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SOME ERGODIC PROPERTIES OF A COMPLEX CONTINUED FRACTION ALGORITHM

by

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Recently R. Kaneiwa, J. Tamura and the author of this paper [1] proved, by making use of a certain kind of continued fraction algrithm for complex numbers, a theorem of Perron on complex Diophantine approximations [2]: For any complex number θ not belonging to the imaginary quadratic field $Q(\sqrt{-3})$ there exist infinitely many integral elements p, q in $Q(\sqrt{-3})$ such that

$$\left|\theta - \frac{p}{q}\right| < \frac{1}{\sqrt[4]{13}\,|q|^2}.$$

If $\theta = \frac{1}{2}(\zeta + \sqrt{\zeta^2 + 4})$, where $\zeta = \frac{1}{2}(1 + \sqrt{-3})$, the constant $\sqrt[4]{13}$ can not be improved.

In this paper we investigate some ergodic properties of the complex continued fraction transformation defined as the remainders of the algorithm in [1].

1. Definition of the algorithm

Every complex number z can be uniquely written in the form $z = u\zeta + v\bar{\zeta}$, where u and v are real and $\bar{w}$ is the complex conjugate of a complex number w. We put

$$[z] = [u]\zeta + [v]\bar{\zeta},$$

where, in the right-hand side, $[x]$ is the largest rational integer not exceeding a real number x. Note that if z is real then $[z]$ becomes the ordinary Gauss's symbol. Now we define a continued fraction algorithm (*) as follows;

$$(*)\begin{cases} T^n z = \dfrac{1}{T^{n-1}z} - \left[\dfrac{1}{T^{n-1}z}\right] \quad (n \geq 1), \quad T^0 z = z - [z], \\ a_n = a_n(z) = \left[\dfrac{1}{T^{n-1}z}\right] \quad (n \geq 1), \quad a_0 = a_0(z) = [z]. \end{cases}$$

These procedures terminate, i.e. $T^n z = 0$ for some $n \geq 0$, if and only if z belongs to $Q(\sqrt{-3})$. Hence every complex number z can be expanded in the form

$$z = a_0 + \frac{1|}{|a_1} + \cdots + \frac{1|}{|a_n + T^n z} \quad (n \geq 0), \qquad (1)$$

provided $T^k z \neq 0$ for all $k < n$.

Let Z_ζ be the ring of all integers in $Q(\sqrt{-3})$ and let N_ζ be the subset of Z_ζ defined by

$N_\zeta = \{ u\zeta + v\bar{\zeta} ;$ u, v non-negative integers with $u+v \geq 1\}$.

We put

$$D = \{ u\zeta + v\bar{\zeta} ;\ u,\ v \geq 0,\ u+v \geq 0 \},$$
$$X = \{ u\zeta + v\bar{\zeta} ;\ 0 \leq u,\ v < 1 \},$$

and

$$Y = D \setminus \{ z ;\ z^{-1} \in X \}.$$

Thus the remainder $T^n z$, in the algorithm (*), is the nth power of the transformation T of X onto itself defined by

$$Tz = \frac{1}{z} - \left[\frac{1}{z}\right] \qquad (z \in X),$$

which is an extension of the real continued fraction transformation

$$Tx = \frac{1}{x} - \left[\frac{1}{x}\right] \qquad (x \in [0,\ 1)\ ).$$

By definition we have

$$|z| \le \frac{2\sqrt{3}}{3} \quad (z \in Y), \tag{2}$$

$$|z| \ge \frac{\sqrt{3}}{2} \quad (z \in D \setminus X), \tag{3}$$

and

$$\{a_0(z)\ ;\ z \in C\} = Z_\zeta ,$$
$$\{a_n(z)\ ;\ z \in C\} = N_\zeta \subset D \setminus X \qquad (n \ge 1) \tag{4}$$

where C is the set of all complex numbers.

Every finite continued fraction

$$\frac{1}{|z_1} + \frac{1}{|z_2} + \cdots + \frac{1}{|z_n}$$

whose partial denominators $z_1, z_2, \cdots, z_n$ belong to $D \setminus \{0\}$ is well-defined, since the fractions z_n^{-1}, $z_{n-1}+z_n^{-1}$, ... are different from zero. (Note that if $z \in D \setminus \{0\}$ then $z^{-1} \in D \setminus \{0\}$ and that if $z, w \in D \setminus \{0\}$ then $z+w \in D \setminus \{0\}$.) Let, more precisely, $z_1, z_2, \ldots, z_n \in D \setminus X$. Then $z_n^{-1} \in Y \setminus \{0\}$ and so $z_{n-1}+z_n^{-1} \in D \setminus X$. Repeating this process we get

$$z_1 + \frac{1}{|z_2} + \cdots + \frac{1}{|z_n} \in D \setminus X \tag{5}$$

and

$$\frac{1}{|z_1} + \frac{1}{|z_2} + \cdots + \frac{1}{|z_n} \in Y \setminus \{0\}. \tag{6}$$

