Parametrices for Degenerate Operators

of Grushin's Type

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Introduction.   Grushin studied the hypoellipticity of an

degenerate operator A of the form given in §1.   In his paper [2]

he used operator valued pseudo-differential operators essentially.

In this note we construct a left parametrix Q for the operator

A as a pseudo-differential operator by symbol calculus instead of

his method.   For the construction of Q, we use the fundamental

solution of parabolic equation studied in [4] and [5].   Also we

discuss estimates for A.

We give main theorems and several examples in §1 .   In § 2

and § 3 the method of construction of Q will be shown.   We devote

§ 4 for prooves of Theorem B and Theorem C .

§ 1.   Main theorems and examples.

In this note we treat operators defined in R^3 for simplicity.
Consider an operator

(1) $\quad A = A(x_1, y, D_{x_1}, D_{x_2}, D_y) = \sum\limits_{\substack{|\alpha| \le m \\ (\sigma, \gamma) \ge (\tau, \alpha) - m}} a_{\alpha, \gamma} x_1^{\gamma_1} y^{\gamma_3} D_{x_1}^{\alpha_1} D_{x_2}^{\alpha_2} D_y^{\alpha_3}$

where $a_{\alpha, \gamma}$ are constants, $D_j = \partial/\partial(i x_j)$ $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\tau = (\tau_1, \tau_2, 1)$,

$\tau_j \ge 1$ $(j=1,2)$, $\gamma = (\gamma_1, 0, \gamma_3)$, $\sigma = (\sigma_1, 0, 1)$, $\min(\tau_1, \tau_2) > \sigma_1 > 0$

and $(\tau, \alpha) = \sum\limits_{j=1}^{3} \tau_j \alpha_j$

Let A_0 be the principal part of A in a sense, that is,

(2) $\quad A = A(x_1, y, D_{x_1}, D_{x_2}, D_{x_3}) = \sum\limits_{\substack{|\alpha| \le m \\ (\sigma, \gamma) = (\tau, \alpha) - m}} a_{\alpha, \gamma} x_1^{\gamma_1} y^{\gamma_3} D_{x_1}^{\alpha_1} D_{x_2}^{\alpha_2} D_y^{\alpha_3}$

We denote by $A(x_1, y, \xi, \eta)$ the differential polynomial correspond

to A_0.

Condition 1. $A_0(x_1, y, D_{x_1}, D_{x_2}, D_y)$ is elliptic for $|x_1| + |y| \ne 0$

i.e.

$$A_0(x_1, y, \xi, \eta) \ne 0$$

for $|x_1| + |y| \ne 0$, $\xi \in R^2$, $\eta \in R^1$, and $|\xi| + |\eta| \ne 0$.

Condition 2. For all $x_1 \in R^1$ and for all nonzero vector $\xi \in R^2$

the equation $A_0(x_1, y, \xi, D_y) v(y) = 0$ has no nonzero solution in $\mathcal{S}(R^1)$.

We get main theorems as follows.

Theorem A. Under condition 1 and condition 2, there exist a neighbourhood Ω of $x_1=y=0$ and a left parametrix Q for A in a class of pseudo-differential operators;

$QA= I+W$ in Ω, where W is a smoothing operator.

By the construction of Q we get estimates for A and A_0.

$$(3)_s \qquad \|u\|_{m+s} \leqq c_K(\|A_0 u\|_s + \|u\|_s) \qquad \forall u \in c_0^\infty(K),$$

$$(4)_s \qquad \|u\|_{m+s)} \leqq c(\|Au\|_{(s)} + \|u\|_{(s)}) \qquad \forall u \in c_0^\infty(\Omega),$$

where $\|\cdot\|_m$ is the usual Sobolev norm, K is any compact set of such that $K \cap \{0,x_2,0\} = \phi$, $\|u\|_{(m)} = \|\Lambda_\tau^m(D)u\|_0$ and Λ_τ is a pseudo-differential operator with symbol $\sum_{j=1}^{2}|\xi_j|^{1/\tau_j} + |\eta| + 1$.

Moreover we get the following statement.

If $Au \in H^s_{loc}(\Omega)$ and $u \in \mathcal{D}'(\Omega)$, then $u \in H^{s+m/\tau}_{loc}(\Omega)$

where $\tau = \max(\tau_1, \tau_2)$.

Remark. If $\tilde{\tau}=1$, then condition 1 means that A is elliptic in case $|x_1| + |y|$ is sufficient small.

We may assume $\tilde{\tau} > 1$ by the above remark.

