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On the explicit formulae of characters for discrete series representations

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Let us consider a simply connected complex simple Lie group G_c and its connected real simple form G. We denote by $\mathfrak{g}_c$, $\mathfrak{g}$ the Lie algebra of G_c, G respectively.

According to a criterion of Harish-Chandra in [3], a square integrable representation ω on G (i.e., one of each matrix coefficients of ω is square integrable with respect to a Haar measure dx on G) exists if and only if G has a compact Cartan subgroup B.

In this paper, we shall study the global characters of square integrable representations which are called the discrete series. Hence we shall assume that G has a compact Cartan subgroup B. For one of each irreducible representation ω of G in the discrete series, we define a distribution H on G by

$$\Theta(f) = \mathrm{Trace} \int_G f(x^{-1})\omega(x)\, dx$$

for all C^∞-functions f on G with compact support.

Then Θ is a tempered invariant eigendistribution on G, and Θ is real analytic on the set of all regular elements in G (cf.[3]).

We select a maximal compact subgroup K of G containing B. Let $\omega | K$ be the restriction of ω to K. Therefore we put, for each irreducible representation π of K. by $|\omega|K:\pi|$ the multiplicity of π occuring in $\omega|K$. Then the restriction of Θ to B is expressed as

$$\Theta\,(b) = \sum_{\mathcal{E}_K} |\omega|K:\pi|\ \mathrm{Trace}\,\pi(b)$$

in the sence of the distributions on B, where $\mathcal{E}_K$ is the set of all inequivalent classes of irreducible representations on K.

We will summarize Harish-Chandra's parametrization in [1], [3] for the characters of representations in the discrete series.

Let us consider Σ the root system of the pair $(\mathfrak{g}_c/\mathfrak{b}_c)$ where $\mathfrak{b}_c$ is the complexification of Lie algebra $\mathfrak{b}$ of B. By P, P^+, P^-, L', we shall denote the set of all positive roots, the set of all noncompact positive roots, the set of all compact positive roots, the set of all regular integral form on $\mathfrak{b}_c$ respectively.

Let Θ be the character of a fixed irreducible representation ω of G in the discrete series. Then

$$\Theta = \varepsilon(\Lambda)(-1)^{|P^+|} \sum_{s\in W(G/B)} \varepsilon(s) \exp s\Lambda \text{ on } B' \quad .....(*)$$

for suitably chosen in L'

where $B' = \{b=\exp H;\ b \in B,\ \alpha(H) \neq 0 \text{ for all } \alpha \text{ in } P\}$,

$W(G/B)$ = the Weyl group of the pair (G/B),

$\varepsilon(s)$ = the signature of s in $W(G/B)$,

and $\varepsilon(\Lambda) = \prod_{\alpha\in P} \operatorname{sgn}(\Lambda,\alpha)$.

Conversely, for each form Λ in L', there exists unique irreducible representation ω of G in the discrete series such that Θ = Trace ω satisfies the above identity (*). Thus one of each discrete character Θ is parametrized by $\Theta = \Theta_\Lambda$ ($\Lambda\in L'$) under the identity (*).

We now state our purpose of this paper. Let us consider a regular dominant integral form Λ on $\mathfrak{b}_c$ and finite dimensional irreducible representation π_Λ of G_c with the highest weight $\Lambda - \frac{1}{2}\sum_{\alpha\in P}\alpha$. Therefore we define a distribution δ_Λ on B as following;

$$\delta_\Lambda(b) = \sum_{s\in W(G/B)\backslash W} \Theta_{s\Lambda} - (-1)^{|P^+|}\,\mathrm{Trace}\;\pi_\Lambda(b),\quad b\in B$$

where W is the Weyl group of the pair $(\mathfrak{g}_c/\mathfrak{b}_c)$.