Let $a_0, a_1, \ldots$ be any sequence of integers in $Q(\sqrt{-3})$ such that $a_n \in N_\zeta$ $(n \ge 1)$. Every finite continued fraction

$$a_0 + \frac{1}{|a_1} + \cdots + \frac{1}{|a_n}$$

has a canonical representation p_n/q_n $(p_n, q_n \in Z_\zeta)$, called nth approximant, in the form of an ordinary fraction. Especially if the sequence $a_0, a_1, \ldots$ is given by the algorithm (*) we call the fraction p_n/q_n the nth approximant of z. Thus from the general theory of finite continued fractions we have the following formulae (7)–(10): (For the proofs see [3].)

$$p_n = a_n p_{n-1} + p_{n-2}, \; q_n = a_n q_{n-1} + q_{n-2} \; (n \geq 1), \qquad (7)$$

$$\frac{1}{\lceil a_n} + \frac{1}{\lceil a_{n-1}} + \cdots + \frac{1}{\lceil a_1} = \frac{q_{n-1}}{q_n} \qquad (n \geq 1), \qquad (8)$$

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1} \qquad (n \geq 0), \qquad (9)$$

where $p_{-1} = 1$, $q_{-1} = 0$, $p_0 = a_0$, $q_0 = 1$. Further if p_n/q_n is the nth approximant of z, then

$$z - \frac{p_n}{q_n} = (-1)^n \left(a_{n+1} + T^{n+1}z + \frac{q_{n-1}}{q_n}\right)^{-1} \frac{1}{q_n^2}. \qquad (10)$$

LEMMA 1. (R. Kaneiwa, I. Shiokawa and J. Tamura [2]) Let $a_0, a_1, \ldots$ be any infinite sequences of integers in $Q(\sqrt{-3})$ such that $a_n \in N$ $(n \geq 1)$ and let p_n/q_n be the nth approximant. Then we have

$$q_n \to \infty \quad \text{as} \quad n \to \infty .$$

For completeness we prove this lemma.

Proof. Suppose, on the contrary, that $q_n \not\to \infty$ as $n \to \infty$. So we can choose an infinite subsequence $\{q_{n_j}\}_{j=1}^{\infty}$ such that $|q_{n_j}| < M$ for all $n \geq 1$, where M is a constant independent of j. But from (2) and (6) we have

$$\left| \frac{p_n}{q_n} \right| < |a_0| + \frac{2\sqrt{3}}{3}$$

and so

$$|p_{n_j}| < \left(|a_0| + \frac{2\sqrt{3}}{3}\right) M,$$

where the right-hand side is also independent of j. It follows from these inequalities that $p_{n_j}/q_{n_j} = p_{n_k}/q_{n_k}$ for some j and k with $j < k$, since the ring of all integers in $Q(\sqrt{-3})$ is discrete. Hence we have

$$\frac{1}{|a_{n_j+1}} + \frac{1}{|a_{n_j+2}} + \cdots + \frac{1}{|a_{n_k}} = 0,$$

which contradicts (7).

LEMMA 2. (ibid.) Let z be any complex number not belonging to $Q(\sqrt{-3})$ and let p_n/q_n be its nth approximant. Then we have

$$z = \lim_{n\to\infty} \frac{p_n}{q_n} .$$

Proof. By (10) as well as (3), (5), (6), (8) we have

$$\left| z - \frac{p_n}{q_n} \right| < \frac{2\sqrt{3}}{3} |q_n|^{-2}$$

which tend to zero as $n \to \infty$.

LEMMA 3. (ibid.) With the same notations as in Lemma 1, the nth approximant p_n/q_n converges to some complex number which belongs to b_0+Y.

Proof. Similar to that of Lemma 2.

Now, by means of Lemma 2, every complex number z can be expanded in a regular continued fraction whose partial denominators $a_n(z)$ are integers in $Q(\sqrt{-3})$;

$$z = a_0(z) + \frac{1}{\lfloor a_1(z)} + \frac{1}{\lfloor a_2(z)} + \cdots .$$

This complex continued fraction expansion is a natural extension of the ordinary real one, since both algorithms coincide when z is real.

2. Admissible sequences and fundamental cells

We put

$$A^{(n)} = \{ a_1(z) \cdots a_n(z) \ ; \ z \in X \} \quad (1 \le n \le \infty)$$

Sequences belonging to $A^{(n)}$ $(1 \le n \le \infty)$ will be called admissible. (Note that Lemma 3 suggests the existance of non-admissible sequences.) By definition if $a_1 \cdots a_n \in A^{(n)}$ then $a_1 \cdots a_{n-1} \in A^{(n-1)}$ and $a_2 \cdots a_n \in A^{(n-1)}$. If $a_1 a_2 \cdots \in A^{(\infty)}$ then $a_1 \cdots a_n \in A^{(n)}$ for all $n \ge 1$. And the conjugate $\bar{a}_1 \bar{a}_2 \cdots$ of some (finite or infinite) admissible sequence is also admissible.

