

A Realization of Riemannian Symmetric Spaces

By Toshio OSHIMA*

§0. Introduction.

The purpose of this paper is to construct an imbedding of every Riemannian symmetric space G/K of non-compact type into a compact real analytic manifold \tilde{X} . Here G is a semi-simple Lie group and K a maximal compact subgroup. Our imbedding has the following properties:

The action of G on \tilde{X} is analytic and the orbital decomposition of \tilde{X} is of normal crossing type in the sense of Remark 6 in §2. Moreover, there appears the Martin boundary in \tilde{X} and the system of invariant differential equations on the symmetric space has regular singularity along the Martin boundary in the sense of Definition 5.1 in [9].

As for realizations of G/K there are several papers [1], [2], [5], [7], [12], [13], [15] and [4], [10], [11], [14]. If the rank of the symmetric space is higher than one, the Martin boundary does not appear in the realizations given by [1], [2], [5], [7], [12], [13], [15] and the orbital decompositions have more complicated geometrical structures than ours. The realizations given by [4], [10], [11], [14] are essentially the same ones called Satake-Furstenberg compactifications. They are only different in the methods of constructions. There exists a realization among Satake-Furstenberg compactifications where the Martin boundary appears. But it is a compactification of G/K as a

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manifold with boundaries and the natural analytic structure around the boundaries is not investigated. In [8] we construct an imbedding \tilde{X}' of G/K to solve S. Helgason's conjecture by using a result in [9]. But it is not sufficient for further investigations because there is only a local action of G on \tilde{X}' . This is a motivation to write this paper. The relation between \tilde{X}' and \tilde{X} is shown in Proposition 11, which says that \tilde{X}' is an open dense submanifold of \tilde{X} .

§1. Notation and preliminaries concerning semi-simple Lie groups

We will use the standard notation \mathbb{Z} , \mathbb{R} and \mathbb{C} for the ring of integers, the field of real numbers and the field of complex numbers, respectively. The set of non-negative integers is denoted by \mathbb{N} and the set of positive real numbers by \mathbb{R}_+ . Lie groups will be denoted by Latin capital letters and their Lie algebras by corresponding small German letters. If G is a Lie group and \mathfrak{g} its Lie algebra, the adjoint representation of G is denoted by Ad (or Ad_G) and the adjoint representation of \mathfrak{g} by ad (or $\text{ad}_{\mathfrak{g}}$).

We will now list some standard notation concerning semi-simple Lie groups used in this paper and subsequent papers. Let G be a connected semi-simple Lie group with finite center Z , \mathfrak{g} the Lie algebra of G and $\langle \cdot, \cdot \rangle$ the Killing form of \mathfrak{g} . Let θ be a Cartan involution of \mathfrak{g} and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ the Cartan decomposition of \mathfrak{g} into the eigenspaces of θ . We also denote by θ the Cartan involution of G corresponding to the Cartan involution θ of \mathfrak{g} . Let σ be a maximal abelian subspace of \mathfrak{p} , σ^* its dual, $\sigma_{\mathbb{C}}^*$ the complexification of σ^* . If $\lambda, \mu \in \sigma_{\mathbb{C}}^*$, let $H_{\lambda} \in \sigma_{\mathbb{C}}$ be determined by $\lambda(H) = \langle H_{\lambda}, H \rangle$ for $H \in \sigma$ and put $\langle \lambda, \mu \rangle = \langle H_{\lambda}, H_{\mu} \rangle$. Let \mathfrak{f} be a Cartan subalgebra of \mathfrak{g} containing σ . Then $\mathfrak{f} = \sigma + \mathfrak{t}$ where $\mathfrak{t} = \mathfrak{f} \cap \mathfrak{k}$. We denote by $\mathfrak{g}_{\mathbb{C}}$ the complexification of \mathfrak{g} and for any subspace \mathfrak{b} of \mathfrak{g} we denote by $\mathfrak{b}_{\mathbb{C}}$ the complex linear subspace of $\mathfrak{g}_{\mathbb{C}}$ spanned by \mathfrak{b} . For any root α of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{f}_{\mathbb{C}})$, we fix a root vector X_{α} corresponding to α . Introducing compatible orders in the space of real valued linear forms on $\sigma + \sqrt{-1}\mathfrak{t}$ and σ , we denote by P_+ the set of non-zero