Theorem B. If $(4)_o$ hold, then we obtain the following estimate for all $x_1 \in R^1$ and nonzero $\xi \in R^2$.

$$\| v \| \leqq c \| A_o(x_1,y,\xi,D_y)v \| \qquad \forall v \in \mathcal{S}(R_y^1)$$

where $\| \cdot \|$ is the norm in $L_2(R_y^1)$.

Owing to the above theorem we get

Theorem C. Assume that A satisfies $(3)_o$ and $(4)_o$. Then, A holds condition 1 and condition 2 .

Now let us give some examples

Example 1. $A=A_o=(-\Delta_y)^\ell + y^{2k}(-\Delta_{x_2})^\ell$ in R^2, where ℓ and k are positive integers. We can take $\tau=(1+k/\ell,1)$, $\sigma=(0,1)$ and m=2 . In this case condition 1 and condition 2 hold.

Example 2. (Grushin [2]). $A=A_o=\dfrac{\partial^2}{\partial y^2} + (y^2+x_1^2)(\dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2})$

$+i\lambda\dfrac{\partial}{\partial x_2}$. In this case we can choose $\tau=(2,2,1)$, $\sigma=(1,0,1)$ and m=2. Condition 1 holds. Condition 2 holds if $|\lambda|<1$ or $Im\lambda\neq0$.

Example 3. $A=A_o=(\dfrac{\partial}{\partial y}-iay^k\dfrac{\partial}{\partial x_2})(\dfrac{\partial}{\partial y}-iby^k\dfrac{\partial}{\partial x_2}) + icy^{k-1}\dfrac{\partial}{\partial x_2}$

in R^2, where k is a odd integer, a,b and c are real constants such that ab<0. In this case $\tau=(k+1,1)$, $\sigma=(0,1)$ and m=2. Condition 1

always holds. Condition 2 holds if and only if

$$\frac{c}{a-b} \not\equiv 0,1 \pmod{k+1}.$$

This result is due to Gilioli and Treves [1].

§ 2. Notations and theorems for pseudo-differential operators

For a pair of real vectors $\rho = (\rho_1, \rho_2, \cdots, \rho_n)$ and $\delta = (\delta_1, \delta_2, \cdots, \delta_n)$ we say $\rho > \delta$ if $\rho_j > \delta_j$ for all j. In this section we fix ρ and δ such that $\rho > \delta \geq 0$.

Definition (cf. [5]). We say that a C^∞-function $\lambda(x,\xi)$ define in $R_x^n \times R_\xi^n$ is a basic weight function when $\lambda(x,\xi)$ satisfies the conditions below:

(5) $\qquad \left| \lambda^{(\alpha)}_{(\beta)}(x,\xi) \right| < A_{\alpha,\beta} \lambda(x,\xi)^{1-(\rho,\alpha)+(\delta,\beta)}$

(6) $\qquad 1 \leq \lambda(x+y,\xi) \leq A \langle y \rangle^{\tau_0} \lambda(x,\xi) \qquad (\tau_0 \geq 0)$,

where $\lambda^{(\alpha)}_{(\beta)}(x,\xi) = (\partial/\partial \xi_1)^{\alpha_1} \cdots (\partial/\xi_n)^{\alpha_n} (-i\partial/\partial x_1)^{\beta_1} \cdots (-i\partial/\partial x_n)^{\beta_n}$

$\lambda(x,\xi)$ and $\langle y \rangle = (|y|^2 + 1)^{1/2}$.

We denote by $S^m_{\lambda,\rho,\delta} (-\infty < m < \infty)$ the set of all C^∞-functions $p(x,\xi)$ defined in $R_x^n \times R_\xi^n$ which satisfies for any α and β

$$\left| p^{(\alpha)}_{(\beta)}(x,\xi) \right| \leq C_{\alpha,\beta} \lambda(x,\xi)^{m-(\rho,\alpha)+(\delta,\beta)}$$

for some constants $C_{\alpha,\beta}$. For a symbol $p(x,\xi) \in S^m_{\lambda,\rho,\delta}$ we define

a pseudo-differential operator by

$$Pu(x) = p(x, D_x)u(x) = \int e^{ix \cdot \xi} p(x, \xi) \hat{u}(\xi) đ\xi ,$$

where $đ\xi = (2\pi)^{-n} d\xi$ and $\hat{u}(\xi)$ denote the Fourier transform of $u(x)$

in $\mathcal{S}$ defined by

$$\hat{u}(\xi) = \int e^{-ix \cdot \xi} u(x) dx.$$

We call a pseudo-differential operator a smoothing operator when

its symbol belongs to $S^m_{\langle \xi \rangle, \rho, \delta}$ for any m.

