathrm{End}(E^{\otimes f})$. As A_f consists exactly of those matrices $\underline{T} = (t_{\underline{\alpha}\underline{\beta}}) \in \mathrm{End}(E^{\otimes f})$ for which $t_{\pi\underline{\alpha},\pi\underline{\beta}} = t_{\underline{\alpha}\underline{\beta}}$ holds $\forall\, \pi \in \mathfrak{S}_f$, we conclude easily (cf. [3], p. 14) that A_f is the commutator of $\varrho(\mathbb{C}[\mathfrak{S}_f])$ in $\mathrm{End}(E^{\otimes f})$. Now Maschke's theorem (cf. [3]), states that $\mathbb{C}[\mathfrak{S}_f]$ is semi-simple, and therefore $\varrho(\mathbb{C}[\mathfrak{S}_f])$ is also semi-simple. Then Schur's commutator theorem (cf. [3]) implies that A_f, as the commutator in $\mathrm{End}(E^{\otimes f})$ of a semi-simple algebra containing the centre $\mathbb{C}$ is semi-simple as well. But then the image $\underline{G}(A_f)$ is semi simple too and hence the representation $\underline{F} : GL(n) \longrightarrow GL(N)$ is completely reducible. In this way, we obtain the theorem

<u>Theorem</u> Every homogeneous representation (and then also every rational representation) of $GL(n,\mathbb{C})$ is completely reducible.

We are now able to calculate the 1-dimensional invariant subspaces of the representation $\underline{F}$ of $GL(n)$. As a matter of fact, Schur's commutator theorem even describes explicitly the simple A_f-modules of the f's tensor representation of $GL(n)$ (or equivalently the minimal left ideals of A_f) in terms of the minimal left ideals of the group ring $\mathbb{C}[\mathfrak{S}_f]$. If c is a generator of a minimal left ideal of $\mathbb{C}[\mathfrak{S}_f]$, then A_f-module $c \cdot E^{\otimes r}$ is a simple A_f-module and all simple A_f-modules are obtained in this way. It suffices, therefore, to determine the minimal left ideals of $\mathbb{C}[\mathfrak{S}_f]$. It is possible to do this explicitly by the method of Young frames; we refer to [3] for a detailed description. In particular, Young's method allows us to describe explicitly the 1-dimensional $GL(n)$-stable subspaces of $E^{\otimes f}$ and then also those of $\mathbb{C}^N$.

We discuss some applications:

a) The simultaneous multilinear invariants of f vectors $x_1,\dots,x_f \in E = \mathbb{C}^n$ appear in the linearisation process from page 6 and are therefore of fundamental interest. The above method yields the following theorem, called the first fundamental theorem of invariant theory:

<u>Theorem</u> Simultaneous multilinear invariants[4] of f vectors of E exist if f is a multiple of n, i.e. $f = g\cdot n$. They are all linear combinations of invariants of the form

$$\left[x_{i_1}\dots x_{i_n}\right]\cdot\left[x_{i_{n+1}}\dots x_{i_{2n}}\right]\cdot \;\dots\cdot\left[x_{i_{f-n+1}}\dots x_{i_f}\right]$$

where $(i_1,\dots,i_f)$ is a permutation of $(1,\dots,f)$ and $\left[z_1,\dots,z_n\right]$ denotes the determinant of the $(n\times n)$-matrix with $z_1,\dots,z_n$ as column vectors. All the invariants have weight g.

b) In applications of invariant theory to elemenary geometry, the simultaneous multilinear invariants of f covariant vectors $x_1,\dots,x_f \in E = \mathbb{C}^n$ and h contravariant vectors $y_1,\dots,y_h \in E^*$, E^* the dual space to E, all with respect to the natural action of GL(n), are needed. The above method yields the following result, called the second main theorem of invariant theory:

<u>Theorem</u> All simultaneous multilinear invariants of the vectors x_i and y_i are linear combinations of products of invariants of the three typs

1) $\left[x_{i_1}\dots x_{i_n}\right]$ of weight 1

4) Invariant = relative invariant, also in the following.

