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Lecture No.2

Analyticity of solutions and boundary value problems in unbounded domains for parabolic systems

A new approach is given here for proving uniqueness theorems, based on the use of apriori estimates of the analytic continuation, with respect to an independent variable, of solutions of some auxiliary systems (see lecture 1). This approach can also be used for the study of the behaviour of solutions of the Cauchy problem and of the boundary value problem, of the fundamental solutions and of the Green functions as $|x| \to \infty$.

Similar uniqueness theorems for the Cauchy problem for parabolic systems in the sense of I. G. Petrovsky [1] were proved before by different methods. (See for example, [2], [3], [4].) More general parabolic systems were considered in the paper [5]. A short note on the uniqueness theorems and the method discribed here is published in [6], [7], [8], the full proofs are given in [9].

Let ω be a domain in the Euclidean space $R^{n+1}_{x,t} = (x_1,\dots,x_n,t)$ bounded by the planes $t=0$, $t=T$, where $T=$ const. > 0. In the domain ω we consider the system of differential equations of the form

$$(1)\quad \sum_{j=1}^{N} \sum_{|\alpha|+2b\beta \leqq s_\ell+t_j} a_{\ell j}^{\alpha\beta}(x,t)\mathcal{D}_x^\alpha \frac{\partial^\beta u}{\partial t^\beta} = 0,\ \ell=1,\dots,N\ ,$$

where $\mathcal{D}_{x_k} = -i\frac{\partial}{\partial x_k}$, $k=1,\dots,n$, $\mathcal{D}_x^\alpha \equiv \mathcal{D}_{x_1}^{\alpha_1}\cdots\mathcal{D}_{x_n}^{\alpha_n}$, $\alpha=(\alpha_1,\dots,\alpha_n)$, $|\alpha| = \alpha_1+\cdots+\alpha_n$, b is a positive integer, $s_1,\dots,s_N,t_1,\dots,t_N$ are integers, such that $s_j \leqq 0$, $t_j \geqq 0$, $j=1,\dots,N$, $\sum_{j=1}^{N}(s_j+t_j) = 2bm$, m is an integer. Let us set

$$\xi^\alpha \equiv \xi_1^{\alpha_1}\cdots\xi_n^{\alpha_n}\quad ,$$

$$\mathcal{L}_0(x,t,\xi,0) \equiv \Big\| \sum_{|\alpha|+2b\beta=s_\ell+t_j} a_{\ell j}^{\alpha\beta}(x,t)\xi^\alpha\sigma^\beta \Big\|\ ,\ \ell,j=1,\dots,N.$$

We suppose that system (1) is uniformly parabolic in ω. This means that the roots $\sigma_1,\dots,\sigma_m$ of the polynomial in the complex variable σ

$$P(x,t,\xi,\sigma) \equiv \det \mathcal{L}_0(x,t,\xi,\sigma)$$

for any $(x,t)\in\omega$ and any $\xi = (\xi_1,\dots,\xi_n)\in R_\xi^n$ satisfy the inequality

$$\operatorname{Re}\,\sigma_s(x,t,\xi) \leqq -\lambda|\xi|^{2b},\ \lambda=\text{const} > 0,\ s=1,\dots,m\ .$$

We denote by $\hat{\mathcal{L}}_0(x,t,\xi,\sigma)$ the matrix $P\mathcal{L}_0^{-1}$, where $\mathcal{L}_0^{-1}(x,t,\xi,\sigma)$ is the reciprocal matrix for $\mathcal{L}_0(x,t,\xi,\sigma)$. Suppose that for $t=0$ the initial conditions are given in the form

$$(2)\quad \sum_{j=1}^{N} \sum_{|\alpha|+2b\beta\leq P_h+t_j} C_{hj}^{\alpha\beta}(x)\mathcal{D}_x^{\alpha} \frac{\partial^{\beta} u_j}{\partial t^{\beta}} = 0, \; h=1,\dots,m$$

where p_h are negative integers, $h=1,\dots,m$. Let conditions (2) be the uniformly complementing initial condition, (see [5]). The complementing initial condition means that the rows of the matrix

$$K_{(x,\sigma)} = C_0(x,0,\sigma)\hat{\mathcal{L}}_0(x,0,0,\sigma) ,$$

where

$$C_0(x,\xi,\sigma) = \| \sum_{|\alpha|+2b\beta=P_h+t_j} C_{hj}^{\alpha\beta}(x)\xi^{\alpha}\sigma^{\beta}\| , \; h=1,\dots,m; \; j=1,\dots,N,$$

are linearly independent moduls σ^m for every $x\in\bar{\omega}\cap\{t=0\}$. Set

$$K(x,\sigma) \equiv \|\sum_{s=0}^{m-1} d_{hj}^{s}(x)\sigma^{s}\| \quad (\mathrm{mod}\ \sigma^m) .$$

The matrix

$$\| d_{hj}^{s}(x)\|$$

has m rows: $h=1,\dots,m$, and mN colums: $s=0,1,\dots,m-1$; $j=1,\dots,N$. Under the complementing initial condition (2), the rank of the matrix $\|d_{h,j}^{s}(x)\|$ will be m. Hence, if M^r, $(r=1,\dots,k)$, denote all the m-rowed minors of $\|d_{h,j}^{s}(x)\|$, not all of the M^r will be zero and, in particular, $\Delta(x) = \max_{1\leq r\leq k} |M^r(x)|$ will not be zero. Set $\Delta_0 = \inf_{x\in R_x^n} \Delta(x)$.