positive roots α such that $\alpha|_{\sigma} \neq 0$, by Σ the set of restricted roots, by Σ^+ the set of restricted positive roots and by $\Sigma = \{\alpha_1, \dots, \alpha_l\}$ the set of restricted positive simple roots. Let ρ denote half the sum of the positive restricted roots with multiplicity, that is, $2\rho = (\sum_{\alpha \in P_+} \alpha)|_{\sigma_{\mathbb{C}}}$. For any root α in Σ , we denote by \mathfrak{g}^{α} the root space in \mathfrak{g} corresponding to α . We put $\mathfrak{n}^+ = \sum_{\alpha \in \Sigma^+} \mathfrak{g}^{\alpha}$ and $\mathfrak{n}^- = \theta(\mathfrak{n}^+)$, then $\mathfrak{n}^+ = \mathfrak{g} \cap \sum_{\alpha \in P_+} \mathbb{C}X_{\alpha}$ and $\mathfrak{n}^- = \sum_{\alpha \in \Sigma^-} \mathfrak{g}^{\alpha}$, where Σ^- denotes the set of negatives of the members in Σ^+ . Let K, A, N^+ and N^- denote the analytic subgroups of G corresponding to $\mathfrak{k}, \mathfrak{a}, \mathfrak{n}^+$ and \mathfrak{n}^- , respectively. Let M denote the centralizer of A in K , M^* the normalizer of A in K and W the factor group M^*/M , the (little) Weyl group. The Weyl group W acts as a group of linear transformations of \mathfrak{a} and also on $\mathfrak{a}_{\mathbb{C}}^*$ by $(w\lambda)(H) = \lambda(w^{-1}H)$ for $H \in \mathfrak{a}$, $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $w \in W$. For any element w in W , we fix its representative m_w in M^* . We put $\mathfrak{a}^+ = \{H \in \mathfrak{a} ; \alpha(H) > 0 \text{ for any } \alpha \text{ in } \Sigma^+\}$, which is called the positive Weyl chamber. Let $A^+ = \exp \mathfrak{a}^+$, $A' = \bigcup_{w \in W} \text{Ad}(m_w)A^+$ and $P = MAN$. Then A' is the totality of regular elements in A , P is a minimal parabolic subgroup of G and there exist the decompositions

$$(1.1) \quad G = \overline{KA^+}K \quad (\text{Cartan decomposition}),$$

$$(1.2) \quad G = KAN^+ \quad (\text{Iwasawa decomposition}),$$

$$(1.3) \quad G = \bigcup_{w \in W} P m_w P \quad (\text{Bruhat decomposition}).$$

Here $\overline{A^+}$ is the closure of A^+ in G and in (1.2) each $g \in G$ can be uniquely written

$$(1.4) \quad g = k(g) \exp H(g) n(g), \quad k(g) \in K, H(g) \in \mathfrak{a}, n(g) \in N^+.$$

Let $U(\mathfrak{g})$ denote the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$, which is naturally identified with $D(G)$, the totality of the left

G -invariant differential operators on G . The number l which equals $\dim \mathfrak{a}$ is called the real rank of G and the rank of the symmetric space G/K . Let $\mathbb{D}(G/K)$ denote the algebra of left G -invariant differential operators on G/K and put $\mathbb{D}(G)^K = \{D \in U(\mathfrak{g}); \text{Ad}(k)D = D \text{ for any } k \in K\}$. Then $\mathbb{D}(G/K)$ is a polynomial ring over \mathbb{C} with l algebraically independent generators and there exists a natural homomorphism of $\mathbb{D}(G)^K$ onto $\mathbb{D}(G/K)$.