2) $[y_{j_1} \dots y_{j_n}]$ of weight -1

3) Scalar products $\langle x_i, y_j \rangle$ of weight 0.

Moreover, they only exist if p-q is divisible by n.

c) The homogeneous invariants of degree h of an n-ary form of degree r can be determined also.

First, a unique symmetric r-linear form Ψ_f associated to a form $f(\underline{x}) = \sum a_\alpha x_1^{\alpha_1} \dots x_n^{\alpha_n}$ of degree r in the variables $\underline{x} = (x_1, \dots, x_n)$ is obtained as follows: Rewrite the form $f(\underline{x})$ as

$$(*) \qquad f(\underline{x}) = \sum \beta(i_1, \dots, i_r)\, x_{i_1} \dots x_{i_r}$$

where the r indices i_ν run from 1 to n independently and where the symmetry condition $\beta(i'_1, \dots, i'_r) = \beta(i_1, \dots, i_r)$ holds if $(i'_1, \dots, i'_r)$ is a permutation of $(i_1, \dots, i_r)$. Then the expression (*) is polarized according to the rule of page 6 to obtain the desired symmetric multilinear form Ψ_f of order r.

If we consider Ψ_f instead of f, the problem becomes determination of the homogeneous invariants of degree h of the multilinear form Ψ_f. We know (see page 7) that this is equivalent to finding the simultaneous multilinear invariants of h symmetric tensors $u_1, \dots, u_r$ of the space $(\mathbb{C}^n)^{\otimes r}$ with respect to the natural action of GL(n). But this again is equivalent to finding the simultaneous multilinear invariants of $h \cdot r$ vectors of $\mathbb{C}^n$, which we are able to do. So we write down a basis for the simultaneous multilinear invariants of $h \cdot r$ vectors of $\mathbb{C}^n$ using the first main theorem, and must then translate and rewrite the obtained invariants in terms of the coordinates of the form f(x). There is a formal symbolic procedure, the famous symbolic method for writing the simultaneous linear invariants of the h.r vectors in a certain way and then for rewriting them in terms of the coefficients of f. We refer to [17] and [3] for details.

What do our considerations give for Cayleys problem of determining the structure of the ring S(n,r) of invariants of n-ary forms of degree r?

Clearly without additional considerations we can only obtain the graded parts of the ring S(n,r); the ring structure is not obtained in this way. But what is then known about the structure of S(n,r)?

Firstly, S(n,r) is, as the fixed ring of $\mathbb{C}[A_\alpha]$ by the action of the group SL(n), an integrally closed integral domain which is finitely generated over $\mathbb{C}$. The last fact was proved in 1890 by Hilbert [9] who showed that every ideal in $\mathbb{C}[A_\alpha]$ is finitely generated (this is Hilbert's Basissatz) and that a finite system of generators of the ideal I of $\mathbb{C}[A_\alpha]$ which is generated by the non-constant invariants, generate the ring S(n,r) over $\mathbb{C}$.

We remark, that for n-ary forms of degree r over an arbitrary field k, the ring $S_k(n,r) = k[A_\alpha]^{SL(n)}$ of invariants, with respect to the obvious action of SL(n) on $k[A_\alpha]$ is also finitely generated. As a matter of fact, by Haboush's recent result [8], for k with any characteristic, SL(n,r) is geometrically reductive and therefore $S(n,r)$ is finitely generated (c.f. [4]).

But what about the explicit structure of the ring $S(n,r)$? Only in a few cases is the structure known explicitly. First, by the classical theory [16], generators and relations are known in the following 3 cases.

(1) $n = 2$, $r \leq 6$

(2) $n = 3$, $r \leq 3$

(3) n-arbitrary, $r = 2$.

In all these cases, the structure of $S(n,r)$ is simple; $S(n,r)$ is essentially a polynomial ring.