Now consider a pseudo-differential operator of parabolic type.

$$L = \frac{\partial}{\partial t} + p(x, D_x)$$

We call an operator $E(t)$ a fundamental solution for L when $E(t)$

satisfies

$$\begin{cases} L \, E(t) = 0 & \text{in} \quad 0 < t < \infty , \\ E(0) = I . \end{cases}$$

By [5], we get the next theorem

Theorem 1. Assume that $p(x, \xi) \in S^{\ell}_{\lambda,}$, satisfies

$$Re \, p(x, \xi) + c \geqq c_o \lambda(x, \xi)^{\ell}$$

for positive constants c and c_o. Then, there exists a fundamental

solution $E(t)$ which belongs to $S^o_{\lambda, \rho, \delta}$ with parameter t and whose

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symbol $e(t;x,\xi)$ has the following expansion

$$e(t;x,\xi) = \sum_{j=0}^{N-1} e_j(t;x,\xi) + r_N(t;x,\xi)$$

where $e_j(t;x,\xi) \in S_{\lambda,\rho,\delta}^{-\omega j}$, $r_N(t;x,\xi) \in S_{\lambda,\rho,\delta}^{\ell-\omega N}$, $\omega=\min(\rho_j-\delta_j)$ and

N is any number such that $\omega N \geq \ell$. Moreover we get $e_0(t;x,\xi)$

$= \exp\left\{-tp(x,\xi)\right\}$ and

$$e_{j(\beta)}^{(\alpha)}(t;x,\xi) = a_{j,\alpha,\beta}(t;x,\xi)e_0(t;x,\xi) \qquad (j\geq 1) ,$$

where $a_{j,\alpha,\beta}(t;x,\xi)$ satisfy

$$\left| a_{j,\alpha,\beta}(t;x,\xi) \right| \leq c_{j,\alpha,\beta} \lambda(x,\xi)^{-\omega j-(\rho,\alpha)+(\delta,\beta)}$$

$$\times \sum_{k=2}^{|\alpha|+|\beta|+2j} \left\{ t \operatorname{Re}p(x,\xi) \right\}^k$$

About the behavior of $e(t;x,\xi)$ for large t, we get

Theorem 2 (Tsutsumi [5]). Let $p(x,\xi)$ satisfy the assumption

of theorem 1 and

$$\operatorname{Re}(p(x,D_x)u, u) \geq c_1 \|u\|^2 \qquad u \in \mathcal{S}(R^n)$$

with a positive constant c_1. Moreover let $\rho>0$ and $\lambda(x,\xi)$ satisfy

$$\lambda(x,\xi) \geq a(|\xi|+|x|+1)^a$$

for some positive constant a . Then the symbol $e(t;x,\xi)$ constructed

in theorem 1 holds the following estimate for any integers j and k

and positive constant ε

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$$\lambda(x,\xi)^j \left| \frac{\partial^k}{\partial t^k} e^{(\alpha)}_{(\beta)} (t+\varepsilon;x,\xi) \right| \leqq C(j,k,\alpha,\beta,\varepsilon) \exp(-c_2 t)$$

$$(t \geqq 0)$$

for any α and β ,

where c_2 is any constant less than c_1 and $C(j,k,\alpha,\beta,\varepsilon)$ is

independent of t .

Now assume that the basic weight function $\lambda(x,\xi)$ is

independent of x and $b^{-1}|\xi|^{k_1} \leqq \lambda(\xi) \leqq b |\xi|^{k_2}$ $(k_1,k_2 > 0)$.

We denote the Sobolev norm $\| \cdot \|_{m,\lambda}$ by

$$\|u\|_{m,\lambda} = \left\| \lambda(D_x)^m u \right\|$$

where $\| \cdot \|$ is the norm in $L^2(R^n)$.

. Let Ω be a open set in R^n and let $p(x,\xi)$ hold the estimate

below

(7) $$\left| p^{(\alpha)}_{(\beta)}(x,\xi) \right| \leqq c_{K,\alpha,\beta} \; \lambda(\xi)^{M-(\rho,\alpha)+(\delta,\beta)} \qquad \forall x \in K$$

for any compact set K in Ω .