For any $a_1 \cdots a_n \in A^{(n)}$ we define

$$X_{a_1 \cdots a_n} = \{ z \in X \ ; \ a_k(z) = a_k, \ 1 \le k \le n \} ,$$

which will be called a fundamental cell of rank n. Thus we have

$$X = \bigcup_{a_1 \cdots a_n \in A^{(n)}} X_{a_1 \cdots a_n}$$

where $X_{a_1 \cdots a_n} \cap X_{b_1 \cdots b_n} = \phi$ if $a_k \ne b_k$ for some k with $1 \le k \le n$; i.e. the set of all fundamental cells of rank n form a partition of X. Besides, for any fixed infinite admissible sequence $a_1 a_2 \cdots$ we find

$$X \supset X_{a_1} \supset \cdots \supset X_{a_1 \cdots a_{n-1}} \supset X_{a_1 \cdots a_n}$$

and (by Lemma 2)

$$\operatorname{diam}(X_{a_1 \cdots a_n}) \to 0 \qquad \text{as} \quad n \to \infty .$$

Hence every Lebesgue measurable subset of X may be approximated with any accuracy by finite unions of mutually disjoint fundamental cells.

For any given $a_1 \cdots a_n \in A^{(n)}$ we difine a function of z by

$$\psi_{a_1 \cdots a_n}(z) = \frac{1|}{|a_1} + \cdots + \frac{1|}{|a_{n-1}} + \frac{1|}{|a_n + z}$$

or (replacing a_n in (7) by a_n+z)

$$= \frac{p_n + p_{n-1} z}{q_n + q_{n-1} z} \qquad (z \in X).$$

Because of the formula (9) the linear transformation $\psi_{a_1 \cdots a_n}$ has the inverse

$$(\psi_{a_1 \cdots a_n})^{-1}(z) = \frac{p_n - q_n z}{-p_{n-1} + q_{n-1} z} \qquad (z \in \psi_{a_1 \cdots a_n}(X)).$$

But the equality (1) can be rewitten in the form

$$z = \psi_{a_1 \cdots a_n}(T^n z) \qquad (z \in X).$$

Hence for any fixed $a_1 \cdots a_n \in A^{(n)}$ the nth power of T restricted on the cell $X_{a_1 \cdots a_n}$ is identical with the inverse of $\psi_{a_1 \cdots a_n}$;

$$T^n z = (\psi_{a_1 \cdots a_n})^{-1}(z) \qquad (z \in X_{a_1 \cdots a_n}). \tag{11}$$

Especially we have for any $a_1 \cdots a_n \in A^{(n)}$

$$X_{a_1 \cdots a_n} = \psi_{a_1 \cdots a_n}(T^n X_{a_1 \cdots a_n}). \tag{12}$$

Now we need some notations : Put

$$U_1 = \{ z \in X ; \left| z + \frac{\sqrt{-3}}{3} \right| > \frac{\sqrt{3}}{3} \},$$

$$U_2 = \{ z \in X ; \ \mathrm{Im}(z) > 0 \},$$

$$U_3 = \{ z \in X ; \ \bar{z} \in U_1, \ \mathrm{Im}(z) > 0 \},$$

and define

$$U_{-j} = \{ \bar{z} ; z \in U_j \} \quad (j = 1, 2, 3).$$

Further we set $U_0 = X$ for notational convenience.

Considering the reciprocals $U_j^{-1} = \{ z ; z^{-1} \in U_j \}$

we obtain (see Fig. 1)

$$X = \psi_\zeta(U_1) \cup \psi_{\bar\zeta}(U_{-1}) \cup (\bigcup_{\substack{a \in N_\zeta \\ a \neq \zeta, \bar\zeta}} \psi_a(X)), \tag{13.0}$$