The next (according to the classical theory) unknown cases are $S(2,8)$ or $S(2,7)$ and $S(3,4)$. They have been treated by Shioda in [15]. For $S(2,8)$, Shioda has determined explicitly a system of generators consisting of 9 homogeneous invariants $J_2,\ldots,J_{10}$ of degree $2,\ldots,10$ and has described 5 basic relations between the J_i, which he states explicitly. All other relations are derived from these basic relations. Moreover Shioda determines the higher Syzygy-moduls of $S(2,8)$ and, as a consequence, finds that $S(2,8)$ is Gorenstein. For the ring $S(3,4)$, Shioda's paper states what a generator system and a basis for the relations should be without giving the proof. The method used by Shioda is the classical one (cf. [16]), consisting of determining the generating function of the ring S

$$h(t,S) = \sum_{d\geq 0} \dim S_d t^d$$

which is a rational function and for $S(2,8)$ equal to $(1+t^8+t^9+t^{10}+t^{18}) \Big/ \prod_{d=2}^{7}(1-t^d)$. Knowing the generating function, one can estimate the minimal number of generators, for S(2,8) the number ist 9. Then it is to a great extent a matter of skill to find generators explicitly by the symbolic method and it is even more difficult to find the basic relations between the choosen generators.

It is of interest that the rings S(2,8) and S(3,4) are related to the moduli space of curves of genus 3. More generally, S(2,2g+2), with $g \geq 2$ an integer, is related to the moduli space of hyperelliptic curves of genus g. To make this precise, we recall that for a hyperelliptic curve X, defined over an arbitrary closed field k of characteristic $\neq 2$, the 1-canonical map $\phi_K: X \longrightarrow \mathbb{P}^1$ is a map onto $\mathbb{P}^1$ of degree 2 with 2g+2 ramification points $P_i = (\alpha_i, \beta_i)$. The homogeneous polynomial $f(x,y) = \prod_{i=1}^{2g+2}(\beta_i x - \alpha_i y)$ associated to X has discriminant $D(f) \neq 0$. Conversely, this determines X, as X is birationally equivalent to the plane curve $Y^2 = \prod_{i=1}^{2g+2}\left(\frac{x}{y} - \frac{\alpha_i}{\beta_i}\right)$, provided all $\beta_i \neq 0$, a condition easily obtained by a change of the coordinate system of $\mathbb{P}^1$. Moreover two such polynomials f and g of degree 2g+2 with discriminant different from 0 determine isomorphic hyperelliptic curves if and only if f and g are equivalent with respect to the action of SL(2) or PGL(1) induced via a coordinate change in $\mathbb{P}^1$. This indicates a fact proven by closer considerations, i.e., that the open affine scheme $\mathrm{Proj}(S(2,2g+2) - \{D = 0\}$ is the coarse moduli space for hyperelliptic curves of genus g in the sense of [14], p. 99 . In particular $\mathrm{Proj}(S(2,8) - \{D = 0\}$ is the coarse moduli for hyperelliptic curves of genus 3 and this facts holds over any algebraically closed field of characteristic $\neq 2$.

To the ring S(3,4), the non-hyperelliptic curves of genus 3 are related as follows: By the Riemann-Roch theorem, the canonical map ϕ_K of such a curve X gives an embedding of X into $\mathbb{P}^2$. The image curve $\phi_K(X)$ has degree 4 in $\mathbb{P}^2$, and two curves X and X' are isomorphic if and only if the curves $\phi_K(X)$ and $\phi_{K'}(X')$ in $\mathbb{P}^2$ are equivalent with respect to the action of PGL(2) (or equivalently with respect to the action of SL(3)). A Zariski open subset of Proj(S(3,4)) is a coarse moduli space for the non-hyperelliptic curves of genus 3.

