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Some remarks on a cohomology for foliated structures.

1. Let L be a topological vector space with a continuous linear map d into itself so that d d = 0, then we see that

$$\mathrm{Im}\, d \subset \mathrm{Ker}\, d, \quad \overline{\mathrm{Im}\, d} \subset \overline{\mathrm{Ker}\, d}$$

and so that

$$\overline{\mathrm{Ker}\, d} = \mathrm{Ker}\, d,$$

therefore we have

$$\overline{\mathrm{Im} d} \subset \mathrm{Ker} d .$$

The following notion is suggested by Prof. K. Shiga,

Definition We set

$$\breve{H}(L, d) = \mathrm{Ker} d/\overline{\mathrm{Im} d}.$$

Let V be a normed vector space with a linear map ∂ into itself so that $\partial \circ \partial = 0$ and denote the conjugate object with $*$. Introducing the weak topology into V^*, we see that ∂^* turns out to be continuous linear.

Then it is obtained that ([S])

Prop. 1. $\dim \bar{H}(V^*, \partial^*) \leqq \dim \bar{H}(V, \partial)$.

In case that ∂ itself is continuous, Prop. 1 is sharpened as follows:

Theorem 1 (Suzuki-Shikata-Sakai)

$$\dim \bar{H}(V^*, \partial^*) = \dim \bar{H}(V, \partial) \leq \dim H(V, \partial)$$

Proof. Take linearly independent elements $a_1 \dots a_m$ in $\mathrm{Ker}\, \partial$ onver $\overline{\mathrm{Im}\, \partial}$, and define V_m to be a subspace in V generated by $a_1 \dots a_m$ and by $\overline{\mathrm{Im}\, \partial}$. Since $V_m/\overline{\mathrm{Im}\, \partial}$ is canonically homeomorphic to the euclidean m space, we can define continuous linear functionals $\tilde{a}_1 \dots \tilde{a}_m$ on Vm so that

$$\tilde{a}_i(a_j) = \delta_{ij}, \quad \tilde{a}_i = 0 \quad \text{on } \overline{\mathrm{Im}\,\partial}\ .$$

The functionals $\tilde{a}_1 \dots \tilde{a}_m$ extend to functionals $a_1^* \dots a_m^*$ on V^* by Hahn-Banach's theorem and satisfy that

$$a_i^*(a_j) = \delta_{ij}, \quad a_i^* = 0 \quad \text{on } \overline{\mathrm{Im}\,\partial}\ .$$

Thus we have that

$$(\partial^* a_i^*)(x) = a_i^*(\partial x) = 0\ , \text{ for any } x \in V,$$
$$(\lim \partial^* \xi_i)(a_j) = \lim (\xi_i(\partial a_j)) = 0, \text{ for } (\lim_i \partial^* \xi_i) \in \overline{\mathrm{Im}\,\partial^*}$$
$$(\Sigma\, C_i a_i^* + \mu)(a_j) = C_j, \quad \text{for } \mu \in \overline{\mathrm{Im}\partial^*}$$

Hence we see that $a_1^* \dots a_m^*$ define linearly independent elements in $H(V^*, \partial^*)$ and that

$$\dim \bar{H}(V^*, \partial^*) \geq \dim \bar{H}(V, \partial)$$

Conversely, take linearly independent elements $\alpha_1 \dots \alpha_m$ in $\mathrm{Ker}\,\partial^*$ over $\overline{\mathrm{Im}\partial^*}$ and define similarly $\alpha_1^* \dots \alpha_m^*$ to be (weak) continuous functionals on V^* satisfying that

$$\alpha_1^*(\alpha_j) = \delta_{ij}, \quad \alpha_i^* = 0 \quad \text{on } \overline{\mathrm{Im}\,\partial^*}\ .$$

Then by Mackey's weak duality theorem, we find $a_1 \dots a_m \in V$ so that

$$\alpha_i^*(x) = x(a_i), \text{ for any } x \in V^*\ ,$$

thus we have that

$$x(\partial a_i) = (\partial^* x)(a_i) = \alpha_i^*(\partial^* x) = 0, \text{ for } x \in V^*$$
$$a_j(\lim \partial b_i) = \lim \alpha_j(\partial b_i) = \lim(\partial^* \alpha_j)(b_i) = 0, \text{ for } \lim \partial b_i \in \overline{\mathrm{Im}\,\partial}$$
$$a_j(\Sigma C_i a_i + \mu) = C_j, \text{ for } \mu \in \overline{\mathrm{Im}\,\partial}\ .$$