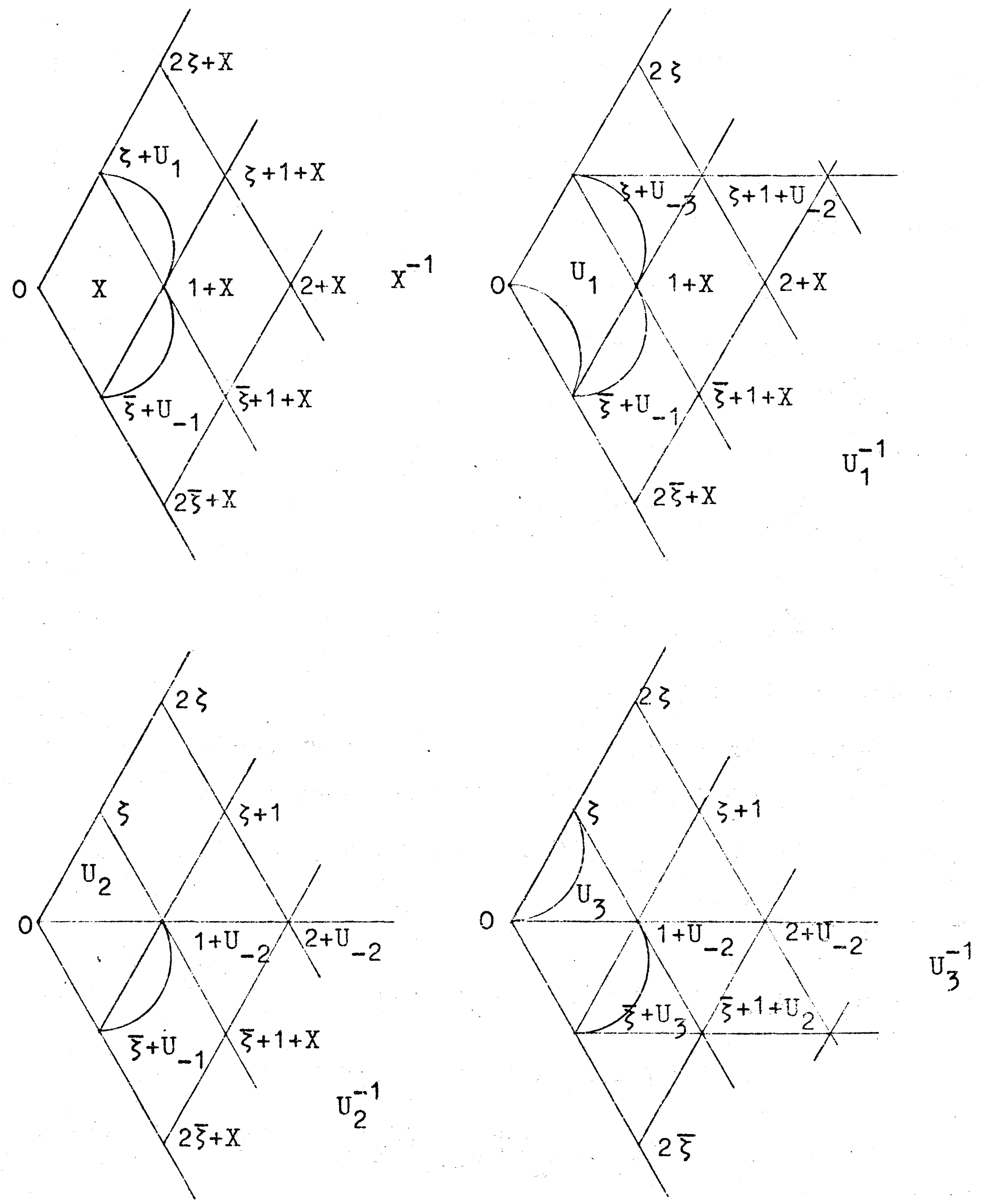
$$U_1 = \psi_\zeta(U_{-3}) \cup \psi_{\bar\zeta}(U_{-1}) \cup (\bigcup_{k=1}^{\infty} \psi_{\zeta+k}(U_{-2})) \cup (\bigcup_{\substack{a \in N_\zeta, a \neq \bar\zeta \\ \mathrm{Im}(a) \leq 0}} \psi_a(X)), \tag{13.1}$$

$$U_2 = \psi_{\bar\zeta}(U_{-1}) \cup (\bigcup_{k=1}^{\infty} \psi_k(U_{-2})) \cup (\bigcup_{\substack{a \in N_\zeta, a \neq \bar\zeta \\ \mathrm{Im}(a) < 0}} \psi_a(X)), \tag{13.2}$$

and

$$U_3 = \psi_{\bar\zeta}(U_3) \cup \bigcup_{k=1}^{\infty} (\psi_k(U_{-2}) \cup \psi_{\bar\zeta+k}(U_2)). \tag{13.3}$$

Taking the complex conjugate of (13.1)–(13.3) we have also the same relations for U_{-1}, U_{-2} and U_{-3} to which we assign (13.-1), (13.-2) and (13.-3) resp.



$2\zeta+X$
$\zeta+U_1$
$\zeta+1+X$
O
X
$1+X$
$2+X$
X^{-1}
$\bar{\zeta}+U_{-1}$
$\bar{\zeta}+1+X$
$2\bar{\zeta}+X$
2ζ
$\zeta+U_{-3}$
$\zeta+1+U_{-2}$
O
U_1
$1+X$
$2+X$
$\bar{\zeta}+U_{-1}$
$\bar{\zeta}+1+X$
U_1^{-1}
$2\bar{\zeta}+X$
2ζ
ζ
$\zeta+1$
U_2
O
$1+U_{-2}$
$2+U_{-2}$
$\bar{\zeta}+U_{-1}$
$\bar{\zeta}+1+X$
U_2^{-1}
$2\bar{\zeta}+X$
2ζ
ζ
$\zeta+1$
U_3
O
$1+U_{-2}$
$2+U_{-2}$
U_3^{-1}
$\bar{\zeta}+U_3$
$\bar{\zeta}+1+U_2$
$2\bar{\zeta}$

Fig. 1

In any case U_j can be written in the form

$$U_j = \bigcup_{a \in M_j} \psi_a(U_k) \tag{14}$$

where M_j is a subset of N_ζ and k $(-3 \leq k \leq 3)$ are chosen uniquely according as j and a. In addition, we note that

$$\psi_a(X) \cap \psi_b(X) = \phi \tag{15}$$

whenever $a \neq b$ $(a,\ b \in N_\zeta)$.