Theorem 3 (Tsutsumi [6]). Let $P = p(x,D_x)$ satisfy (7) and

$$\|u\|^2_{m,\lambda} \leqq c_K (\|Pu\| + \|u\|_{m',\lambda}) \qquad u \in C^\infty_0(K).$$

where $M \geqq m \geqq 0$, $m > m'$ and K is any compact set in Ω . Then, for

every K and every integer N one can find C so that when $x \in K$ and $\xi \in R^n$

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$$\lambda(\xi)^{2m}\|\psi\|^2 \leqq c \int \left| \sum_{|\alpha+\beta|<N} p_{(\beta)}^{(\alpha)}(x,\xi) \lambda(\xi)^{(0,\alpha-\beta)} y^\beta D_y^\alpha \psi/\alpha! \, \beta! \right|^2 dy$$

$$+ \lambda(\xi)^{2M-2\varepsilon_0 N+d_1} \left(\sum_{|\alpha+\beta|\leqq N+d_2} \int |y^\beta D_y^\alpha \psi|^2 dy \right) \quad, \quad \psi \in C_0^\infty(R^n)$$

Here $\Theta=(\Theta_1,\Theta_2,\cdots,\Theta_n)$ is chosen such that $\varepsilon_0 = \min(\beta_j-\Theta_j, \Theta_j-\delta_j, \Theta_j)>0$

and both d_1 and d_2 are constants independant of N .

§ 3. Outline of construction of a left parametrix Q.

We introduce some notations.

$$h=h(x_1,y)=(|x|^{1/\tau_1}+ |y|)$$

$$\mathcal{Y}=\mathcal{Y}(x_1,y,\xi,\eta)=\sum_{j=1}^2 h(x_1,y)^{\tau_j-1} |\xi_j| + |\eta|$$

$$\mu=\mu(\xi)=\sum_{j=1}^2 |\xi_j|^{1/\tau_j} + 1$$

$$\chi=\chi(\xi,\eta)=\sum_{j=1}^2 |\xi_j| + |\eta| + 1$$

We denote by $S^m(\mathcal{Y},\mu,\chi)$ $(-\infty<m<\infty)$ the set of all C^∞-functions

$p(x_1,y,\xi,\eta)$ defined in $R^1_{x_1}\times R^1_y\times R^2_\xi\times R^1_\eta$ which satisfy

(8) $$\left| p_{(\beta)}^{(\alpha)}(x_1,y,\xi,\eta)\right| \leqq c_{\alpha,\beta} (\mathcal{Y}+\mu)^{m-|\beta|} \mu^{|\beta|+\sigma_1|\beta_1|} \chi^{-|\alpha_1|-|\alpha_2|}$$

for any $\alpha=(\alpha_1,\alpha_2,\alpha_3)$, $\beta=(\beta_1,0,\beta_3)$. We define a psuedo-differntial

operator $Pu(x_1,x_2,y)$ with a symbol $\sigma(P)=p(x_1,y,\xi,\eta)\in S^m(\mathcal{Y},\mu,\chi)$ by

$$Pu(x_1,x_2,y)=\int e^{i(x_1\xi_1+x_2\xi_2+y\eta)}p(x_1,y,\xi,\eta)\hat{u}(\xi,\eta)d\xi d\eta$$

for $u\in \mathcal{S}(R^3)$.

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We get the following lemma for operators of this class.

Lemma 1. (Tsutsumi [6]) (i)If P_j belongs to $S^{m_j}(\gamma,\mu,\chi)$ $(j=1,2)$ then $P_1 P_2$ belongs to $S^{m_1 + m_2}(\gamma,\mu',\chi)$ and

$$\sigma(P_1 P_2)(x_1,y,\xi,\eta) - \sigma(p_1(x_1,y,\xi,D_y)p_2(x_1,y,\xi,D_y))$$

belongs to $S^{-\delta_0}(\gamma,\mu,\chi)$, where $\delta_0 = \min(\tau_1,\tau_2)-\sigma_1$.

(ii)If P belongs to $S^0(\gamma,\mu,\chi)$, then P is a bounded operator in $L^2(R^3)$

(iii)$P \in S^m(\gamma,\mu,\chi)$ has pseudo-local propety, i.e. For any $\varphi,\psi \in C^\infty(R^3)$

 P is a smoothing operator when $\mathrm{supp}\varphi \cap \mathrm{supp}\psi = \phi$.

The main result of this section is the proposition below.

Proposition 1. Under condition 1 and condition 2 there exist a neighbourhood $\Omega' \subset R^2$ of $x_1=y=0$ and $\tilde{A}(x_1,y,\xi,\eta) \in S^m(\gamma,\mu,\chi)$ which satisfy the following properties.

(i)We can construct a left parametrix $Q \in S^{-m}(\gamma,\mu,\chi)$ for $\tilde{A}$, i.e.

 $Q\tilde{A} = I + W$, where W is a smoothing operator .