Besides the paper of Shioda, the papers [11] of Igusa on curves of genus 2, and [7] of Geyer on the structure of the ring S(2,2g+2) and the moduli variety for hyperelliptic curves of genus g, should be mentioned. Both papers deal mainly with the reduction of these rings modulo p and show that if $p > 2g+2$, the ring in characteristic p is the reduction of the corresponding ring in characteristic 0. For small characteristics (with respect to g), particularities appear.

Summing up, we can say that classical invariant theory does not contribute much explicitness to the theory of moduli.

Professor Shioda, with whom I had several interesting conversations on this matter, expressed the situation as follows: If you have to describe generators and relations of a ring of invariants of the type S(n,r), then if the structure of S(n,r) is simple, you will find the structure. However if you are unlucky and the structure of S(n,r) is complicated, you will not be able to do anything.

So much for the relationship of classical invariant theory to moduli theory and to present algebraic geometry. In the 19th century, as was already stated, elementary geometry was closely related to invariant theory. This we describe next.

"Projective geometry ist all geometry" Cayley has stated in [1], then Felix Klein introduced his Erlanger Programm in 1872.

What does that mean? First, we discuss Cayley's statement. Projective geometry (over the complex or real number field k) deals with points and forms. A <u>sentence of projective geometry</u> in $\mathbb{P}^n$ involves finitely many points $(\xi^{(i)}) = (\xi_0^i,\dots,\xi_n^i)$ and finitely many forms $f^{(\nu)} = \sum a_\alpha^{(\nu)} \underline{x}^\alpha$ and must be independent from the coordinate system and therefore invariant with respect to the action of PGL(n). In analytic terms, such a sentence is given by a rational function $F(\xi^{(i)}, a_\alpha^{(\nu)})$ in the coordinates of the points and forms which satisfies the following properties.

1) F is homogeneous of degree 0, i.e. $F(\lambda^i\xi^i, \varrho^\nu a_\alpha^{(\nu)}) = F(\xi^i, a_\alpha^{(\nu)})$, $\forall \lambda^i, \varrho^\nu \in k-\{0\}$. Equivalently, the values of F depend only on the points and the forms and not on their representing tuples.

2) F is a relative invariant with respect to the action of GL(n+1), i.e. $F(\sigma(\xi^i), \sigma(a_\alpha^{(\nu)})) = (\det\sigma)^m F(\xi^i, a_\alpha^{(\nu)})$, $\sigma \in GL(n+1)$, with m an integer, called the weight of F.

A rational function F as above is called a <u>homogeneous invariant</u>. The homogeneous absolute invariants have meaning in the geometry of the $\mathbb{P}^n$.

A <u>theorem in $\mathbb{P}^n$</u> is a (polynomial) identity involving finitely many homogeneous invariants or, in Klein's terminology, a syzygy between homogeneous invariants. In this way, the invariant theory of GL(n+1) determines the geometry of the $\mathbb{P}^n$.

The following simple example makes the situation more precise. Consider $\mathbb{P}^1/k$ with (x,y) as homogeneous coordinates. Let $P_i = (x_i,y_i)$, $i = 1,2,3,4$ be 4 points in P^1. Then $\Delta_{ij} = \begin{vmatrix} x_i & y_i \\ x_j & y_j \end{vmatrix}$ is an invariant of weight 1 which is not homogeneous in the coordinates x_i,y_i. Δ_{ij} has therefore no geometric meaning in projective geometry. If it happens that $\Delta_{ij} = 0$ then this statement is homogeneous and has a geometric meaning, namely that the points P_i and P_j' are the same.

To obtain homogeneous invariants from the Δ_{ij}, we have to consider more then three points. Four points lead to the rational fraction $\frac{\Delta_{12}\cdot\Delta_{34}}{\Delta_{14}\,\Delta_{32}}$ which is an absolute invariant and the well known cross ratio of the points $P_1,\ldots,P_4$.