For the uniformly complementing initial condition on $\overline{\omega} \cap \{t=0\}$ the constant Δ_0 is positive.

For systems which are parabolic in the sense of I.G.Petrovsky, the initial condition (2) lead to the Cauchy data, (see [5]).

We introduce the space $C^{s,b}(A)$ of functions u with the norm

$$\| u \|_A^{s,b} = \sup_A \sum_{|\alpha|+2b\beta \leqq s} \left| \mathcal{D}_x^\alpha \frac{\partial^\beta u}{\partial t^\beta} \right| .$$

In order to study system (1) in ω with the initial conditions (2) for $t=0$ we consider in the domain $\Omega = \omega \times \{ |x_0| < \infty \}$ of the space $R^{n+2}_{x_0,x,t} = (x_0,x_1,\dots,x_n,t)$ an auxiliary system of the form

$$(3) \qquad \sum_{j=1}^{N} \sum_{|\alpha|+2b\beta \leqq s_\ell + t_j} a_{\ell j}^{\alpha\beta}(x,t) \mathcal{D}_x^\alpha \left(\frac{\partial}{\partial t} + \mathcal{D}_{x_0}^{2b} \right)^\beta v_j = 0, \qquad \ell=1,\dots,N,$$

where $\mathcal{D}_{x_0} = -i\frac{\partial}{\partial x_0}$, with the initial conditions for $t=0$ of the form

$$(4) \qquad \sum_{j=1}^{N} \sum_{|\alpha|+2b\beta \leqq P_h + t_j} C_{hj}^{\alpha\beta}(x) \mathcal{D}_x^\alpha \left(\frac{\partial}{\partial t} + \mathcal{D}_{x_0}^{2b} \right)^\beta v_j = 0 , \qquad h=1,\dots,m.$$

It is easy to see that the system (3) is uniformly parabolic in Ω by virtue of the uniform parabolicity of the system (1) in ω. In addition, the initial conditions (4) is the uniformly complementing initial conditions on $\overline{\Omega} \cap \{t=0\}$ for the system (3).

The further investigation of the problem (1)-(2) is based on the theorem concering the analyticity, with respect to the independent variable x_0, of solutions of the problem (3)-(4).

The proof of the analyticity, with respect to x_0, of solutions $v=(v_1,\dots,v_N)$ of the problem (3)-(4) uses the apriori estimate, proved in the paper [5] and the Morrey-Nirenberg method.

Let us set

$$\Omega_R^\tau = \{x_0, x, t;\ |x_0| < R;\ |x| < R,\ 0 < t < \tau\},$$

$$\omega_R^\tau = \{x, t;\ |x| < R,\ 0 < t < \tau\}$$

$$Q_\delta(A) = \{x_0, y_0, x, t;\ (x_0, x, t) \in A,\ |y_0| < \delta\}.$$

Theorem 1. Suppose that $v = (v_1,\dots,v_N)$ is a solution of the system (3) in Ω_{R+1}^τ with the initial conditions (4) for $t=0$. Suppose that

$$(5) \qquad \| a_{\ell j}^{\alpha\beta} \|_{\Omega}^{-s_\ell+1,b} + \| c_{hj}^{\alpha'\beta'} \|_{t=0}^{-P_h+1,b} \leq M$$

for $\ell, j = 1,\dots,N$, $h=1,\dots,m$, $|\alpha|+2b\beta \leq s_\ell + t_j$, $|\alpha'|+2b\beta' \leq P_h + t_j$, $M = \text{const}$. Then any solution $v = (v_1,\dots,v_N)$ can be extended into the domain $Q_\delta(\Omega_R^\tau)$ as an analytical vector-function of the complex variable $x_0 + iy_0$ and the following estimate holds:

$$(6) \qquad \sum_{j=1}^{N} \sup_{Q_\delta(\Omega_R^\tau)} \sum_{|\alpha|+2b\beta \leq t_j - 1} \left| \mathcal{D}_x^\alpha \frac{\partial^\beta v_j}{\partial t^\beta} \right| \leq C_1 \sum_{j=1}^{N} \sup_{\Omega_{R+1}^\tau} |v_j|,$$

where δ, C_1 are positive constants dependent only on n, N, m, b, λ, M, Δ_0, (they do not depend on τ and R).