(ii)$(A-\tilde{A})(x_1,y,\xi,\eta) = 0$ if $(x_1,y) \in \Omega'$.

By this proposition and lemma 1 one can prove theorem A, noting that Λ_τ belongs to $S^1(\gamma,\mu,\chi)$ and that Q is elliptic for $|x_1| + |y| \neq 0$, i.e. The symbol $q(x_1,y,\xi,\eta)$ of Q satisfy $|q| \geq c(|\xi| + |\eta| + 1)^{-m}$ for $|x_1| + |y| \neq 0$.

We need several steps to show proposition 1.

Lemma 2. For all α and β we have

$$\left| \partial_\eta^\alpha \partial_y^\beta A_0(x_1,y,\xi,\eta) \right| \leq C_{\alpha,\beta} (\nu+\mu)^{m-|\alpha|} \mu^{|\beta|} \qquad \text{for } |\xi| \geq c > 0$$

Moreover if we fix a $C_0^\infty(R^1)$-function φ, then $\varphi(|x_1|^2+|y|^2)A_0(x_1,y,\xi,\eta)$ belongs to $S^m(\nu,\mu,\chi)$ and $\varphi(|x_1|^2+|y|^2)(A-A_0)(x_1,y,\xi,\eta)$ satisfies (3) for $C_{\alpha,\beta} h(x_1,y)^{\varepsilon_1}$ ($\varepsilon_1 > 0$) instead of the constants $C_{\alpha,\beta}$ in (8).

Lemma 3. If condition 1 hold , then we get for some constant $c > 0$

$$c^{-1} \left| A_0(x_1,y,\xi,\eta) \right| \leq \nu(x_1,y,\xi,\eta) \leq c \left| A_0(x_1,y,\xi,\eta) \right| .$$

Lemma 4.(Grushin [2]). We find a constant $c_1 > 0$ such that

$$\| u \| \leq c_1 \| A_0(x_1,y,\xi,D_y)u \| \qquad \forall u \in \mathcal{S}(R^1)$$

for any ξ when $|\xi|=1$.

Set $\lambda_z(y,\eta)$ with parameters $z=(x_1,\xi)$ by

$$\lambda_z(y,\eta) = \left(|A_0(x_1,y,\xi,\eta)|^2 + 1 \right)^{1/2m}$$

then by lemma 2 and lemma 3, we get

Lemma 5. If $|\xi|=1$, then we get

(i) $\lambda_z(y,\eta)$ is a basic weight function which holds (5) and (6) for $A_{\alpha,\beta}$ and A independant of parameters z.

(ii) $|A_0(x_1,y,\xi,\eta)|^2 + 1 \geq \lambda_z(y,\eta)^{2m}$

(iii) $\left| \partial_\eta^\alpha \partial_y^\beta A_0(x_1,y,\xi,\eta) \right| \leq C_{\alpha,\beta} \lambda_z(y,\xi)^{m-|\alpha|}$

where $C_{\alpha,\beta}$ are independant of z .

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Now, by lemma 5 we can apply theorem 1 for $L_z = L_{x_1, \xi}$ ($|\xi| = 1$)

defined by

$$L_z = L_{x_1, \xi} = \frac{\partial}{\partial t} + p(x_1, y, \xi, D_y) ,$$

where $p(x_1, y, \xi, D_y) = A_0(x_1, y, \xi, D_y) A_0^*(x_1, y, \xi, D_y)$ and $A_0^*(x_1, y, \xi, D_y)$

is the adjoint operator of $A(x_1, y, \xi, D_y)$.

Let $e(t; y, D_y; z) \in S^0_{\lambda_z, 1, 0}$ be the fundamental solution for L_z with

parameters z. ($|\xi| = 1$).

For any ξ ($|\xi| \geq 1$) we get

Lemma 6. If condition 1 and condition 2 hold, then there exists

the fundamental solution $u(t; y, D_y; z)$ for L_z which admits the expansion

below for $N \geq 2m$,

$$u(t; y, \eta; z) = \sum_{j=0}^{N-1} u_j(t; y, \eta; z) + v_N(t; y, \eta; z)$$

$$\left| \partial_\eta^\alpha \partial_y^\beta u_j(t; y, \eta; z) \right| \leq C_{j, \alpha, \beta} (\nu + \mu)^{-j - |\alpha|} \mu^{j + |\beta|} \exp \left[-c_0 (\nu + \mu)^{2m} t \right]$$

$$\left| \partial_\eta^\alpha \partial_y^\beta v_N(t; y, \eta; z) \right| \leq C_{N, \alpha, \beta} (\nu + \mu)^{-N - |\alpha|} \mu^{N + |\beta|} ,$$

(9) $\quad \left| \partial_\eta^\alpha \partial_y^\beta u(t; y, \eta; z) \right| \leq C'_{\alpha, \beta} (\nu + \mu)^{-k - |\alpha|} \mu^{k + |\beta|} \exp \left[-c_0 \mu^{2m} t \right]$

for $\mu^m t \geq \ell > 0$ and all k.