LEMMA 4. Let $n \geq 1$ and let $a_1 \cdots a_n \in A^{(n)}$. Then we have

$$X_{a_1 \cdots a_n} = \psi_{a_1 \cdots a_n}(U_j) \tag{16}$$

and so

$$T^n X_{a_1 \cdots a_n} = U_j \tag{17}$$

for some j $(-3 \leq j \leq 3)$.

Proof. By induction on n. First we prove (16). If $n = 1$ (16) follows from (13.0). Suppose that (16) hold for all $a_1 \cdots a_n \in A^{(n)}$. Then we have for any $a_1 \cdots a_{n+1} \in A^{(n+1)}$

$$\begin{aligned} X_{a_1 \cdots a_{n+1}} &= \{ z \in X_{a_1 \cdots a_n};\ a_{n+1}(z) = a_1(T^n z) = a_{n+1} \} \\ &= \{ \psi_{a_1 \cdots a_n}(w);\ w \in U_j,\ a_1(w) = a_{n+1} \} \\ &= \psi_{a_1 \cdots a_n}(\psi_{a_{n+1}}(U_k)), \quad \text{(by (14),(15))} \\ &= \psi_{a_1 \cdots a_{n+1}}(U_k), \end{aligned}$$

where j is defined by $U_j = T^n X_{a_1 \cdots a_n}$ and k chosen uniquely in (14). Now (17) follows from (12) and (16).

Let E be any subset of X. Then by Lemma 4 we have for any $a_1 \cdots a_n \in A^{(n)}$

$$T^{-n}E = \{ z \in X ; T^n z \in E \}$$
$$= \bigcup_{a_1 \cdots a_n \in A^{(n)}} \{ z \in X_{a_1 \cdots a_n} ; T^n z \in E \cap U_j \}$$
$$= \bigcup_{a_1 \cdots a_n \in A^{(n)}} \psi_{a_1 \cdots a_n}(E \cap U_j), \quad U_j = T^n X_{a_1 \cdots a_n} \qquad (18)$$

3. Estimates of the Lebesgue measure

Let m be the Lebesgue measure on the complex plane and let $\mathcal{B}$ be the σ-field of all measurable subsets of X. Then we have for any $a_1 \cdots a_n \in A^{(n)}$ and $E \in \mathcal{B}$

$$m(\psi_{a_1 \cdots a_n}(E)) = \iint_E |\psi'_{a_1 \cdots a_n}(z)|^2 dxdy, \quad z = x+iy. \qquad (19)$$

But using (9) we find

$$\psi_{a_1 \cdots a_n}(z) = (-1)^n (q_n + q_{n-1} z)^{-2}$$

and so

$$|\psi'_{a_1 \cdots a_n}(z)|^2 = |q_n|^{-4} \left| 1 + \frac{q_{n-1}}{q_n} z \right|^{-4} \qquad (20)$$

Hence we have

$$3^{-4} < |q_n|^4 \, |\psi'_{a_1 \cdots a_n}(z)|^2 < 3^4 \qquad (21)$$

and

$$3^{-4} < |q_n|^{-4} \, |(\psi^{-1}_{a_1 \cdots a_n})'(z)|^2 < 3^4, \qquad (22)$$

since (from (2), (3), (6), (8))

$$3^{-1} < \frac{\sqrt{3}}{2} \le \left| 1 + \frac{q_n}{q_n} z \right| \le 1 + \frac{2\sqrt{3}}{3} < 3.$$

Taking account of the fact that $3^{-2} < m(U_j) < 1$ $(-3 \le j \le 3)$ we have from (19) and (21)

$$3^{-6} < |q_n|^4\, m(X_{a_1\cdots a_n}) < 3^4 \quad (a_1\cdots a_n \in A^{(n)}) \tag{23}$$

We write

$$S(n) = \sum_{a_1\cdots a_n \in A^{(n)}} |q_n|^{-4}$$

Then for any $n \geq 1$ we have

$$3^{-5} < S(n) < 3^6 \tag{24}$$

Indeed it follows from (23) that

$$3^4\, S(n) > \sum_{A^{(n)}} m(X_{a_1\cdots a_n}) = m(X) > 3^{-1}$$

and

$$3^{-6}\, S(n) < m(X) < 1.$$

By means of Lemma 4 the set $A^{(n)}$ of all admissible sequences can naturally be divided into seven subsets ; we put

$$A_j^{(n)} = \{\, a_1\cdots a_n \in A(n) \ ;\ T^n X_{a_1\cdots a_n} = U_j \} \ (-3 \leq j \leq 3),$$

then we have

$$A^{(n)} = \bigcup_{j=-3}^{3} A_j^{(n)} .$$

By (13.j) $(-3 \leq j \leq 3)$ we have for any $n \geq 1$ the following relations;