For an element w in W , we define subalgebras \mathfrak{n}_w^+ , \mathfrak{u}_w^+ and \mathfrak{u}_w^- of \mathfrak{g} by

$$(1.5) \quad \begin{aligned} \mathfrak{n}_w^+ &= \mathfrak{n}^+ \cap \text{Ad}(m_w) \mathfrak{n}^+, & \mathfrak{u}_w^+ &= \mathfrak{n}^+ \cap \text{Ad}(m_w) \mathfrak{n}^- \\ \mathfrak{u}_w^- &= \text{Ad}(m_w^{-1}) \mathfrak{u}_w^+ = \mathfrak{n}^- \cap \text{Ad}(m_w^{-1}) \mathfrak{n}^+. \end{aligned}$$

We put $N_w^+ = \exp(\mathfrak{n}_w^+)$, $U_w^+ = \exp(\mathfrak{u}_w^+)$ and $U_w^- = \exp(\mathfrak{u}_w^-)$, then they are closed simply connected subgroups of G and

$$(1.6) \quad N^+ = N_w^+ U_w^+ = U_w^+ N_w^+, \quad N_w^+ \cap U_w^+ = \{1\}.$$

The Killing form defines a Euclidian inner product on \mathfrak{a}^* and $\alpha_i \in \bar{\Sigma}$ ($i = 1, \dots, l$) defines the reflection $w_{\alpha_i}: \lambda \mapsto \lambda - 2 \frac{\langle \lambda, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} \alpha_i$ on \mathfrak{a}^* . We can naturally identify W with the reflection group generated by $w_{\alpha_1}, \dots, w_{\alpha_l}$. Let $w = w_1 \cdots w_n$ is the minimal expression for $w \in W$ as a product of reflections with respect to the roots in $\bar{\Sigma}$, then the length $L(w)$ of w is said to be n . Let Θ be the subset of $\bar{\Sigma}$ and W_Θ be the subgroup of W generated by the reflections with respect to the elements in Θ . We note here that the number of the subsets of $\bar{\Sigma}$ equals 2^l . We put

$$(1.7) \quad \begin{aligned} \langle \Theta \rangle^+ &= \Sigma^+ \cap \sum_{\alpha_i \in \Theta} \mathbb{R} \alpha_i, \\ W(\Theta) &= \{w \in W; w^{-1} \langle \Theta \rangle^+ \subset \Sigma^+\}. \end{aligned}$$

Then every element w in W can be written in one and only one way in the form (cf. Proposition 1.1.2.13 in [16])

$$(1.8) \quad w = w_{\Theta} w(\Theta), \quad w_{\Theta} \in W_{\Theta}, w(\Theta) \in W(\Theta).$$

Let w^* denote the unique element in W such that $\text{Ad}(w^*) \pi^+ = \pi^-$.

Then $L(w^*) \geq L(w)$ for any $w \in W$ and $L(w^*) = L(w)$ means $w = w^*$.

Let w_{Θ}^* and $w^*(\Theta)$ denote the elements in W_{Θ} and $W(\Theta)$, respectively, such that $w^* = w_{\Theta}^* w^*(\Theta)$. Put $P_{\Theta} = \bigcup_{w \in W_{\Theta}} P m_w P$. Then

P_{Θ} constitute the parabolic subgroups containing P when Θ runs through the subsets of $\bar{\Psi}$. We define subalgebras $\sigma_{\Theta}, \sigma(\Theta),$

$\pi_{\Theta}^{\pm}, \pi^{\pm}(\Theta), \mathfrak{m}_{\Theta}$ and $\mathfrak{m}_{\Theta}(K)$ of \mathfrak{g} by

$$(1.9) \quad \begin{aligned} \sigma_{\Theta} &= \{ H \in \sigma ; \alpha(H) = 0 \text{ for every } \alpha \text{ in } \Theta \}, \\ \sigma(\Theta) &= \{ H \in \sigma ; \langle H, X \rangle = 0 \text{ for any } X \text{ in } \sigma_{\Theta} \}, \\ \pi_{\Theta}^+ &= \sum_{\lambda \in \Sigma^+ - \langle \Theta \rangle^+} \mathfrak{g}^{\lambda}, \quad \pi_{\Theta}^- = \theta(\pi_{\Theta}^+), \\ \pi^+(\Theta) &= \sum_{\lambda \in \langle \Theta \rangle^+} \mathfrak{g}^{\lambda}, \quad \pi^-(\Theta) = \theta(\pi^+(\Theta)), \\ \mathfrak{m}_{\Theta} &= \mathfrak{m} + \pi^+(\Theta) + \pi^-(\Theta) + \sigma(\Theta), \\ \mathfrak{m}_{\Theta}(K) &= \mathfrak{m}_{\Theta} \cap \mathfrak{k}. \end{aligned}$$