Proof. For $|\xi| = 1$ by lemma 4 and theorem 2 , one can find a

constant C independent of z such that

$$\max_{|\alpha| + |\alpha'| + |\beta| + |\beta'| < \ell, y \in R^l, \eta \in R^l} \sup \left| y^\alpha \eta^{\alpha'} D_y^\beta D_\eta^{\beta'} e(t + \varepsilon; y, \eta; z) \right| \leq C \exp \left[-c_2 t \right] \quad (t \geq 0)$$

for any $\varepsilon > 0$, ℓ and $c_2 < c_1$.

$A_0(x_1, y, \xi, \eta)$ is quasihomogeneous in the sense

$$A_0(\lambda^{-\sigma_1}x_1, \lambda^{-1}y, \lambda^{\tau_1}\xi_1, \lambda^{\tau_2}\xi_2, \lambda\eta) = \lambda^m A_0(x_1, y, \xi_1\xi_2\eta)$$

then $u(t;y,\eta;z)$ is given by

(10) $\qquad u(t;y,\eta;x_1,\xi_1,\xi_2) = e(\tilde{\mu}^{2m}t;\tilde{\mu}y,\tilde{\mu}^{-1}\eta;\tilde{\mu}^{\sigma_1}x_1, \tilde{\mu}^{-\tau_1}\xi_1,\tilde{\mu}^{-\tau_2}\xi_2)$

where $\tilde{\mu}=\tilde{\mu}(\xi)$ is the positive root of the equation $\sum_{j=1}^{2}\tilde{\mu}^{-2\tau_j}\xi_j^2 = 1$.

By (10) and theorem 1 we get the assertion.

Put $r(x_1,y,\xi,\eta) = r(y,\eta;x_1,\xi) = \int_0^\infty u(t;y,\eta;x_1,\xi)dt$ for $|\xi|\geq 1$.

Then $R(x_1,\xi) = r(y,D_y;x_1,\xi)$ is a left and right inverse of

$p(x_1,y,\xi,D_y)$. Let us define a pseudo-differential operator $K(x_1,\xi)$

with parameters by

$$K(x_1,\xi) = R(x_1,\xi)A_0(x_1,y,\xi,D_y) \qquad .$$

Then for $|\xi|\geq 1$,

Lemma 7. $K(x_1,\xi)$ is a left inverse of $A_0(x_1,y,\xi,D_y)$ with a

symbol $k(y,\eta;x_1\xi) = k(x_1,y,\xi,\eta)$ which admits for any α and β

$$\left|\partial_\eta^\alpha \partial_y^\beta k(y,\eta;x_1,\xi)\right| \leq c_{\alpha,\beta}(\nu+\mu)^{-m-|\alpha|}\mu^{|\beta|} \quad .$$

Moreover $\varphi(|x_1|^2+|y|^2)k(x_1,y,\xi,\eta)$ belongs to $S^{-m}(\nu),\mu,\chi)$ for any

$\varphi \in C_0^\infty(R^1)$.

Proof. It is sufficient to show that $r(y,\eta;x_1,\xi)$ satisfies

$$\left|\partial_\eta^\alpha \partial_y^\beta r(y,\eta;x_1,\xi)\right| \leq c'_{\alpha,\beta}(\nu+\mu)^{-2m-|\alpha|}\mu^{|\beta|}$$

and that $\varphi(|x_1|^2+|y|^2)r(x_1,y,\xi,\eta)$ belongs to $S^{-2m}(\nu,\mu,\chi)$, by lemma 2.

We write

$$r(y,\eta;x_1,\xi)=\int_0^{\mu^{-m}(\xi)}u(t;y,\eta;x_1,\xi)dt + \int_{-\mu^{-m}(\xi)}^{\infty}u(t;y,\eta;x_1,\xi)dt$$

$$= I_1 + I_2 .$$

For I_1 we fix N such that $N \geqq 2m$ and use the expansion in lemma 6.