$$\begin{aligned} A_0^{(n)} = \{\, a_1\cdots a_n \in A^{(n)};\ & a_1\cdots a_{n-1} \in A_0^{(n-1)},\ a_n \neq \zeta, \bar{\zeta}\ ; \\ & \text{or } a_1\cdots a_{n-1} \in A_1^{(n-1)},\ a_n \neq \bar{\zeta},\ \mathrm{Im}(a_n) \leq 0\ ; \\ & \text{or } a_1\cdots a_{n-1} \in A_{-1}^{(n-1)},\ a_n \neq \zeta,\ \mathrm{Im}(a_n) \geq 0\ ; \\ & \text{or } a_1\cdots a_{n-1} \in A_2^{(n-1)},\ a_n \neq \bar{\zeta},\ \mathrm{Im}(a_n) < 0\ ; \\ & \text{or } a_1 \quad a_{n-1} \in A_{-2}^{(n-1)},\ a_n \neq \zeta,\ \mathrm{Im}(a_n) > 0 \,\} \end{aligned} \tag{25.0}$$

$$A_1^{(n)} = \{ a_1 \cdots a_n \in A^{(n)};\ a_1 \cdots a_{n-1} \in A_0^{(n-1)} \cup A_{-1}^{(n-1)} \cup A_{-2}^{(n-1)},$$
$$a_n = \zeta \ \} \qquad (25.1)$$

$$A_2^{(n)} = \{ a_1 \cdots a_n \in A^{(n)};\ a_1 \cdots a_{n-1} \in A_{-1}^{(n-1)} \quad A_3^{(n-1)},\ a_n - \bar{\zeta} \in N\ ;$$
$$\text{or } a_1 \cdots a_{n-1} \in A_{-2}^{(n-1)} \quad A_{-3}^{(n-1)},$$
$$a_n \in N \ \} \qquad (25.2)$$

$$A_3^{(n)} = \{ a_1 \cdots a_n \in A^{(n)};\ a_1 \cdots a_{n-1} \in A_{-1}^{(n-1)} \cup A_3^{(n-1)},$$
$$a_n = \bar{\zeta} \ \} \qquad (25.3)$$

and

$$A_{-j}^{(n)} = \{ \bar{a}_1 \cdots \bar{a}_n;\ a_1 \cdots a_n \in A_j^{(n)} \} \quad (j=1,2,3) \qquad (25.-j)$$

where N is the set of all positive integers.

We write

$$S_j(n) = \sum_{a_1 \cdots a_n \in A_j^{(n)}} |q_n|^{-4} \quad (-3 \leq j \leq 3).$$

Thus we have

$$S_j(n) = S_{-j}(n) \quad (-3 \leq j \leq 3). \qquad (26)$$

and

$$S(n) = \sum_{j=-3}^{3} S_j(n). \qquad (27)$$

LEMMA 5. For any $n \geq 3$ we have

$$S_j(n) > 3^{-12} \quad (-3 \leq j \leq 3)$$

Proof. From (7) and the inequality

$$3^{-1} \ |a_n| < \left| a_n + \frac{q_{n-2}}{q_{n-1}} \right| < 3 \ |a_n|$$

we find for any $a_1 \cdots a_n \in A^{(n)}$

$$3^{-1} \ |a_n||q_{n-1}| < |q_n| < 3 \ |a_n||q_{n-1}| \ . \qquad (28)$$

By (25.0), (26) and (28) we have

$$3\, S_0(n) > \sum_{a \in N_\zeta \setminus \{\zeta, \bar{\zeta}\}} |a|^{-4} \sum_{a_1 \cdots a_{n-1} \in A_0^{(n-1)}} |q_{n-1}|^{-4}$$
$$+ 2 \sum_{\substack{a \in N_\zeta \setminus \{\bar{\zeta}\} \\ \mathrm{Im}(a) \le 0}} |a|^{-4} \sum_{a_1 \cdots a_{n-1} \in A_1^{(n-1)}} |q_{n-1}|^{-4}$$
$$+ 2 \sum_{\substack{a \in N_\zeta \setminus \{\bar{\zeta}\} \\ \mathrm{Im}(a) < 0}} |a|^{-4} \sum_{a_1 \cdots a_{n-1} \in A_2^{(n-1)}} |q_{n-1}|^{-4}$$
$$> S_0(n-1) + 2S_1(n-1) + 2\,|\bar{\zeta} + 1|^{-4}\, S_2(n-1)$$
$$> 3^{-2}(S_0(n-1) + S_1(n-1) + S_2(n-1)).$$

Hence we have

$$S_0(n) > 3^{-3}(S_0(n-1) + S_1(n-1) + S_2(n-1)).$$

In the same way we obtain

$$S_1(n) > 3^{-1}(S_0(n-1) + S_1(n-1) + S_2(n-1)),$$
$$S_2(n) > 3^{-3}(S_1(n-1) + S_2(n-1) + S_3(n-1)),$$

and

$$S_3(n) > 3^{-1}(S_1(n-1) + S_3(n-1))$$

(using (25.1) - (25.3).) It follows from these inequalities with (24), (26), (27) that

$$S_0(n) > 3^{-6} \sum_{j=0}^{3} S_j(n-2) > 3^{-7} S(n) > 3^{-12}.$$

Similary we have for any $n \ge 3$

$$S_j(n) > 3^{-12} \quad (-3 \le j \le 3).$$

4. Invariant measure and ergodicity

THEOREM 1. Let E be any measurable subset of X such that $T^{-1}E = E$. Then $m(E) = 0$ or 1.

