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On the regularity of arithmetical additive function.

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A. Definitions and notations.

N is the set of all the integers $0, 1, 2, \ldots$; $N^* = N \setminus \{0\}$. If g is a prime, "$g^\alpha || n$" means "g^α divides n and $g^{\alpha+1}$ does not divide n". If $\alpha \in N$, we define an additive function δ_α by:

$$\delta_0(n) = \begin{cases} 1 & \text{if } 2|n \\ 0 & \text{if } 2\nmid n, \end{cases} \quad \text{and} \quad \text{if } \alpha > 0, \quad \delta_\alpha(n) = \begin{cases} \frac{1}{2} & \text{if } 2^{\alpha+1}|n \\ -\frac{1}{2} & \text{if } 2^\alpha||n \\ 0 & \text{if } 2^\alpha \nmid n. \end{cases}$$

f is additive (resp. completely additive) means: $f : N \to C$, and $f(mn) = f(m) + f(n)$ if $(m, n) = 1$ (resp. $f(mn) = f(m) + f(n)$.)

B. Results: Let f and g be arithmetical additive functions.

I. Suppose there exist $a, b \in N^*$, $(a, b) = 1$, $\varepsilon = \pm 1$ (fixed), $\ell \in C$ such that:

$$\lim_{x \to +\infty} \frac{1}{x} \sum_{n \leq x} |f(an + \varepsilon b) - g(n) - \ell| = 0 \quad ([H_I])$$

I.1. If a is even, then, there exist a completely additive function g' such that:

$$\lim_{x\to+\infty} \frac{1}{x} \sum_{n\leqslant x} |g'(kn+\varepsilon) - g'(kn)| = 0 \quad \text{if} \quad k \geqslant a,$$

and h, an additive function, such that:

$$h(n) = h((n, b)), \text{ (i.e. } h(p^r) = 0 \text{ if } r \geqslant 1,\ p \nmid b,$$
$$\text{and } h(p^r) = h(p^\alpha) \text{ if } r \geqslant \alpha,\ p^\alpha || b,\ \alpha \geqslant 1)$$

and we have:

$$g(n) = g'(n) + h(n) \quad \text{if} \quad n \in \mathbb{N}^*,$$
$$f(n) = g'(n) + h(n) \quad \text{if} \quad (n, a) = 1,$$
$$\ell = g'(a).$$

I.2. If a is odd, and suppose $2^\alpha || b$, $\alpha \geqslant 0$, then, there exist g' and h satisfying to the conditions stated before, and a constant λ such that:

$$g(n) = g'(n) + h(n) + \lambda\delta_\alpha(n) \quad \text{if} \quad n \in \mathbb{N}^*$$
$$f(n) = g'(n) + h(n) - \lambda\delta_\alpha(n) \quad \text{if} \quad (n, a) = 1.$$

Moreover, we have :

$$\lambda = 2\, g(2^{\alpha+1}) - g(2^{\alpha+2}) - g(2^\alpha)$$
$$\ell = g'(a) \quad \text{if} \quad \alpha > 0$$
$$\ell = g'(a) - \lambda \quad \text{if} \quad \alpha = 0.$$

II. If, in $[H_1]$, $a = 1$, then I.1 and I.2 hold with $g'(n) = c \log n$, where c is a constant.

III. If there exist $a, b \in N^*$, $(a, b) = 1$, $\varepsilon = \pm 1$ (fixed), $\ell \in C$ such that:

$$f(an + \varepsilon b) - g(r) - \ell = o(1), \quad (n \to +\infty),$$

then I.1. and I.2. hold with $g'(n) = c \log n$, where c is a constant.

IV. If there exist $a, b \in N^*$, $(a, b) = 1$, $\varepsilon = \pm 1$ (fixed) such that

$$f(an + \varepsilon b) - g(n) = O(1),$$

then, there exists a constant d such that

$g(n) - d \log n$ is bounded on N^*,

$f(n) - d \log n$ is bounded on $\{n \in N^* \; ; \; (n, a) = 1\}$.

Proofs will be given later.

C. Conclusion.

1. It can be remarked that II, III, IV are necessary and sufficient conditions. Concerning I, it seems probable that the following conjecture is true: "If h is completely additive, and there exist $k > 1$, $\varepsilon = \pm 1$ (fixed), such that

$$\lim_{x \to +\infty} \frac{1}{x} \sum_{n \leq x} |h(km + \varepsilon) - h(km)| = 0, \text{ then } h(n) = c \log n."$$

2. The condition "(a, b) = 1" can be removed, because if f is additive, then $(f(kn) - f(k)) = f_k(n)$ is additive too;

So, if $a = a_1\delta$, $b = b_1\delta$, $(a_1, b_1) = 1$, we have:

$$\begin{aligned} f(an + \varepsilon b) - g(n) - \ell &= (f(\delta(a_1 n + \varepsilon b_1)) - f(\delta)) - g(n) - (\ell - f(\delta)) \\ &= f_\delta(a_1 n + \varepsilon b_1) - g(n) - (\ell - f(\delta)), \end{aligned}$$

and we have the same hypothesis as in B.