The syzygy

$$(*)\qquad \Delta_{12}\Delta_{34} + \Delta_{13}\Delta_{42} + \Delta_{14}\Delta_{23} = 0$$

leads to a theorem in P^1 which has its geometric interpretation in the well known relation between the 6 values of the cross ratios of the points $P_1,\ldots,P_4$ depending on the order of the points. To indicate this, we divide (*) by the last summand and obtain

$$\frac{\Delta_{12}\cdot\Delta_{34}}{\Delta_{14}\cdot\Delta_{32}} = 1 - \frac{\Delta_{13}\cdot\Delta_{24}}{\Delta_{14}\cdot\Delta_{23}}\ .$$

This completes our description of projective geometry by invariant theory.

Now, we move on to Cayley's statement that projective geometry is all geometry. How are affine geometry, Euclidean geometry and the non-Euclidean geometries a part of projective geometry?

Let $x_0,\dots,x_n$ be a coordinate system in $\mathbb{P}^n$. Let $H_\infty = \{x_0 = 0\}$ and $A^n = \mathbb{P}^n - H_\infty$ be the complement of the hyperplane H_∞. A^n is the affine space of dimension n embedded in $\mathbb{P}^n$ with H_∞ as infinite hyperplane.

Cayley noted the principle that the statements and theorems of the affine space A^n on a geometric configuration, which consists of finitely many points of A^n and finitely many polynomials or hypersurfaces of A^n, are the projective statements and theorems on the associated projective configuration extended by the hyperplane H_∞. (Note, that to a polynomial $f(x_i)$ of degree r in the affine coordinates $\overline{x}_i = \frac{x_i}{x_0}$, $i = 1,\dots,n$, one may associate the form $f(\overline{x}_i)\cdot x_0^r = f(x_0,\dots,x_n)$ of degree r in the variables x_i. Every point of A^n is considered as a point of $\mathbb{P}^n$.) The statements and theorems of Euclidean geometry on a geometric configuration of $A^n \subset \mathbb{P}^n$ are the projective statements of the associated projective configuration which is extended by the quadratic hypersurface of H_∞, defined by $x_0 = 0, \sum_{i=1}^{n} x_i^2 = 0$.

The following examples make this principle clearer.

First we examine affine geometry.

Consider, in A^n, n+1 points $P_i = (1, \frac{p_{i1}}{p_{i0}}, \dots, \frac{p_{in}}{p_{i0}})$, $i = 0,\dots,n$. Then

$$V(P_0,\dots,P_n) = \frac{1}{n!}\begin{vmatrix} 1 & \frac{p_{01}}{p_{00}} & \frac{p_{02}}{p_{00}} & \dots & \frac{p_{0n}}{p_{00}} \\ 1 & \frac{p_{11}}{p_{10}} & \frac{p_{12}}{p_{10}} & \dots & \frac{p_{1n}}{p_{10}} \\ \vdots & & & & \\ 1 & \frac{p_{n1}}{p_{n0}} & \frac{p_{n2}}{p_{n0}} & \dots & \frac{p_{nn}}{p_{n0}} \end{vmatrix} = \frac{1}{n!\, p_{00} \cdots p_{n0}} \begin{vmatrix} p_{00} & \dots & p_{0n} \\ \vdots & & \\ p_{n0} & \dots & p_{nn} \end{vmatrix}$$

is the well known expression for the volume of the n-symplex determined by the points P_i. Now, the determinant $\begin{vmatrix} p_{00} & \cdots & p_{0n} \\ \vdots & & \\ p_{n0} & \cdots & p_{nn} \end{vmatrix}$ is clearly an invariant with respect to GL(n+1).

Furthermore the fraction $\frac{1}{p_{00} \cdots p_{n0}}$ is an invariant, as it is the inverse of the product of the values of the linear form x_o at the point P_i. $V(P_i)$ is therefore a rational invariant which is also homogeneous of degree 0 with respect to the points $P_i = (p_{i0}, \ldots, p_{in})$, but homogeneous of degree $-(n+1)$ with respect to the coefficients $(1,0,\ldots,0)$ of the linear form x_o. We find that V has no geometric meaning in $\mathbb{P}^n$. This is not surprising for any measurement needs in general a unit measure. In other words, we should fix a non degenerated symplex $\langle Q_o,\ldots,Q_n\rangle$ in A^n and consider the fraction $V(P_i) = \frac{V(P_i)}{V(Q_i)}$ which is an absolute projective invariant and belongs then to affine geometry. This fraction is the volume of the n-symplex $\langle P_o,P_1,\ldots,P_n\rangle$ normalized by the symplex $\langle Q_o,\ldots,Q_n\rangle$.