We apply (9) taking $k=2m$ for I_2 . For derivatives with respect

to parameters x_1 and ξ, we can write for example

$$r_{\xi_j}(y,D_y;x_1,\xi)=-r(y,D_y;x_1,\xi)p_{\xi_j}(x_1,y,\xi,D_y)r(y,D_y;x_1,\xi)$$

Then, by lemma 2 and symbol calculus of pseudo-differential

operator with parameters $z=(x_1,\xi)$ and also noting that $R(x_1,\xi)$

has pseudo-local property with respect to y, one get the assertion.

By lemma 2 we can construct $\widehat{A}_0(x_1,y,\xi,\eta)\in S^m(\mathcal{Y},\mu,\chi)$ such that

$|\widehat{A}_0(x_1,y,\xi,\eta)|\geqq c\chi^m$ for $|x_1|+|y|\geqq 1$ and $\widehat{A}_0(x_1,y,\xi,\eta)=A_0(x_1,y,\xi,\eta)$

if $|x_1|+|y|\leqq 1$. Now let us construct a pseudo-differential

operator $Q_0=q_0(x_1,y,D_{x_1},D_{x_2},D_y)$ which is a left parametrix for

$\widetilde{A}_0(x_1,y,D_{x_1},D_{x_2},D_y)$ by using $k(x_1,y,\xi,\eta)$.

Put $q_0'(x_1,y,\xi,\eta)=\mathscr{P}_1(|x_1|^2+|y|^2)\mathscr{P}_2(|\xi|^2)k(x_1,y,\xi,\eta) +\mathscr{P}_1(|x_1|^2+|y|^2)\mathscr{P}_1(|\xi|^2)$

$\mathscr{P}_2(|\xi|^2+|\eta|^2)k_1(x_1,y,\xi,\eta) +\mathscr{P}_2(|x_1|^2+|y|^2)k_1(x_1,y,\xi,\eta)$.

Here $k_1(x_1,y,\xi,\eta)=A_0(x_1,y,\xi,\eta)^{-1}$, $\mathscr{P}_1\in C_0^{\infty}(\mathbb{R}^1)$ such that $\mathscr{P}_1(t)=1$ $|t|\leqq 1$

$\varphi_1(t)=0$ $|t|\geq 2$ and $\varphi_2=1-\varphi_1$. It is clear that $q_0'(x_1,y,\xi,\eta)$ belongs $s^{-m}(\nu,\mu,\chi)$.

(11) $\qquad q_0'(x_1,y,\xi,D_y)\widetilde{A}_0(x_1,y,\xi,D_y)-I \in s^{-\delta_0'}(\nu,\mu,\chi)$

where $\delta_0'=\min(\delta_0,1)>0$ using (i) of lemma 1 .

From (11) one can get a left parametrix $Q_0\in s^{-m}(\nu,\mu,\chi)$.

For any $\varepsilon>0$ there exist Ω' and $\widetilde{A}(x_1,y,\xi,\eta)\in s^m(\nu,\mu,\chi)$ such that

$$\widetilde{A}(x_1,y,\xi,\eta) = A(x_1,y,\xi,\eta) \qquad \text{if } (x_1,y)\in\Omega'$$

(12) $\qquad \left|(\widetilde{A}-\widetilde{A}_0)\binom{\alpha}{\beta}(x_1,y,\xi,\eta)\right| \leq \varepsilon \ (\nu+\mu)^{m-|\alpha|} \mu^{\sigma_1|\beta_1|+|\beta_3|} \chi^{-|\alpha|-|\alpha_2|}$

$$\text{for } \alpha=(\alpha_1,\alpha_2,\alpha_3) \text{ and } \beta=(\beta_1,0,\beta_3) .$$

If ε is sufficient small, then by (12) there exists $R_0\in s^0(\nu,\mu,\chi)$ the inverse of $I+Q_0(\widetilde{A}-\widetilde{A}_0)$. Set $Q=R_0Q_0$, then Q is a required left parametrix .

§ 4. Prooves of theorem B and theorem C.

At first we show theorem B.

Define a weight function $\lambda(\xi,\eta)$ by

$$\lambda(\xi,\eta) = \varphi(\xi,\eta)\left(\sum_{j=1}^{2}|\xi_j|^{1/\tau_j}+ |\eta|^{1/\bar{\tau}}\right) + 1$$

where $\varphi\in C^\infty(R^3)$ such that $\varphi(\zeta)=1$ $(|\zeta|\geq 1)$, $\varphi(\zeta)=0$ $(|\zeta|\leq 1/2)$ and $\bar{\tau}=\max(\tau_1,\tau_2)$. Then by $(4)_0$ it is clear that

(13) $\qquad \|u\|_{m,\lambda}\leq C(\|Au\|+\|u\|) \qquad \forall u\in C_0^\infty(\Omega) .$

It is easy to see that $\widetilde{A}(x_1,y,\xi,\eta)$ belongs to $S^m_{\lambda,\rho,\delta}$ if $\rho=(\tau_1,\tau_2,\tau)$

and $\delta=(0,0,0)$. Now from (13) we can apply theorem 3 in §1 for

$\rho=0$,choosing θ_j such that $\theta_j=(\tau_j+\sigma_j)/2$ (j=1,2) and $\theta_3=1$.