By The identity (*), $\delta_\Lambda \equiv 0$ on B'. Moreover, since the discrete representations are all infinite dimensional, $\delta_\Lambda \not\equiv 0$ on B. Hence δ_Λ is a singular distribution on B, because B' is open dense in B. We shall, in this paper, give a characterization of and obtain a global formula of the character $\sum_{s\ W(G/B)\backslash W} \Theta_{s\Lambda}$ under the following assumptions.

(A1): G contains a compact Cartan subgroup B.

(A2): All of noncompact roots in Σ have the same length with each other.

For this purpose, we will state more precisely descriptions. We define the generating function $\Phi_{s,\Lambda}$ ($s \in W$) on B_c as followings;

$$\Phi_{s,\Lambda} = \prod_{\alpha \in P^-(s)} (1-\exp -\alpha)\exp(s\Lambda+s\rho) / \prod_{\alpha \in P^+(s)} (1-\exp\alpha)$$

where $\rho = \frac{1}{2} \sum_{\alpha \in P} \alpha$, $P^+(s)=\{\alpha \in P^+ \cup -P^+; s^{-1}\alpha > 0\}$, and $P^-(s)=\{\alpha \in P^- \cup -P^-; s^{-1}\alpha > 0\}$.

Then $\Phi_{s,\Lambda}$ is holomorphic on the complex domain

$D_s = \{b = \exp H \in B_c;\ |\exp\alpha(H)| < 1$ for each α in $P^+(s)\}$.

The functions $\Phi_{s,\Lambda}$ ($s \in W$) are concerned with Blattner's conjecture. This phenomenon is stated as followings. Let L be the set of all integral form on $\mathfrak{b}_c$ and $Q_s(\mu)$ ($\mu \in L$) be the partition function on L, which is defined by

$$1 / \prod_{\alpha \in P^+(s)} (1-\exp\alpha) = \sum_{\mu \in L} Q_s(\mu) \exp\mu .$$

Then $\Phi_{s,\Lambda}$ is expressed as

$$\Phi_{s,\Lambda} = \sum_{\mu \in L} Q_s(\mu) \prod_{\alpha \in P^-(s)} (1-\exp -\alpha) \exp(s\Lambda+s\rho+\mu).$$

We now choose an element u in W satisfying $u^{-1}\alpha > 0$ for each α in P^-.

Since $\Delta_K(b^s) = \varepsilon(s)\Delta_K(b)$ $(\Delta_K = \exp\frac{1}{2}\sum_{\alpha\in P^-}\alpha \prod_{\alpha\in P^-}(1-\exp-\alpha))$
and $sP^+ \subseteq P^+ \cup -P^+$ for all $b \in B$, $s \in W(G/B)$.

we get

$$\sum_{s\in W(G/B)} \overline{\Phi}_{su,\Lambda} = \sum_{\mu\in L}\sum_{s\in W(G/B)} Q_{su}(\mu)\varepsilon(s)\Delta_K \exp(su\Lambda+s\rho^+(u)+\mu)$$

where $\rho^+(u) = \frac{1}{2}\sum_{\alpha\in P^+(u)}\alpha$.

Moreover by $P^+(su) = sP^+(u)$, the above equation is rewriten as

$$\sum_{s\in W(G/B)} \Phi_{su,\Lambda} = \sum_{\mu\in L}\sum_{s\in W(G/B)} Q_u(\mu)\Delta_K \exp(su\Lambda+s\rho^+(u)+s\mu)$$

$$= \sum_{\mu\in L}\sum_{s\in W(G/B)} \varepsilon(s)Q_u(s\mu-u\Lambda-\rho^+(u)) \exp\mu.$$