Let $A_{\Theta}, A(\Theta), N_{\Theta}^{\pm}, N^{\pm}(\Theta), M_{\Theta,0}$ and $M_{\Theta}(K)_0$ denote the connected analytic subgroups of G corresponding, respectively, to $\sigma_{\Theta},$

$\sigma(\Theta), \pi_{\Theta}^{\pm}, \pi^{\pm}(\Theta), \mathfrak{m}_{\Theta}$ and $\mathfrak{m}_{\Theta}(K)$. Then $A_{\Theta} N_{\Theta}^{\pm}$ is a closed solvable subgroups of G and we have the direct decomposition

$$(1.10) \quad A = A_{\Theta} A(\Theta)$$

and the semi-direct decomposition

$$(1.12) \quad N^{\pm} = N_{\Theta}^{\pm} N^{\pm}(\Theta).$$

We put $M_{\Theta} = M M_{\Theta,0}$ and $M_{\Theta}(K) = M M_{\Theta}(K)_0$, then the group $M_{\Theta} A_{\Theta}$ is the centralizer of σ_{Θ} in G , $M_{\Theta}(K) = K \cap M_{\Theta}$ and we have the decompositions (cf. §1.2.4 in [16])

$$(1.12) \quad M_{\Theta} = M_{\Theta}(K) A(\Theta) N^{\pm}(\Theta) \quad (\text{Iwasawa decomposition}),$$

$$(1.13) \quad P_{\Theta} = M_{\Theta} A_{\Theta} N_{\Theta}^+ \quad (\text{Langlands decomposition}),$$

$$(1.14) \quad P_{\Theta} = M_{\Theta}(K) A N^+,$$

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$$(1.15) \quad G = \bigcup_{w \in W(\mathbb{H})}^{-1} N^+ m_w P_{\mathbb{H}} \quad (\text{disjoint union}),$$

$$(1.16) \quad G = \bigcup_{w \in W(\mathbb{H})}^{-1} U_w^+ m_w P_{\mathbb{H}} \quad (\text{disjoint union}).$$

The decompositions (1.12), (1.13) and (1.14) give analytic diffeomorphisms of the product manifolds $M_{\mathbb{H}}(K) \times A(\mathbb{H}) \times N^{\pm}(\mathbb{H})$, $M_{\mathbb{H}} \times A_{\mathbb{H}} \times N_{\mathbb{H}}^+$, $M_{\mathbb{H}}(K) \times A \times N^+$ onto $M_{\mathbb{H}}$, $P_{\mathbb{H}}$ and $P_{\mathbb{H}}$, respectively, and if w is in $W_{\mathbb{H}}$, the map $(u, p) \mapsto u m_w p$ defines an analytic diffeomorphism of the product manifold $U_w^+ \times P_{\mathbb{H}}$ onto the submanifold $N^+ m_w P_{\mathbb{H}}$ of G . Here we note

$$(1.17) \quad \text{Ad}(m_w^{-1}) U_w^+ \subset N_{\mathbb{H}}^- = \text{Ad}(m_{w^*(\mathbb{H})}) U_{w^*(\mathbb{H})}^+^{-1} \quad \text{for } w \in W(\mathbb{H}).$$

Hence G is the union of the open submanifold $N_{\mathbb{H}}^- P_{\mathbb{H}}$ and submanifolds of lower dimensions.

§2. A realization of symmetric spaces in compact manifolds

In this section we will construct a compact manifold \tilde{X} such that G acts analytically on G and that the open G -orbits are isomorphic to symmetric spaces. To investigate all the G -orbits appeared in \tilde{X} , we prepare the following lemma.