Next we turn to Euclidean geometry, and consider first the angle ω of two hyperplanes $\sum_{i=0}^{n} a_i x_i = 0$ and $\sum_{i=0}^{n} b_i x_i = 0$ (the hyperplanes are in A^n, i.e. different from H_∞); their projective extensions are as above. We must consider the geometric configuration consisting of the two linear forms $\sum_{i=0}^{n} a_i x_i$, $\sum_{i=0}^{n} b_i x_i$ with coordinates (a_i), (b_i) and the infinite hyperquadric $\phi(x_i) = 0 \cdot x_o^2 + x_1^2 + \ldots x_n^2$. The expression of this quadric in hyperplane coordinates $u_o,\ldots,u_n$ of the $\mathbb{P}^n$ $[u_o,\ldots,u_n$ dual to $x_o,\ldots,x_n]$ is $\phi^*(u_i) = 0 \cdot u_o^2 + u_1^2 + \ldots + u_n^2$.

The values of the quadratic form $\phi^*(u_i)$ at the linear form (a_i) and (b_i), i.e. $\phi^*(a_i) = \sum_{i=1}^{n} a_i^2$ respectively $\phi^*(b_i) = \sum_{i=1}^{n} b_i^2$ are absolute invariants. Also the polar form $0\cdot a_0 b_0 + \sum_{i=1}^{n} a_i b_i$ is an absolute invariant. Therefore $\cos\omega = \frac{\sum_{i=1}^{n} a_i b_i}{\sqrt{\phi^*(a_i)\cdot\phi^*(b_i)}}$ is an invariant. Moreover $\cos\omega$ is homogeneous of degree 0 with respect to the coordinates (a_i), (b_i) and also the coefficients $(0,1,\ldots,1)$ of the quadric form $0\cdot u_0^2+u_1^2+\ldots+u_n^2$. Hence, $\cos\omega$ is an absolute homogeneous invariant of the two hyperplanes and the quadric $\phi(x_i)$ which belongs therefore by Cayley's principle to Euclidean geometry.

The distance between two points $P = (1,\frac{p_1}{p_0},\ldots,\frac{p_n}{p_0})$ and $Q = (1,\frac{q_1}{q_0},\ldots,\frac{q_n}{q_0})$ is determined by $r = \sqrt{\sum\left(\frac{p_i}{p_0}-\frac{q_i}{q_0}\right)^2} = \frac{\sqrt{\sum_{i=1}^{n}(q_0p_i-p_0q_i)^2}}{p_0q_0}$.

We analyse this expression.

First one checks that

$$r^2 = \begin{vmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & p_0 & q_0 \\ 0 & 1 & 0 & & & & \vdots & \vdots \\ 0 & 0 & 1 & & & & & \\ \vdots & & & \ddots & & & & \\ 0 & & & & 1 & & & \\ 0 & & & & & 1 & p_n & q_n \\ p_0 & p_1 & p_2 & \cdots & & p_n & 0 & 0 \\ q_0 & q_1 & & \cdots & & q_n & 0 & 0 \end{vmatrix} : \left\{ \begin{vmatrix} 0 & 0 & \cdots & 0 & 0 & p_0 \\ 0 & 1 & & & & \vdots \\ \vdots & & \ddots & & & \\ 0 & & & 1 & & \\ 0 & & & & 1 & p_n \\ p_0 & p_1 & \cdots & & p_n & 0 \end{vmatrix} \cdot \begin{vmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & q_0 \\ 0 & 1 & & & & & \vdots \\ \vdots & & & \ddots & & & \\ 0 & & & & 1 & & \\ 0 & & & & & 1 & q_n \\ q_0 & & \cdots & & & q_n & 0 \end{vmatrix} \right\}$$