Defining the Blattner's number $b_{u,\Lambda}(\mu)$ $(\mu\in L)$ and a subset L^+ in L by

$$b_{u,\Lambda}(\mu) = \sum_{s\in W(G/B)} \varepsilon(s)Q_u(s\mu-u\Lambda-\rho^+(u)),$$

and $L_+ = \{\mu\in L;\ (\mu,\alpha) > 0$ for each compact positive root $\alpha\}$,

then

$$\sum_{s\in W(G/B)} \overline{\Phi}_{su,\Lambda} = \sum_{\mu\in L_+} b_{u,\Lambda}(\mu) \sum_{s\in W(G/B)} \varepsilon(s)\exp s\mu$$

$$= \sum_{\mu\in L_+} b_{u,\Lambda}(\mu)\ (\Delta_K)^2 \ \mathrm{Trace}\ \pi_\mu$$

where π_μ is the finite dimensional irreducible representation of K with the highest weight $\mu - \frac{1}{2}\sum_{\alpha\in P^-}\alpha$.

There was conjectured that for the representation $\omega = \omega(u\Lambda)$ with its character $\Theta_{u\Lambda}$,

$$|\omega(u\Lambda)|K:\pi| = \begin{cases} b_{u,\Lambda}(\mu) & \text{if } \pi = \pi_\mu,\ \mu\in L_+ \\ 0 & \text{otherwise} \end{cases}$$

This conjecture implies

$$\sum_{s\in W} \Phi_{s,\Lambda}(b) = (-1)^{|P^-|} |\Delta_K(b)|^2 \sum_{s\in W(G/B)\backslash W} \Theta_{s\Lambda}(b) \quad .....(**).$$

According to these observations, we arrive the following situation; it will be suggested that a calculation of explicit relation between $\sum_{s\in W} \Phi_{s,\Lambda}(b)$ and Trace $\pi_\Lambda(b)$ $(b \in B)$ enable us to clear the gap δ_Λ.

As far as the autor knows, the several results of multiplicity formulae of characters in the discrete series (Blattner's conjecture) have been obtained in the following papers; [14], [15] (W. Schmid), [10] (R. Hotta and K. R. Parthasarathy). [4] (H. Hecht and W. Schmid), [17] (N. Wallach).

In the amount of these papers, Blattner's conjecture was completely solved by [4], [15] and by [17] for all real semisimple matrix groups with compact Cartan subgroups. Especially, the explicit multiplicity formulae of characters of discrete series representations were obtained by [4] and [15].

However, we need not apply the multiplicity theorem in [4], [15] to our arguments. Our main results will be stated after the following preparations.

Definition; a subset F in P^+ is strongly orthogonal if and only if F satisfies that for each of two distinct roots α, β in F, $\alpha = -\beta$ and $\alpha \pm \beta \notin \Sigma$.

Definition; two strongly orthogonal system F_1, F_2 in P^+ are conjugate if and only if there exists t in $W(G/B)$ such that $tF_1 \cup -tF_1 = F_2 \cup -F_2$.

By Γ_0, we denote a complete representative of all nonconjugate strongly orthogonal system in P^+ under $W(G/B)$. Choosing a system F^0 in Γ_0 satisfying $|F^0|$ = real rank of $\mathfrak{g}$, then we can assume that $F \subseteqq F^0$ for all F in Γ_0 under the condition A1, A2. Let $B(F)$ $(F \in \Gamma_0)$ be the subgroup of B, which is defined by $B(F) = \{b = \exp H \in B; \alpha(H) = 0 \text{ for each } \alpha \text{ in } F\}$. Let us consider $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ the Cartan decomposition of $\mathfrak{g}$, here $\mathfrak{k}$ is the Lie algebra of K. Then for each α in F, there exist X_α, $X_{-\alpha}$ in $\mathfrak{g}_c$ such that $ad(H)X_{\pm\alpha} = \pm\alpha(H)X_{\pm\alpha}$ $(H \in \mathfrak{b}_c)$, $\sqrt{-1}\,(X_\alpha + X_{-\alpha})$, $(X_\alpha - X_{-\alpha}) \in \mathfrak{p}$. Therefore we put $\mathfrak{a}_R(F)$, $\mathfrak{z}_c(F)$, $\mathfrak{g}_c(F)$, $\mathfrak{b}(F)$ $(\equiv \mathfrak{a}_I(F))$, $\mathfrak{a}(F)$, and $\mathfrak{b}_c(F)$ by the followings;

