数理解析研究所講究録
第 282 巻 1976 年 26-31

On a flow on the 3-torus

by

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1. Definitions. Let π_t be a flow on the phase space X. We denote by $O(\pi_t, P)$ the orbit of π_t through the point $P \in X$.

Definition 1. A homeomorphism h of X is said to be *orbit preserving* (for π_t), if $h(O(\pi_t, P)) = O(\pi_t, h(P))$ for every $P \in X$. If π_t has no periodic orbits, then for every $P \in X$ and $t \in R$ there exists a number $\tau = \tau(t, P)$ such that $h(\pi_t(P)) = \pi_\tau(h(P))$. When $\tau(t, P)$ is an increasing function in t for each P, we call h a *positive* orbit preserving homeomorphism. In the case when h is decreasing, it is said to be *negative*.

Definition 2. π_t is said to be a *homogeneous* flow, if for any two points $P, Q \in X$, there exists an orbit preserving homeomorphism which takes P to Q.

Definition 3. π_t is said to be *a.p.* (*almost periodic*), if the family of homeomorphisms $\{\pi_t\}$ is equi-continuous.

2. Results. We consider the system of differential equations

$$\text{(1)} \qquad \frac{dx}{dt} = 1 , \quad \frac{dy}{dt} = \gamma , \quad \frac{dz}{dt} = f(x, y) ,$$

where we assume the followings.

Assumptions.

(A) $f(x, y)$ has the period 1 in each variables.

(B) $f(x, y) > 0$ for all (x, y).

(C) γ is an irrational number but not quadratically algebraic.

Because of the periodicity of $f(x, y)$, the system (1) determines a flow φ_t on the 3-torus $T^3 = R^3/Z^3$. Our main results are the following two theorems.

Theorem 1. If φ_t is not a.p., then it is a minimal flow.

Theorem 2. If φ_t is not a.p., then it is not a homogeneous flow.

Remarks.

(i) There really exist an irrational number γ and a real analytic function $f(x, y)$ which make φ_t a non-a.p. flow.

(ii) If γ is quadratically algebraic and $f(x, y) \in C^3$, then φ_t is a.p..

(iii) A linear minimal flow on T^3 is homogeneous, so this non-a.p. flow cannot be linearized even if the time is changed.

In the following we shall give an outline of the proof of Theorem 2.

3. Sketch of the proof of Theorem 2.

First we prove

Lemma 1. The following conditions are equivalent.

(a) φ_t is a.p..

(b) $H_y(x) = \int_0^x [f(s, y+\gamma s) - f(s, \gamma s)]ds$ is a bounded function for each y.

(c) $\sum_{(m,n)\neq(0,0)} \left| a_{m,n} / (e^{2\pi i(m+n\gamma)} - 1) \right|$ converges, where $f(x, y) = \sum a_{m,n} \exp 2\pi i(mx + ny)$.

Proof. The system (1) can be integrated as

$$x = x_0 + t \ , \quad y = y_0 + \gamma t \ , \quad z = z_0 + \int_0^t f(x_0+s, y_0+\gamma s)\, ds$$

Since $f(x_0+s, y_0+\gamma s)$ is an a.p. function in s, this lemma can be proved by the following well-known theorem.

Theorem. For the indefinite integral of an a.p. function to be a.p., it is necessary and sufficient that it is bounded.

Definition 4. An *integral* of φ_t is a foliation $\mathcal{F}$ of T^3 such that $\varphi_t(L) = L$ for any leaf L of $\mathcal{F}$ and any $t \in R$.

Let $\mathcal{F}_0$ be the foliation which is determined by the Pfaffian equation $\gamma\, dx - dy = 0$. Then we obtain

Lemma 2. If φ_t is not a.p., then $\mathcal{F}_0$ is the only integral of

Proof. Using the minimality of φ_t and the theory of the rotation number, we can prove that if there exists another integral, then the restriction of φ_1 to $\{x = 0\}$ is isomorphic to a rotation of the 2-torus and so φ_t is a.p..

Let ψ_t be the flow on the 2-torus which is determined by the first two equations of system (1). Let C be a section of ψ_t, then we can define the rotation number of the returning-map on C.

Let C and C' be two sections of φ_t , and α and α' be the rotation numbers for C and C' respectively. Then we have

Lemma 3. Suppose γ is irrational but not quadratically algebraic. Then $\alpha \equiv \pm\alpha'$ (mod 1) if and only if C and C' are homotopic to each other.