Proof. We assume that $m(E) > 0$. By (17) and (18) we find for any $a_1 \cdots a_n \in A^{(n)}$

$$E \cap X_{a_1 \cdots a_n} = T^{-n}E \cap \psi_{a_1 \cdots a_n}(U_j)$$
$$= \psi_{a_1 \cdots a_n}(E \cap U_j), \quad U_j = T^n X_{a_1 \cdots a_n}.$$

From this as well as (19), (21), and (23) we have

$$m(E \cap X_{a_1 \cdots a_n}) \geq 3^{-4}|q_n|^{-4}\, m(E \cap U_j)$$
$$\geq 3^{-8}\, m(X_{a_1 \cdots a_n}) \min\{m(E \cap U_3), m(E \cap U_{-3})\}. \tag{29}$$

But (13.3) and (18) implies that

$$E \cap U_3 = T^{-1}E \cap U_3 \supset \psi_{\overline{\xi}+1}(E \cap U_2) \cup \psi(E \cap U_{-2}).$$

Beside for any measurable subset F of U_2 we have by (19) and (20)

$$m(\psi_1(F)) = \iint_F |1 + z|^{-4} dxdy$$
$$> \iint_F |\overline{\xi} + 1 + z|^{-4} dxdy = m(\psi_{\overline{\xi}+1}(F)).$$

Hence

$$m(E \cap U_3) > m(\psi_{\overline{\xi}+1}(E \cap U_2)) + m(\psi_{\overline{\xi}+1}(E \cap U_{-2}))$$
$$= m(\psi_{\overline{\xi}+1}(E)) > 3^{-4}\ |\overline{\xi} + 1|^{-4} m(E) = 3^{-6} m(E). \tag{30}$$

Similary we have

$$m(E \cap U_{-3}) \geq 3^{-6} m(E) \tag{31}$$

By (29), (30) and (31) the inequality

$$m(E \cap F) \geq 3^{-14} m(E) m(F). \tag{32}$$

hold for all fundamental cell F, and so for any measurable set F in X. Thus, putting $F = X \setminus E$ in (32), we have

$$m(E)m(X \setminus E) = 0$$

which implies $m(E) = 1$.

THEOREM 2. There exists an unique, T-invariant probability measure μ equivalent to Lebesgue measure such that the inequalities

$$3^{-15}\frac{m(E)}{m(X)} \leq \mu(E) \leq 3^{10}\frac{m(E)}{m(X)}, \qquad (33)$$

hold for all $E \in \mathcal{B}$.

Proof. To prove the existance it is enough to show that the inequalities

$$3^{-15}m(E) \leq m(T^{-n}E) \leq 3^{10}m(E), \ (n \geq 0) \qquad (34)$$

hold for all $E \in \mathcal{B}$. (see F. Schweiger [5] §6–§7). By (18) (19), (21) and (24) we have

$$m(T^{-n}E) \leq \sum_{A^{(n)}} m(\psi_{a_1\ldots a_n}(E))$$

$$\leq 3^4 m(E)S(n) \leq 3^{10}m(E)$$

To prove the left-hand side inequalities in (34), we suppose fiast that $E \subset U_3$. Then, by (18),(19), (21) and Lemma 5, we have

$$m(T^{-n}E) \geq \sum_{j=0}^{3} \sum_{A_j^{(n)}} m(\psi_{a_1\ldots a_n}(E))$$

$$\geq 3^{-4}m(E)\sum_{j=0}^{3} S_j(n) \geq 3^{-15}m(E),$$

as required. Similary for any $E \subset U_2 \setminus U_3$

$$m(T^{-n}E) = \sum_{j=0}^{2} \sum_{A_j^{(n)}} m(\psi_{a_1\ldots a_n}(E)) \geq 3^{-15}m(E).$$

Besides the left-hand side of the inequalities (34) is also true for any subset E of U_{-2} or U_{-2} U_{-3}. As a result (34) holds for any subset E of X, since

$$E = (E \cap U_3) \cup (E \cap (U_2 \setminus U_3)) \cup (E \cap U_{-2}) \cup (E \cap (U_{-2} \setminus U_{-3})).$$

By Theorem 1 the T-invariant probability measure is uniquely given by the limit

$$\mu(E) = \frac{1}{m(X)} \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} m(T^{-k}E), \quad E \in \mathcal{B}. \tag{35}$$

(see also F. Schweiger [5].) And so (33) follows from (34) and (35).

THEOREM 3. T is ergodic with respect to μ ; i.e. for any $f \in L^1(X)$ we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k z) = \int_X f(z) d\mu, \text{ a.e.}$$

Proof. Follows from Theorem 1, 2 and Birkhoff's individual ergodic theorem.