$\mathfrak{a}_R(F) = \sum_{\alpha \in F} \sqrt{-1}\, R(X_\alpha + X_{-\alpha})$, $\mathfrak{z}_c(F)$ = the centralizer of $\mathfrak{a}_R(F)$ in $\mathfrak{g}_c$,

$\mathfrak{g}_c(F)$ = the orthogonal complement of $\mathfrak{a}_R(F)$ in $\mathfrak{z}_c(F)$ with respect to the Killing form on $\mathfrak{g}_c$,

$\mathfrak{b}(F)$ = the Lie algebra of $B(F)$, $\mathfrak{a}(F) = \mathfrak{a}_I(F) + \mathfrak{a}_R(F)$, and

$\mathfrak{b}_c(F)$ = the complexification of $\mathfrak{b}(F)$ in $\mathfrak{g}_c$.

Then $\mathfrak{g}_c(F)$ is a reductive subalgebra of $\mathfrak{g}_c$ with Cartan subalgebra $\mathfrak{b}_c(F)$. Moreover, the positive (noncompact positive) root system of $(\mathfrak{g}_c(F)/\mathfrak{b}_c(F))$ can be identified with the set

$P(F) = \{\alpha \in P;\ (\alpha,\beta) = 0,\ \beta \in F\}$ (resp. $P^+(F) = \{\alpha \in P^+; \alpha \pm \beta \notin \Sigma$ for all in $F\}$. We now consider, for a fixed system F in Γ_0, a singular distribution $X_{s\Lambda}(F;b)$ on B, which is given by

$$\langle X_{s\Lambda}(F;b), f(b)\rangle = \frac{1}{|W(G/B)|} \sum_{u \in W(G/B)} \int_{B(F)} \sum_{t \in W(\mathfrak{g}_c(F)/\mathfrak{b}_c(F))} \varepsilon(ut) \times$$

$$|\Delta_{K(F)}(b)|^2 f(b) \frac{\exp uts\Lambda(\log b)}{\prod_{\alpha \in P(F)} (1-\exp -\alpha(\log b))} db$$

for all C^∞-functions f on B, where db is the Haar measure on B(F) normalized as $\int_{B(F)} db = 1$,

$\Delta_{K(F)} = \exp(\frac{1}{2} \sum_{\alpha \in P^-(F)} \alpha) \prod_{\alpha \in P^-(F)} (1-\exp -\alpha)$, $P^-(F) = P(F) - P^+(F)$.

and $W(\mathfrak{g}_c(F)/\mathfrak{b}_c(F))$ = the Weyl group of $(\mathfrak{g}_c(F)/\mathfrak{b}_c(F))$.

Our first main result is stated below.

Theorem I. Suppose G fulfills the conditions A1, A2. Then, for the functionals $\Phi_{s,\Lambda}$ $(s \in W)$, $X_{s\Lambda}(F;b)$ on B, we have

$$\sum_{s \in W} (-1)^{|P^-|} \Phi_{s,\Lambda}(b) = \sum_{\substack{s \in W(F)\backslash W \\ s^{-1}F > 0}} \varepsilon(s)(-1)^{|P^+|+|F|} X_{s\Lambda}(F;b)$$

for all b in B, where $W(F) = \{s \in W;\ sF \subseteq F \cup -F\}$.

Let us consider, for a fixed system F in Γ_0, the Cartan subgroup A(F) of G, which is corresponded to $\mathfrak{a}(F)$. Let Z(F) be the centralizer of $\mathfrak{a}_R(F)$ in G. Then $\mathfrak{g} \cap \mathfrak{z}_c(F)$ is the Lie algebra of Z(F).

Morover, there exists a unique parabolic subgroup $Q(F)$ of G such that the reductive part coincides with $Z(F)$. Therefore we define a representation $\pi^F_{s\Lambda}$ of G, which is induced from a finite dimensional irreducible representation of $Q(F)$.

