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ON THE TRANSEVERSALITY CONDITIONS IN ELECTRICAL CIRCUITS

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1. Introduction.

In S. Smale's formulation of electrical circuit theory ([1]) and in T. Matsumoto's extension of it ([2]), the transeversality conditions of the characteristic manifolds and the Kirchhoff spaces are standing hypotheses. Of course, accoding to Thom's transeversality theorem we can make them transeversal by arbitrarily small perturbations of the characteristic manifolds. But two problems [redacted] remain[redacted]. First, is it possible that the perturbations of the characteristic manifolds are realized in electrical circuits? Second, according to the temperature or to the pressure and so on, the characteristic manifolds are always perturbed and hence the transeversality may be destroyed.

Therefore, the following problem proposed by T. Matsumoto is natural.

Problem. By adding small capacitances parallel to the given circuit (or/and small inductances series), can we make the characteristic manifold and the Kirchhoff space transeversal ?

In this note, under a certain weak condition we will give the affirmative answewr to the above problem.

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2. Statements of results.

As in [1], we assume that the electrial circuit is represented by an oriented graph G. Let C_j (C^j) be the real j-chains (j-cochains) of G, j=0,1. Then the currents and the voltages in the branches of the circuit can be thought as elements of C_1 and C^1 respectively.

The characteristic manifold $\Lambda=\Lambda(G)$ representing the characteristics of the elements (possibly including non-linear coupled resistors and so on ([2])) is a $2b-\rho$ dimensionnal smooth submanifold of $C_1 \times C^1$, where b is the number of the branches and ρ corresponds to the number of the resistors. Note that $\rho \leq b$.

The Kirchhoff's laws restrict the possible states to a b-dimensional linear subspace called the Kirchhoff space, $K=\mathrm{Ker}\,\partial \times \mathrm{Im}\,\partial^*$ of $C_1 \times C^1$ where ∂ (∂^*) is the boundary (coboundary) operator,

$$\partial : C_1 \longrightarrow C_0, \qquad (\partial^* : C^0 \longrightarrow C^1).$$

Now, we define the controlledness of the characteristic manifold, extending the case of an independent element.

Definition. The characteristic manifold Λ is called voltage-controlled (current controlled) if there exists a smooth function F: $C_1 \times C^1 \longrightarrow R^{\rho}$ such that the partial derivative with respsctive to the first factor $D_1F: R^b \longrightarrow R^{\rho}$ (with respective to the second factor $D_2F: R^b \longrightarrow R^{\rho}$) has the maximum rank ρ and $\Lambda = F^{-1}(0)$.

Definition. The characteristic submanifold Λ is called locally controlled if Λ is locally voltage-controlled or current-controlled.

The results are the followings.

Theorem. If the characteristic manifold $\Lambda = \Lambda(G)$ is voltage-controlled (current-controlled), by adding small capacitances parallel (small inductances series) to G we can get a new circuit G′ for which $\Lambda' = \Lambda(G')$ and $K' = K(G')$ are transeversal.

Corollary. If the characteristic manifold Λ is locally controlled, by adding small capacitances and inductances appropriately we can get a circuit G′ for which Λ' and K' are transeversal.

3. Preliminaries from circuit theory.

We recall what we need from circuit theory. (cf. Rohler[3].)

Let V be a linear subspace of C_1 such that $C_1 = \mathrm{Ker}\,\partial \oplus V^{\perp}$ where $V^{\perp}$ is the orthogonal complement of V (with respect to the usual metric in $C_1 = R^b$). The space V is linearly isomorphic to the space $\mathrm{Ker}\,\partial$. We denote the into isomorphism $V \longrightarrow \mathrm{Ker}\,\partial \subset C$ by ι. Let $p: C_1 \longrightarrow V$ be the projection map along $\mathrm{Ker}\,\partial$ (i.e. $p(\mathrm{Ker}\,\partial) = 0$). Then we obtain the following commutative diagrams:

$$\begin{array}{ccccccc} 0 & \longrightarrow & V & \xrightarrow{\iota} & C_1 & \xrightarrow{\partial} & C_0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longleftarrow & V^* & \xleftarrow{\iota^*} & C^1 & \xleftarrow{\partial^*} & C^0 \end{array}, \qquad \begin{array}{ccccccc} 0 & \longrightarrow & V & \xrightarrow{\iota} & C_1 & \xrightarrow{p} & V^{\perp} \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longleftarrow & V^* & \xleftarrow{\iota^*} & C^1 & \xleftarrow{p^*} & (V^{\perp})^* \end{array},$$

where V^* is the dual space of V, ι^* and p^* are the dual maps of ι and p respectively and the vertical maps are the natural isomorphisms. Since the rows of the above two diagrams are exact, Kirchhoff's laws can be represented as follows.

(KCL) $\mathring{i} \in \mathrm{Ker}\,p$. (KVL) $v \in \mathrm{Ker}\,\iota^*$.

Now, we give the space V explicitly. First, we take a (maximal) tree T of G. We assume G to be connected for simplicity. Then T is characterized as follows.

1.) Each node of G is a node of T.

2.) T is connected.

3.) T contains no loop.