As an application of Theorem 3, we have

$$\lim_{n\to\infty} (a_1(z) + \cdots + a_n(z))^{\frac{1}{n}} = e^{\alpha}, \text{ a.e.}$$

where

$$\alpha = \int_X \log a_1(z)\, d\mu .$$

(Note that $f(z) = \log a_1(z) \in L^1(X)$, since the series $\sum_{a \in N_\zeta} a^{-4} \log a$ is convergent.)

5. Exactness

A measure-preserving transformation T on a normalized measure space $(X, \mathcal{B}, \mu)$ is said to be exact if

$$\bigcap_{n=0}^{\infty} T^{-n}\mathcal{B} = \{ \emptyset, X \},$$

or equivalently, if for every set E of positive measure with the measurable images TE, T^2E, ... the relation

$$\lim_{n \to \infty} \mu(T^n E) = 1 \tag{36}$$

holds.(see V.A. Rohlin [4])

THEOREM 4. The transformation T is exact.

The proof requires the following

LEMMA 6. Let $\varepsilon > 0$ and let E be any measurable set such that

$$\mu(U_j \setminus E) < \varepsilon$$

for some j $(-3 \leq j \leq 3)$. Then

$$\mu(TE) > 1 - 3^{31}\varepsilon .$$

Proof of Lemma 6. It is clearly enough to consider only the case $j = \pm 3$. We may assume further that $j = 3$, since the following arguments are available for the conjugate case $j = -3$. First, by (13.3), we note that

$$\psi_1(U_{-2}) \cup \psi_{\bar{\xi}+1}(U_2) \subset U_3. \tag{37}$$

But, by (33) and (22) with $a_n = 1$, we have

$$\begin{aligned} \mu(T(\psi_1(U_{-2}) \setminus E) &\leq 3^{10} m(X)^{-1} m(T(\psi_1(U_{-2}) \setminus E)) \\ &\leq 3^{14} m(X)^{-1} m(\psi_1(U_{-2}) \setminus E) \leq 3^{29} \mu(\psi_1(U_{-2}) \setminus E). \end{aligned} \tag{38}$$

In the same way, (using (22) with $a_n = \bar{\xi}+1$)

$$\mu(T(\psi_{\bar{\xi}+1}(U_2) \setminus E)) \leq 3^{31} \mu(\psi_{\bar{\xi}+1}(U_2) \setminus E)). \tag{39}$$

Hence it follows from (37), (38), and (39) that

$$\begin{aligned} &\mu(\ T((\psi_1(U_{-2}) \cup \psi_{\bar{\xi}+1}(U_2)) \setminus E)\ ) \\ &\leq 3^{31} \mu(\ (\psi_1(U_{-2}) \cup \psi_{\bar{\xi}+1}(U_2)) \setminus E\ ) \\ &\leq 3^{31} \mu(U_3 \setminus E) \leq 3^{31} \varepsilon\ . \end{aligned} \tag{40}$$

Therefore, by (37), (40) and (13.3), we obtain

$$\mu(TE) \geq \mu(T((\psi_1(U_{-2}) \cup \psi_{\xi+1}(U_2)) \cap E))$$
$$\geq \mu(T(\psi_1(U_{-2}) \cup \psi_{\xi+1}(U_2))) - \mu(T((\psi_1(U_{-2}) \cup \psi_{\xi+1}(U_2)) \setminus E))$$
$$> 1 - 3^{31}\varepsilon .$$

Proof of Theorem 3. We prove (36). Let $E \in \mathcal{B}$ given arbitrary. (Note that, by the definition of T, $E \in \mathcal{B}$ if and only if $TE \in \mathcal{B}$.) Let $\varepsilon > 0$. Then there exists a fundamental interval $F = X_{a_1 \ldots a_n}$ such that

$$m(F \setminus E) < 3^{-50}\varepsilon\, m(F). \tag{41}$$

Otherwise, the inequality

$$m(F \setminus E) \geq 3^{-50}\varepsilon\, m(F)$$

holds for all fundamental interval F, and so it holds also for arbitrary measurable set F. Putting $F = E$ we have $m(F) = 0$; a contradiction.

Now by Lemma 4, (33), (11), (22), (23) and (41) .

$$\mu(T^nF \setminus T^nE) \leq \mu(T^n(F \setminus E))$$
$$\leq 3^{11} m(T^n(F \setminus E)) \leq 3^{15} |q_n|^4 m(F \setminus E)$$
$$\leq 3^{19} m(F)^{-1} m(F \setminus E) < 3^{-31}\varepsilon \quad . \tag{42}$$

Noticing that $T^nF = U_j$ for some j by Lemma 4, we have from (42) and Lemma 6

$$\mu(T^{n+1}E) > 1 - \varepsilon .$$

Since $\mu(E)$, $\mu(TE)$, $\mu(T^nE)$, ... is increasing, the relation (36) is proved.

As a general property of exact transformations (see V.A. Rohlin [4]) we have

Corollary. The transformation T is mixing of all degrees. In particular T is strongly mixing; i.e. for any E, $F \in \mathcal{B}$ we have

$$\lim_{n \to \infty} \mu(T^{-n}E \cap F) = \mu(E)\,\mu(F).$$

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