and that therefore r^2 is relative invariant of the two points P,Q and the form $\phi(x_i)$. Next, r^2 is homogeneous of degree 0 in the coordinates of the points P and Q but homogeneous of degree -1 in the coordinates $(0,1,\ldots,1)$ of the form $\phi(x_i)$. Also r^2 has weight -2 (notice, every determinant has weight 2) and is therefore not an absolute invariant. The numerical value of r^2 has no meaning in projective geometry which is again not surprising. As a matter of fact, one can measure the distance between two points only if a unit distance is chosen. We must consider quotients of invariants of the above type to obtain absolute homogeneous invariants. If we fix two different points E_o, E_1 in A^n, which will determine the unit distance, and consider the fraction $R(P,Q) = \frac{r(P,Q)}{r(E_o,E_1)}$ as distance between the points P and Q, then $R(P,Q)$ belongs by Cayley's principle to Euclidean geometry.

The described projective treatment of Euclidean geometry using the quadratic form $0 \cdot x_o^2 + x_1^2 + \dots + x_n^2$, suggested to Cayley that he considers an arbitrary non-degenerate real quadratic form in P^n and, using this quadratic form define a quasi-metric and then the angle and the length as in the case of Euclidean geometry. This is what Klein calls Cayley's "projective Maß-bestimmung". Later, (cf. [13]), Klein showed that by this principle of Cayley a new foundation of the non-Euclidean geometries is possible. For simplicity we describe this for the two-dimensional (real) non-Euclidean geometries. These are obtained by taking a coordinate system (x_o, x_1, x_2) in $P^2/\mathbb{C}$ and then a real non-degenerate quadratic form $\phi = ax_o^2 + x_1^2 + x_2^2$, $a \in \mathbb{R}$, $a \neq 0$. ($\phi = 0$ is called the absolute quadric of the geometry.) The non-Euclidean angle φ of two lines in $\mathbb{P}^2$ which are not tangent to $\phi = 0$ is defined by $(ght_1t_2) = e^{2i\varphi}$, where t_1 and t_2 denote the tangents on $\phi = 0$ through the intersection point $g \cap h$ and (ght_1t_2) is the cross ratio of the four lines.[5] The non-Euclidean distance d of two distinct points P and Q not on $\phi = 0$ is defined by $(PQA_1A_2) = e^{\pm 2ikd}$, where A_1, A_2 are the

[5] By this definition also the angle in Euclidean geometry can be obtained. If g,h are two lines in A^2 which intersect in P and i_1, i_2 are the lines in A^2 intersection the infinite line $x_o=0$ in the zeros $(0,1,-i)$ and $(0,1,\ i)$ of the quadric $x_o=0$, $x_1^2+x_2^2=0$, then $e^{2i\varphi} = (g,h,i_1,i_2)$ gives the angle. Compare [18] p.348.

intersection points of the line PQ with $\phi = 0$. $k \neq 0$ is a factor which can be chosen suitably, allowing one to introduce unit distance . k has to be chosen real and > 0 if $a > 0$, and purely imaginary if $a < 0$ to obtain a real distance, at least for certain points of $\mathbb{P}^2$. If $a < 0$ the geometry of J. Bolyai and N.I. Lobatschewsky is obtained. Klein calls this geometry the hyperbolic geometry, as opposed to the elliptic geometry which corresponds to the case $a > 0$ and which was introduced by Riemann in his Habilitationsvortrag [19] . This notation of Klein has nothing to do with the fact that $\phi = 0$ is an ellipse or hyperbola. The points of the 2-dim. hyperbolic geometry or the geometry of Bolyai-Lobatschewsky consist of the real points in $\mathbb{P}^2$, i.e the points with real coordinates with respect to the chosen coordinate system (x_0, x_1, x_2) of $\mathbb{P}^2$, which are in the interior of $\phi = 0$. The interior, here, is characterized by the fact that the tangents to $\phi = 0$ are imaginary. The lines of the hyperbolic geometry consist of the parts of the ordinary real lines in $\mathbb{P}^2$ which are in the interior of $\phi = 0$. Every such line has two "infinite points", the intersection points with $\phi = 0$. Two lines have angle 0 if and only if the intersection point is on $\phi = 0$. Hence, in Bolyai-Lobatschewski's geometry,for every line g and every point $P \notin g$ there exist exactly two lines l_1 and l_2 through P which are parallel to g, i.e. have angle 0 with g.