If we remove a branch of T, then T is disconnected into two parts, and hence the set of nodes of G is also partitioned into two disjoint sets. (This makes a fundamental cut-set.) Note that each pair of nodesof G is connected uniquely through only branches of T. We call the subgraph of G which consists of the branches not contained in T cotree or link and denote it by L. Put $V=C_1(L)$, then $V^{\perp}=C_1(T)$. Hence $C_1=\mathrm{Ker}\,\partial \oplus V^{\perp}$. And we obtain the following maps:

$$C^1(L) \xleftarrow{\imath^*} C^1, \qquad C_1 \xrightarrow{p} C_1(T).$$

We call the matrices B and Q representing $\imath^*$ and p with respect to the natural basis of $C_1=C_1(G)$ the fundamental loop matrix and the fundamental cut-set matrix respectively.

Finally, the Kirchhoff's laws are represented as follows:

$$\text{(KCL)} \quad Q\,\mathring{i}=0, \qquad \text{(KVL)} \quad B\,v=0.$$

By the way, $QB^t=0$ and $BQ^t=0$, because $p\circ\imath=0$ and $\imath^*\circ p^*=0$.

Now, we note that the Kirchhoff space K is

$$\mathrm{Ker}Q \times \mathrm{Ker}B \ (=\mathrm{Ker}\,\imath^* \times \mathrm{Ker}\,p),$$

and the map

$$p^*: C_1(L) \times C^1(T) \longrightarrow K$$

is an isomorphism. Therefore the currents of link branches and the voltages of tree branches $(\mathring{i}_L, v_T) \in C_1(L) \times C^1(T)$ can be thought as coordinates of K.

3. Proofs.

Let $\Lambda=\Lambda(G)$ and $K=K(G)$ represent the characterristic manifold and the Kirchhoff space for G respectively. Since Λ is a $(2b-\rho)$-dimensional smooth submanifld of $C_1 \times C^1 (=R^{2b})$, there exist a neighborhood U of x in R^{2b} and a smooth function $F: R^{2b} \longrightarrow R^{2b}$ such that

$$\mathrm{rank}(JF)_x=\rho \quad \text{and} \quad \Lambda \cap U=F^{-1}(0),$$

where (JF)x represents the Jacobian matrix of F at x. By the definition of transeversality, the following holds clearly.

Proposion. Λ and K are transeversal if and only if the derivative of $F|_K: K \longrightarrow R^{\rho}$, $D(F|_K): R^b \longrightarrow R^{\rho}$ is onto map.

Now, we fix a tree T of G. Note that the matrices B and Q have the following forms ([3]).

$B=[I;A]$, $\quad Q=[-A^t;I]$,

where I is the identity matrix.

Put $\mathring{i}=(\mathring{i}_L, \mathring{i}_T) \in C_1(L) \times C_1(T)=C_1(G)$,

$$v=(v_L, v_T) \in C^1(L) \times C^1(T)=C^1(G),$$

then

$$J(F|_K)=J(F\circ(\iota \times p^*))=JF\circ J(\iota \times p^*)=[J_{\mathring{i}}F,\ J_v F]\begin{bmatrix} B^t & 0 \\ 0 & Q^t \end{bmatrix}$$

$$=[J_{\mathring{i}_L}F,\ J_{\mathring{i}_T}F,\ J_{v_L}F,\ J_{v_T}F]\begin{bmatrix} I & 0 \\ A^t & 0 \\ 0 & -A \\ 0 & I \end{bmatrix}$$

$$=[J_{\mathring{i}_L}F+(J_{\mathring{i}_T}F)A^t,\ \ -(J_{v_L}F)A+J_{v_T}F]. \quad --- \quad (*)$$

We need the following Lemma.

Lemma. By adding small capacitances parallel to the given circuit G, we can obtain the new circuit G′ for which there exists a tree T′ such that the link L′ of T′ contains G.

Proof. Take a tree T of G. And add a small capacitance to each branch of T, then we obtain the new circuit G′ and the subgraph T′ consisting of the elements added is clearly a tree of G′. The link L′ corresponding to T′ contains G. This proves the lemma. (Alternatively, we can obtain the new circuit G′ in the lemma by adding one new node and connecting it to each node of G.)

Proof of Theorem. It is sufficient to show that the new characteristic manifold $\Lambda(G')$ and the new Kirchhoff space K(G′) are transeversal at any point. Note that the function F′ representing $\Lambda(G')$ is essentially the same as F, since we have added only capacitances. The function F′ is independent on the currents of the branches of T′. Hence,

$$J_{\mathring{i}_{T'}}F'=0, \qquad \text{and} \qquad \text{rank}(J_{\mathring{i}}F)=\text{rank}(J_{\mathring{i}_{L'}}F').$$

By the assumption of the theorem, the rank$(J_{\mathring{i}}F)$ is ρ at any point. Therefore, according to (*) and the proposition, we see that $\Lambda(G')$ and K(G′) are transeversal at any point. This proves the half of the theorem. The rest of the theorem is proved dually by adding small inductances series to link branches.

Finally, the corollary is deduced from the theorem, since the transeversality is local and independent of the choice of the particular tree.

References

[1] S. Smale, On the mathematical foundation of electrical circuit theory, J. Differential Geometry, 7, pp193-210,(1972).

[2] T. Matsumato, On the dynamics of electrical networks, J. Differential Equations, 21, pp179-196, (1976).

[3] R. Rohler, Electric circuit theory, McGrow-Hill, 1970.

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