Proof. By a direct calculation.

Lemma 4. Suppose that φ_t is not a.p.. If there is a positive (negative) orbit preserving homeomorphism h' by which (0,0,0) goes to $(0,y_0,0)$, then there exists a positive (negative) orbit preserving homeomorphism h such that $h(0,0,0) = (0,y_0,0)$ and $h(\{x = 0\}) = \{x = 0\}$.

Proof. By Lemma 2 and 3, we can see that $h'(\{x = 0\})$ is homotopic to $\{x = 0\}$. Hence we can constract a positive orbit preserving homeomorphism σ such that $\sigma(h'(\{x = 0\})) = \{x = 0\}$ and it fixes the point $(0,y_0,0)$. Then $h = \sigma \circ h'$ is the desired homeomorphism.

Let β be a real number defined by

$$\beta = \lim_{n\to\infty} \frac{1}{n}\int_0^n f(\xi, y+\gamma\xi)\,d\xi .$$

β is independent of y , and is a topological invariant as the rotation number.

Lemma 5. Suppose that β/γ is 0 or irrational, and that φ_t is not a.p.. If there is a positive orbit preserving homeomorphism which carries (0,0,0) to $(0,y_0,0)$, then $H_{y_0}(x)$ is bounded. And if there is a negative orbit preserving homeomorphism, then

$$\hat{H}_{y_0}(x) = \int_0^x f(s,\gamma s)\,ds + \int_0^{-x} f(s, y_0+\gamma s)\,ds$$

is bounded.

Proof. By Lemma 4, we can find an positive (negative) orbit preserving homeomorphism h such that $h(\{x=0\}) = \{x=0\}$ and $h(0,0,0) = (0,y_o,0)$.

If h is positive and β/γ is 0 or irrational, then we can see that

$$(2) \qquad \tilde{h}\circ\tilde{\varphi}_1 = \tilde{\varphi}_1\circ\tilde{h} \quad \text{on} \quad \{x=0\}$$

and $\tilde{h}$ satisfies

$$(3) \qquad \begin{cases} \tilde{h}(0,y+1,z) - \tilde{h}(0,y,z) = (0,1,0) \\ \tilde{h}(o,y,z+1) - \tilde{h}(o,y,z) = (0,0,1) \end{cases}$$

where $\tilde{h}$ and $\tilde{\varphi}_1$ is the lifts of h and φ_1 respectively. On the other hand, if we choose a suitable lift $\tilde{\varphi}_1$ of φ_1, then

$$(4) \qquad z^n(y,z) = z + \int_0^n f(\xi, y+\gamma\xi)\,d\xi$$

where $\tilde{\varphi}_1^{\,n}(o,y,z) = (0, y^n, z^n)$. From (2), (3) and (4), it follows that $H_{y_o}(x)$ is bounded if h is positive and $h(0,0,0) = (0,y_o,0)$.

In the case when h is negative, we have that $\tilde{h}\circ\tilde{\varphi}_1 = \tilde{\varphi}_1^{-1}\circ\tilde{h}$ and

$$\begin{cases} \tilde{h}(0,y+1,z) - \tilde{h}(0,y,z) = (0,-1,0) \\ \tilde{h}(0,y,z+1) - \tilde{h}(0,y,z) = (0,0,-1) \end{cases}$$

Therefore it can be proved that $\hat{H}_{y_o}(x)$ is bounded.

Proof of Theorem 2. Suppose that β/γ is 0 or irrational. By Lemma 1 and 5, if φ_t is not a.p., we can choose a point $(0,y_o,0)$ so that there exists no positive orbit preserving homeomorphism by which $(0,0,0)$ goes to $(0,y_o,0)$. Hence we have only to prove that no negative orbit preserving homeomorphism carries $(0,0,0)$ to $(0,y_o,0)$. For this purpose, according to Lemma 5, it is sufficient to show the following lemma.

Lemma 6. If $H_{y_o}(x)$ is not bounded, then $\hat{H}_{y_o}(x)$ is also not bounded.

This lemma can be proved by means of the almost periodicity of $f(\xi, y+\gamma\xi)$.

In the case when β/γ is rational, considering the map $\varphi_{1/\gamma}$ on $\{y = 0\}$, we can prove the theorem by the similar way.

Reference

I. Ishii, On a non-homogeneous flow on the 3-dimensional torus, Funkcial. Ekvac., 17 (1974), 231-248.