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Lecture No. 1

On the analyticity of solutions of partial differential equations.

It is known that all solutions of linear elliptic equations and systems with analytic coefficients are analytic functions (see [1], [2], and survey in [3]). If the coefficients are constant, only elliptic systems have this property. It is of interest to discover other classes of equations and systems with variable coefficients, all of whose solutions are analytic functions, as well as classes which do not have this property. Some results in this direction have been obtained in [4]-[8]. Here a new approach is given to study these questions.

First, we shall establish a priori estimates in the complex domain for the solutions of differential equations and systems with only analytic solutions. Using these estimates we then exhibit classes of equations and systems which possess nonanalytic solutions. We can get such a priori estimates as a consequence of the following general theorem.

Let Ω be a bounded domain in $R^{n+1} = (x_0, x_1, \ldots, x_n)$.

Theorem 1. Suppose that the Banach space B consists of distributions $u \in \mathcal{D}'(\Omega)$ and the convergence of a sequence in B implies its convergence in $\mathcal{D}'(\Omega)$. Suppose that for a domain $G \Subset \Omega$ and for $u \in B$ there exists a domain $Q_{\delta,j}(G)=\{x,y_j;x\in G,|y_j|<\delta\}$ such that u can be extended in $Q_{\delta,j}(G)$ as an analytic function $u(x_0,\dots,x_j+iy_j,\dots,x_n)$ of x_j+iy_j and u and $\frac{\partial u}{\partial x_k}$, $k=0,1,\dots,n$, are bounded by modulus in $Q_{\delta,j}(G)$, $\delta=\text{const}>0$, (δ can depend on u). Then there exist such constant $\delta_0>0$ and $C_0>0$ that for any $u\in B$ the following inequality is valid:

$$\sup_{Q_{\delta_0,j}(G)} |u| \leqq C_0\|u\|_B. \tag{1}$$

This theorem can be proved, using Baire's category theorem (see [9]).

The theorem 1 is also valid for a Banach vector space B^N.

Let us consider a linear system of partial differential equations with analytic coefficients

$$L(u) = 0 \quad \text{in} \quad \Omega \subset R^{n+1},\ u=(u_1,\dots,u_N). \tag{2}$$

We say that a weak solution $u\in(\mathcal{D}'(\Omega))^N$ of system (2) is an analytic function with respect to x_j in Ω, if for any domain $G\Subset\Omega$ there exists a constant $\delta(G)$ such that u can be extended in $Q_{\delta,j}(G)$ as an analytic function $u(x_0,\dots,x_j+iy_j,\dots,x_n)$ of x_j+iy_j and $u,\frac{\partial u}{\partial x_k}$, $k=0,1,\dots,n$, are bounded by modulus in $Q_{\delta,j}(G)$. The class of analytic functions with respect to x_j in Ω we denote by $A_j(\Omega)$.

Using theorem 1, we describe a class of linear first order systems of partial differetial equations and a class of linear partial differetial equations of the order m

with analytic coefficients, which have at least one nonanalytic solution. For this purpose we construct a family of analytic with respect to x_j solutions which do not satisfy the estimate (1) with some norm $\|u\|_{B^N}$. It means that such an equation or a system have at least one nonanalytic solution (see [10], [11], [12], [13]). For example, the equation, studied by M. Baouendi and C. Goulaouic in [6].

$$u_{x_0x_0} + x_0^2u_{x_1x_1} + u_{x_3x_3} = 0 \quad \text{in} \quad R^3$$

has a family of solutions

$$u_\rho(x) = \exp\{i\rho x_1 - \tfrac{1}{2}\rho x_0^2 + \sqrt{\rho}x_2\}.$$

These solutions $u_\rho(x)$ do not satisfy the inequality (1) for $j = 0$, if Ω contains points with the coordinate $x_0 = 0$ and if ρ is sufficiently large, $\|u\|_B \equiv \sup_\Omega|u|$. For the first order differential equation of the form

$$\frac{\partial}{\partial x_0} + \sum_{k=1}^{n} a^k(x)\frac{\partial u}{\partial x_k} + c(x)u = 0 \qquad (3)$$

in the same way it can be proved that if $\operatorname{Im} x_0a^j(x_0,x'+iy')$ does not change the sign in a neighbourhood of the origin of the space $R^{2n+1} = (x_0,x',y')$, then equation (3) has a non-analytical solution with respect to x_j in a neighbourhood of the origin of $R^{n+1} = (x_0,x')$. For Mizohata's equation [4]

$$\frac{\partial u}{\partial t} + it^s\frac{\partial u}{\partial x} = 0 \qquad (4)$$

it means that if s is odd, then equation (4) has nonanalytic solution in any neighbourhood of the origin. For systems with constant coefficients it can be proved that if a system has a characteristic direction, then such a system has nonanalytic solutions. For second order equations the following theorem is valid (see [14]).

Theorem 2. Suppose that the equation

$$\sum_{k,j=1}^{n} a^{kj}(x)u_{x_k x_j} + \sum_{k=1}^{n} b^k(x)u_{x_k} + c(x)u = 0 \qquad (5)$$

has real analytic coefficients in a neighbourhood Ω of the origin of $R^n_x = (x_1,\dots,x_n)$ and for $x \in \Omega$ and $\xi \in R^n_\xi$

$$\sum_{kj=1}^{n} a^{kj}(x)\xi_k\xi_j \geqq 0,$$

$\sum_{kj}^{n} |a^{kj}(0)| \neq 0$, there exists a real analytic function $H(x)$ such that $H(0) = 0$, grad $H(0) \neq 0$,

$$\sum_{k,j=1}^{n} a^{kj}(x)H_{x_k}(x)H_{x_j}(x) = 0$$

for any $x \in \Omega$, if $H(x) = 0$. Then the equation (5) has a non-analytic solution in any neighbourhood of $x = 0$. In the paper [14] for equation (5) with $n = 2$ the conditions are given when this equation has only analitic solutions.

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