数理解析研究所講究録
第 282 巻 1976 年 45-48

Boundedness and Convergence of Solutions of Duffing's Equation

Kenichi Shiraiwa

We shall discuss boundedness of solutions of the equation

(1) $x'' + f(x)x' + g(x) = e(t)$ $\quad (\ ' = d/dt \ )$

under suitable conditions. Furthermore, we shall discuss asymptotic stability of a periodic solution and convergence of solutions for the equation

(2) $x'' + cx' + g(x) = e(t)$,

where c is a positive constant and e(t) is a periodic function.

This work is motivated by the work of H.Kawakami [2], which gives some numerical computations on the equation $x'' + kx' + x^3 = B \cos t$ for positive constants k and B. There are also very interesting results by C.Hayashi, Y.Ueda and H.Kawakami [1]. Also, our paper heavily depends on the work of W.S.Loud [3].

Theorem 1 In the equation (1) we assume the following conditions (a), (b) and (c).

(a) There exists a solution of (1) under any initial condition.

(b) There exist positive constants c and E such that

$$f(x) \geqq c \quad \text{and} \quad |e(t)| \leqq E .$$

(c) g(x) is a differentiable function satisfying the following conditions (i), (ii) and (iii).

(i) g(x) is bounded on any finite interval.

(ii) $g'(x) \geqq 0$

(iii) $\lim_{x\to\infty} g(x) > E$ and $\lim_{x\to-\infty} g(x) < -E$.

By the condition (c), g(x) is a monotone increasing function, and there exist numbers x_1 and x_2 $(x_1<x_2)$ such that

$$g(x_1) = -E \quad \text{and} \quad g(x_2) = E .$$

Let x(t) be any solution of (1). Then there exists a number t_0 such that

$$x_1 - 4E/c^2 \leqq x(t) \leqq x_2 + 4E/c^2 \quad \text{and}$$
$$|x'(t)| \leqq 4E/c \quad \text{for any } t \geqq t_0 .$$

Our proof is similar to that of Theorem 1 of W.S.Loud [3]. He assumed that $g'(x) \geqq b$ for some positive constant b in his paper and got an additional information.

Corollary In addition to the conditions (a), (b) and (c) of Theorem 1, we assume the following two conditions,

(d) f(x) and e(t) are continuous, and f(x) satisfies the local Lipschitz condition.

(e) e(t) is periodic of period τ($\tau>0$).

Then the equation (1) has a periodic solution of period τ .

The equation (2) is a special case of (1), and it is equivalent to the following system of equations.

$$(3) \quad \begin{cases} x' = y \\ y' = -cy - g(x) + e(t) \end{cases}$$

Theorem 2 Assume the following conditions A(i), A(ii) and A(iii).

A(i) e(t) is a continuous periodic function of period τ($\tau>0$), and E is a positive constant such that $|e(t)| \leqq E$.

A(ii) g(x) is of class C^1 such that

$$g'(x) \geqq 0, \quad \lim_{x\to\infty} g(x) > E, \quad \lim_{x\to-\infty} g(x) < -E, \quad \text{and } g'(x)=0$$

only on a countable subset of the real numbers.

A(iii) c is a positive constant.

Now, let n be a positive number and let $x = \phi_1(t)$, $y = \phi_2(t)$ be a non-constant periodic solution of period $n\tau$ for the equation (3). Suppose that $|\phi_1(t)| \leqq \beta$ for all t and $c^2 > H(\beta)$, where $H(\beta) = \sup\{g'(x)\}; -\beta \leqq x \leqq \beta\}$.

Then the periodic solution $x = \phi_1(t)$, $y = \phi_2(t)$ is asymptotically stable.

This is a generalization of Loud [3].

Corollary 1 Assume the above conditions A(i), A(ii) and A(iii). Let $A = \max\{|x_1 - 4E/c^2|, |x_2 + 4E/c^2|\}$ where $g(x_1) = -E$ and $g(x_2) = E$. Further, assume that $c^2 > H(A) = \sup\{g'(x); -A \leqq x \leqq A\}$.

Then every non-constant periodic solution of period $n\tau$ (n a positive integer) of the equation (3) is asymptotically stable.

Corollary 2 In addition to the assumptions of Corollary 1, we assume that $e(t)$ is non-constant.

Then there exists a non-constant periodic solution $x = \psi_1(t)$, $y = \psi_2(t)$ of period τ for the equation (3) such that any periodic solution of period $n\tau$ (for a suitable positive integer n) for the equation (3) coincides with solution $x = \psi_1(t)$, $y = \psi_2(t)$.

Theorem 3 Under the same assumption of Corollary 2 of Theorem 2, there exists a unique periodic solution $x = \psi_1(t)$,

$y = \psi_2(t)$ of period τ for the equation (3) such that for any solution $x = x(t)$, $y = y(t)$ of (3) the following equalities hold.

$$\lim_{t\to\infty} |x(t) - \psi_1(t)| = \lim_{t\to\infty} |y(t) - \psi_2(t)| = 0$$

This also generalizes the results of Loud [3].

Details will appear elsewhere.

References

[1] C.Hayashi, Y.Ueda and H.Kawakami : Transformation Theory as Applied to the Solution of Non-Linear Differential Equations of Second Order, Int. J. Non-Linear Mechanics, 4 (1969), 235-255

[2] H.Kawakami : Qualitative Study on the Solutions of Duffing's Equation, Thesis (1973), Kyoto University.

[3] W.S.Loud : Boundedness and Convergence of Solutions of $x'' + cx' + g(x) = e(t)$, Dnke Math. J.24. (1957), 63-72

Department of Mathematics
College of General Education
Nagoya University