Then for any N

$$\lambda(\xi,0)^{2m}\|\psi\|^2 \leq c \left\{ \int \left| \sum_{|\alpha+\beta|<N} A^{(\alpha)}_{(\beta)}(x_1,y,\xi,0)\, \lambda(\xi,0)^{(|\alpha_1|-|\beta_1|)\theta_1+(|\alpha_2|-|\beta_2|)\theta_2+|\alpha_3|-|\beta_3|} \right. \right.$$

(14)

$$\left. \left. \xi^\beta D^\alpha_\xi \psi(\varsigma)/\alpha!\beta! \right|^2 d\varsigma + \lambda(\xi,0)^{2m\tau-2\varepsilon_0 N+d_1} \sum_{|\alpha+\beta|\leq N+d_2} \|\xi^\beta D^\alpha_\xi \psi\|^2 \right\}$$

Now take N such that $2m\tau-2\varepsilon_0 N+d_1=2m+\varepsilon_0$ .Note that (14) is

true for $\mu(\xi)$ instead of $\lambda(\xi,0)$ and put $x_1=t^{-\sigma_1}x^0_1, y=0$ and

$\xi_j=t^{\tau_j}\xi^0_j$ (j=1,2) such that $\mu(\xi^0)=1$. Then quasihomogeneity of

$\mu(\xi)$ and $A^{(\alpha)}_{(\beta)}(x_1,y,\xi,0)$ we get the following estimate

$$\|\psi\|^2 \leq c \left\{ \int \left| \sum_{|\alpha+\beta|<N} A_0 {\scriptstyle \begin{pmatrix}0,0,\alpha\\0,0,\beta\end{pmatrix}}(x^0_1,0,\xi^0,0)\varsigma^\beta D^\alpha_\varsigma\psi/\alpha!\beta! \right|^2 d\varsigma \right\}$$

$$+ O(t^{-r})$$

where $r=\min(\varepsilon_1,(\tau_j-\sigma_j)/2)$.

Note that $\sum_{|\alpha+\beta|<N} A_0 {\scriptstyle \begin{pmatrix}0,0,\alpha\\0,0,\beta\end{pmatrix}}(x^0_1,0,\xi^0,0)\varsigma^\beta D^\alpha_\varsigma \psi/\alpha!\beta! = A_0(x^0_1,\varsigma,\xi^0,D_\varsigma)\psi$ and

quasihomogeneity of $A_0(x_1,y,\xi,\eta)$ we get the assertion of theorem B.

To show theorem C we take $\lambda'(\xi,\eta)=(\sum_{j=1}^{2}|\xi_j|^2+|\eta|^2+1)^{1/2}$.

Then by $(3)_0$ and theorem 3 we get for $(x_1,y)\in K$

$$\chi(\xi,\eta)^m \leqq c_K |A_o(x_1,y,\xi,\eta)| \qquad \text{for } |\xi|+|\eta| \geqq c > 0$$

By quasihomogeneity of $A_o(x_1,y,\xi,\eta)$, it is elliptic for $|x_1|+|y| \neq 0$.

References

[1] Gilioli, A. & Treves,F.: An example in the solvability theory of linear PDE'S, Amer. J. Math., 96(1974), 366-384.

[2] Grushin, V.V. : Hypoelliptic differential equations and pseudodifferential operators with operator-valued symbols, Mat.Sb., 88(1972), 504-521.

[3] Hörmander, L. : Pseudo-differential operator and hypoelliptic equations, Singular Integrals , Prc.Symposia Pure Math., 10(1967), 138-183.

[4] Tsutsumi, C. : The fundamental solutuin for a degenerate parabolic pseudo-differential operator, Proc. Japan Acad. 50(1974), 11-15.

[5] Tsutsumi, C. : The fundamental solution for a parabolic pseudo-differential operator and parametrices for degenerate operators, Proc. Japan Acad., 51(1975). 103-108.

[6] Tsutsumi, C. : Parametrices for degenerate operators of Grushin's type, in preparation .