Hence we also see that $a_1 \dots a_m$ define linearly independent elements in $\bar{H}(V, \partial)$ and that

$$\dim \bar{H}(V^*, \partial^*) \leq \dim \bar{H}(V, \partial)$$

finishing the proof . As a corollary to Th. 1, we have that

Corollary 1,

$$\dim H(V^*, \partial^*) \geq \dim \bar{H}(V, \partial).$$

2. Let M be a smooth manifold with two foliated structures F, G which are dual to each other. The flat norm $\| \ \|^b$ in the cosheaf $\Psi_q(R^s)$ of q-chain in R^s ($s=\dim F$) ~~space~~ extends naturally to that in the cosheaf $\Psi_q(F)$ through the (local) identification of the leaf of F to R^s. Let $\hat{\Psi}_q$ denote the completion of $\Psi_q = \Psi_q(F)$ under the flat norm, then the Čech chain $C_p(\{U\}; \Phi_q)$ ($\Phi_q = \Psi_q$ or $\hat{\Psi}_q$) for a finite open covering $\{U\}$ turns out to be a normed linear space with the norm given by $\|\alpha\| = \sup \{ \| \alpha_{j_0 \cdots j_p} \|^b / (j_0 \cdots j_p) : \text{p-Čech simplex} \}$ for $\alpha \in C_p(\{U\}; \Phi_q)$.

Since the completion $\hat{\Psi}_q$ and Ψ_q have the same dual space F^q, we have that

Prop. 2 For the sheaf F^q of flat q-cochain on R^s, it holds that

$$\dim \bar{H}_{\Phi}(C^p(\{U\}; F^q), \Delta^*) = \dim \bar{H}(C_p(\{U\}; \Phi_q), \Delta)$$

where $\bar{H}_{\Phi}$ indicates the difference of the weak topology in F^q.

Corollary 3

$$\dim \bar{H}_{\Psi}(C^p(\{U\}; F^q), \Delta^*) \leq \dim H(C_p(\{U\}; \Psi_q), \Delta))$$

$$\dim \bar{H}_{\hat{\Psi}}(C^p(\{U\}; F^q), \Delta^*) \leq \dim H(C_p(\{U\}; \hat{\Psi}_q), \Delta)) ,$$

Corollary 4

$$\dim \bar{H}(C_p(\{U\}, \Psi_q), \Delta) \leq \dim H_{\Psi}(C^p(\{U\}, F^q), \Delta^*)$$

$$\dim \bar{H}(C_p(\{U\}, \hat{\Psi}_q), \Delta) \leq \dim H_{\hat{\Psi}}(C^p(\{U\}, F^q), \Delta^*)$$

Taking $H_{p,q}$ (resp $\hat{H}_{p,q}$) to be (resp. the completion of) the cosheaf of (p,q)-chains, the Weil's chain gives isomorphisms between homologies of cosheaves;

Prop. 3. For a finite simple (adimissible) covering U it holds that

$$L(C_p(\{U\}, \Phi_q) : \Delta) = L(\Phi_{p,q}; \partial x),$$

where $L = H$ or $\bar{H}_{\Phi}$,

$\Phi_q = \Phi_q$ or $\hat{\Phi}_q$, $\Phi_{p.q} = H_{p.q}$ or $\hat{H}_{p.q}$.

Since we know that

$$\dim H(H_{p,o}; \partial_x) = \#\left\{\begin{array}{c}\text{independent p-cycles}\\ \text{in the leaf of } F\end{array}\right\},$$

we have that

Corollary 4

$$\#\{\text{indep. p-cycles in the leaf}\} = \dim H(C_p(\{U\} : \Phi_o), \Delta)$$
$$\geq \dim \bar{H}_{\Psi}(C^p(\{U\} : F^o), \Delta)$$

Introducing the notion of pseudo closed cycles in the leaf, we also have that

$$\dim H(\hat{H}_{p,o} : \partial_x) = \#\left\{\begin{array}{c}\text{indep. pseudo closed cycles}\\ \text{in the leaf of } F\end{array}\right\}$$

and that

Corollary 5

$$\#\{\text{indep.pseudo p=cycles in the leaf}\} \geq \dim \bar{H}_{\hat{\Psi}}(C^p(\{U\}; F^o), \Delta)$$

[S] Y. Shikata On the cohomology of bigradesed forms associated with foliated structures.