For the simplicity of our notations, we will identify Σ, W with the root system of $(\mathfrak{g}_c/\mathfrak{a}_c(F))$, the Weyl group of $(\mathfrak{g}_c/\mathfrak{a}_c(F))$ by using the Cayley transform. Putting

$$T_\Lambda = \sum_{F\in\Gamma_0} \sum_{\substack{s\in W(F)\backslash W \\ s^{-1}F>0,\ s^{-1}P(F)>0}} \varepsilon(s)\ \mathrm{Trace}\ \pi^F_{s\Lambda},$$

then the distribution T_Λ on G is an extention of $\sum_{s\in W} \Phi_{s,\Lambda}$ to G. Speaking more precisely,

$$\sum_{s\in W} \Phi_{s,\Lambda} = (-1)^{|P^-|} |\Delta_K|^2\ T_\Lambda \ \text{on } B \qquad \ldots\ldots(***).$$

We notice that the equation (***) is corresponded to (**).

Theorem II. Under the same assumptions as in Theorem I,

$$\varepsilon_R(F;h)\Delta(F;h)\ T_\Lambda(h)$$

$$= \sum_{s\in W} \sum_{u\in V_0(F)\backslash W(G/A(F))} \prod_F \varepsilon(s^{-1}\alpha)\exp s\Lambda(\log h_I)\ \times$$

$$\prod_{\alpha\in F} \exp -|\alpha(\log(h_R)^u)|\ |(s\Lambda,\alpha)|/|\alpha|^2$$

for all regular elements $h = h_I h_R$ ($h_I \in \underline{A(F)\cap K}$, $h_R \in A(F)\cap \exp p$) where $W(G/A(F))$ = the Weyl group of $(G/A(F))$,

$$V_0(F) = \{s \in W(G/A(F));\ sF \subseteq F \cup -F\},$$

$$\Delta(F;h) = \exp(\tfrac{1}{2}\sum_{\alpha\in P}\alpha)(\log h)\prod_{\alpha\in P}(1-\exp-\alpha(\log h)),$$

$$\mathcal{E}_R(F;h) = \prod_{\substack{\alpha\in P \\ \alpha\not\equiv 0 \text{ on } \mathfrak{a}_R(F)}}(1-\exp-\alpha(\log h)),$$

and $\varepsilon(\alpha)$ = the signature of root α .

Theorem III. We keep the same conditions as in Theorem I. Then $T_\Lambda = \sum_{s\in W(G/B)\backslash W}\Theta_{s\Lambda}$. Consequently, the right hand side in the equation of Theorem I obtains a global formula for the character $\sum_{s\in W(G/B)\backslash W}\Theta_{s\Lambda}$.

The formulae in Theorem I, II will be calculated out by using the certain properties for the groups $W(F)$, $V_0(F)$ ($F\in\Gamma_0$). The relation in Theorem III can be proved by using Theorem II, and by using the uniqueness of tempered invariant eigen-distributions in [1], the explicit formulae of $\Theta_{s\Lambda}$ ($s\in W$) on B' (cf. [3]).

For the global character formulae of discrete series representations, there were known the following cases; real rank one groups in [2], pp.120-122 (Harish-Chandra), indefinite unitary groups, $S_p(n.R)$ in[8] ,[9] (T. Hirai) respectively. On the other hand, [14](W. Schmid) has given the explicit formulae of characters for discrete series representations on split real Cartan subgroup of $S_p(n,R)$. This calculation is based on the relation of c) in Theorem (4.15),[15] (W. Schmid).

The same relation as in Theorem III have been observed by [12] for real rank one case, and by [18](G. Zuckerman) for real rank one case, indefinite unitary groups. In essence, our direction of this paper is similar to the one in [12], [18]. Harish-Chandra has given general principle for these relations. However, the explicit relations are not completely known in general.

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