Now we prove the uniqueness theorem for the initial value problem (1), (2) in the class of growing functions using the analiticity theorem 1.

Theorem 2. Suppose that the coefficients of the problem (1)-(2) satisfy the condition (5) and $u = (u_1,\dots,u_N)$ is a solution of the problem (1), (2) in ω and $u_j \in C^{t_j+1,b}(\tilde{\omega})$ for any bounded subdomain $\tilde{\omega}$ of the domain ω, $j=1,\dots,N$. If

$$(7) \qquad \| u_j \|^{0,b}_{\omega^\tau_R} \leqq \exp\{\delta_1 |R|^q\} ,$$

where $q = \frac{2b}{2b-1}$, $\delta_1 = \text{const}$, $j=1,\dots,N$, then $u \equiv 0$ in ω for $0 \leqq t \leqq T_0 \leqq \tau$, $T_0 = \text{const}$.

Proof. Consider in Ω the auxiliary system (3) with the initial conditions (4). It is easy to see that if u is a solution of the problem (1), (2), then $v(x_0,x,t) = u(x,t)\times \exp\{i\mu x_0-\mu^{2b}t\}$, for any $\mu \in R^1$, is a solution of the problem (3), (4). For this solution the estimate (6) is valid. Therefore, for $R=1,2,\dots$ we have

$$(8) \qquad \sum_{j=1}^{N} \| v_j \|^{0,b}_{Q_\delta(\Omega^\tau_R)} \leqq C_1 \sum_{k=1}^{N} \| v_k \|^{0,b}_{\Omega^\tau_{R+1}} .$$

From (8) it follows that

$$(9) \qquad \sum_{j=0}^{N} \| e^{-\mu^{2b}t} u_j \|^{0,b}_{\omega^\tau_R} \leqq C_1 e^{-\delta\mu} \sum_{k=1}^{N} \| e^{-\mu^{2b}t} u_k \|^{0,b}_{\omega^\tau_{R+1}} .$$

Using successively the estimate (9) for $R = R_0$, $R_0+1, \dots, R_0+s$, we obtain

$$\sum_{j=0}^{N} \| e^{-\mu^{2b}t} u_j \|^{0,b}_{\omega^\tau_{R_0}} \leq \exp\{s(\ln C_1 - \delta\mu)\} \sum_{k=1}^{N} \| e^{-\mu^{2b}t} u_k \|^{0,b}_{\omega^\tau_{R_0+s}} ,$$

and also

$$(10) \quad \sum_{j=0}^{N} \| u_j \|^{0,b}_{\omega^\tau_{R_0}} \leq \exp s(\ln C_1 - \delta\mu) + \tau\mu^{2b}\} \sum_{k=1}^{N} \| u_k \|^{0,b}_{\omega^\tau_{R_0+s}}$$

Using the condition (7), we deduce from (10) that

$$(11) \quad \sum_{j=0}^{N} \| u_j \|^{0,b}_{\omega^\tau_{R_0}} \leq C_2 \exp\{s(\ln C_1 - \delta\mu) + \tau\mu^{2b} + \delta_1 (R_0+s)^q\},$$

$$C_2 = \text{const.}$$

Then we set $\mu = s^{\frac{1}{2b-1}}$. From (11) it follows that

$$(12) \quad \sum_{j=0}^{N} \| u_j \|^{0,b}_{\omega^\tau_{R_0}} \leq C_2 \exp\{s \ln C_1 - \delta s^{\frac{2b}{2b-1}} + \tau s^{\frac{2b}{2b-1}} + \delta_1 (R_0+s)^{\frac{2b}{2b-1}} \}.$$

If $\tau+\delta_1 < \delta$, then the right hand side of the inequality (12) tends to zero as $s \to \infty$. Therefore, in this case $\|u_j\|^{0,b}_{\omega^\tau_{R_0}} \equiv 0$. It means that $u_j \equiv 0$ in ω , since R_0 is an arbitrary number. If the inequality $\tau+\delta_1 < \delta$ is not valid, then we change the variables for system (3) with initial conditions (4)

$$x_0 = \rho x_0' , \quad x = \rho x' , \quad t = \rho^{2b} t', \quad v_j = \rho^{t_j} v_j' ,$$

where $0 < \rho = \text{const.} \leq 1$. It can be proved that in the new variables we have the same δ in the inequality (6), but δ_1 can be arbitrarily small, if ρ is the inequality (6), but δ_1 can be arbitrarily small, if ρ is sufficiently small. It proves the theorem 2 for $T_0 < \delta$.

In the same way the uniqueness of the solution of the boundary value problem in an unbounded domain can be proved in a class of growing functions and also the behaviour of solutions of parabolic systems as $|x| \to \infty$ can be studied (see [6], [7], [8], [9]).

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