The real elliptic 2-dimensional geometry of Riemann consists of all real points in $\mathbb{P}^2$(with respect to the chosen coordinate system) and of all real lines. No line of this geometry contains an infinite point, i.e. a point on $\phi = 0$ and no two lines have angle 0. Exercise No 19 in [18], § 33, shows that the 2-dimensional elliptic geometry is essentially the spherical geometry on the real 2-sphere.

Historically, the various elementary geometries were developed at the beginning of the 19th century in the spirit of Euklid in an axiomatic way (cf. [13] and the literature there). As already stated, it was Klein who first realized the connection between these geometries and that Cayley's principle of projective measurement gives a foundation for the elementary geometries. There is still another principle of a group theoretic nature which allows a foundation of elementary geometry within projective geometry. This principle was stated by Klein in his Erlanger Programm.

Cayley considered exclusively invariants with respect to all projective transformations but of an extended geometric configuration. Klein states that the various geometries can also be obtained according to the rule described on page 16 for projective geometry, if one considers instead certain subgroups of PGL(n) and the invariant theory of these subgroups. The subgroup of a particular geometry consists of those elements of PGL(n) which map the added object into itself. Affine geometry in $A^n = \mathbb{P}^n - H_\infty$, for example, is obtained according to this principle by the invariant theory of the "affine group" which consists of the projective transformation of $\mathbb{P}^n$ mapping H_∞ into itself. Euclidean geometry in $A^n = \mathbb{P}^n - H_\infty$ is obtained via this principle as follows. Consider all real projective transformations of the P^n (real with respect to the chosen coordinate system $x_o,\ldots,x_n$) which map the absolute $\phi^{(n-2)} = \left\{x_1^2 + x_2^2 + \ldots + x^2 = 0,\ x_o = 0\right\}$ (and then also the hyperplane $H_\infty = \left\{x_o = 0\right\}$) into itself. They form the group of similarity transformations. The subgroup generated by those similarity transformations which are involutions is the orthogonal group. By the rule of page 16 the invariant theory of this group gives the Euclidean geometry.

The group which characterizes the non-Euclidean geometry determined by the quadric $\phi^{(n-1)} = 0$ is the group of all real projective transformations which leaves $\phi^{(n-1)} = 0$ invariant. The corresponding invariant theory gives the non-Euclidean geometry. We refer to [13] and Pickert's book [18], especially chapter 33 and exercises 16, 17 and 18 of this chapter for more details.

In general, Klein's Erlanger Programm states that every subgroup of PGL(n) determines a geometry via the corresponding invariant theory.

<u>Final remark</u> That Cayley's principle of projective measurement and Klein's group theoretic principle in his Erlanger Programm always yield the same geometries, as Klein states in [12], p.465, is not obvious. It is clear that the notions of angle and length are obtained according to both principles. It is also clear that every geometric statement and theorem according to Cayley's principle, is one in the sense of Klein. But it is not obvious that the converse is true and it would be interesting to see

which part of an elementary geometry according to Klein can be obtained by Cayley's method. For the subgroups G of GL(n) for which both principles are equivalent, the invariant theory of GL(n) contains the invariant theory of G which appears in connection with the considered geometries and is very powerful.

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