Lemma 1. Put $P_{\mathbb{H}}(K) = M_{\mathbb{H}}(K) A_{\mathbb{H}} N_{\mathbb{H}}^+$. Then $P_{\mathbb{H}}(K)$ is a closed subgroup of G and there exist the decompositions

$$(2.1) \quad G = \bigcup_{w \in W(\mathbb{H})} {}^{-1} m_w U_w^- N^-(\mathbb{H}) A(\mathbb{H}) P_{\mathbb{H}}(K) \quad (\text{disjoint union}),$$

$$(2.2) \quad G = \bigcup_{w \in W} m_w N^- A(\mathbb{H}) P_{\mathbb{H}}(K).$$

If $w \in W(\mathbb{H})^{-1}$, the map $(u_w, n, a, p) \mapsto u_w n a p$ defines an analytic diffeomorphism of the product manifold $U_w^- \times N^-(\mathbb{H}) \times A(\mathbb{H}) \times P_{\mathbb{H}}(K)$ onto the submanifold $U_w^- N^-(\mathbb{H}) A(\mathbb{H}) P_{\mathbb{H}}(K)$ in G . And G is a union of the open dense submanifold $N^- A(\mathbb{H}) P_{\mathbb{H}}(K)$ and submanifolds of lower dimensions.

Proof. To show $P_{\mathbb{H}}(K)$ is a group we need only verify $ma = am$, $aN_{\mathbb{H}}^+ a^{-1} \subset N_{\mathbb{H}}^+$ and $\text{Ad}(m)N_{\mathbb{H}}^+ \subset N_{\mathbb{H}}^+$ for $m \in M_{\mathbb{H}}(K)$ and $a \in A_{\mathbb{H}}$. But they clearly follow from (1.9) and the definition of $M_{\mathbb{H}}(K)$. The groups $M_{\mathbb{H}}(K)$, $A_{\mathbb{H}}$ and $N_{\mathbb{H}}^+$ are closed in K , A and N^+ , respectively. Therefore $P_{\mathbb{H}}(K)$ is closed in G because of the Iwasawa decomposition (1.2). Next we note that $M_{\mathbb{H}} = N^-(\mathbb{H}) A(\mathbb{H}) M_{\mathbb{H}}(K)$ (cf. (1.12)). Then (1.5), (1.13) and (1.12) imply that in (1.16)

$$\begin{aligned} U_w^+ m_w P_{\mathbb{H}} &= m_w (N^- \cap m_w^{-1} N^+ m_w) N^-(\mathbb{H}) A(\mathbb{H}) M_{\mathbb{H}}(K) A_{\mathbb{H}} N_{\mathbb{H}}^+ \\ &= m_w U_w^- N^-(\mathbb{H}) A(\mathbb{H}) P_{\mathbb{H}}(K) \end{aligned}$$

for $m_w \in W(\mathbb{H})^{-1}$ and that

$$\begin{aligned} N_{\mathbb{H}}^- P_{\mathbb{H}} &= N_{\mathbb{H}}^- N^-(\mathbb{H}) A(\mathbb{H}) M_{\mathbb{H}}(K) A_{\mathbb{H}} N_{\mathbb{H}}^+ \\ &= N^- A(\mathbb{H}) P_{\mathbb{H}}(K). \end{aligned}$$

This proves the rest part of Lemma 1.

q.e.d.

Remark 2. Suppose $\Theta = \mathbb{F}$. Then $W_{\Theta} = W$, $W(\Theta) = \{1\}$, $M_{\Theta} = G$, $M_{\Theta}(K) = G \cap K = K$, $A_{\Theta} = \{1\}$, $A(\Theta) = A$, $N_{\Theta}^{\pm} = 1$, $N^{\pm}(\Theta) = N^{\pm}$, $P_{\Theta} = G$, $P_{\Theta}(K) = K$ and (2.1) is reduced to $G = N^{-}AK$ (Iwasawa decomposition). On the other hand, suppose $\Theta = \phi$. Then $W_{\Theta} = \{1\}$, $W(\Theta) = W$, $M_{\Theta} = M$, $M_{\Theta}(K) = M$, $A_{\Theta} = A$, $A(\Theta) = \{1\}$, $N_{\Theta}^{\pm} = N^{\pm}$, $N^{\pm}(\Theta) = \{1\}$, $P_{\Theta} = P_{\Theta}(K) = MAN^{+} = P$ and (2.2) equals $G = \bigcup_{w \in W} m_w N^{-}P$.

If C is a Lie group and \mathfrak{L} is its Lie algebra, we identify \mathfrak{L} with the totality of left invariant vector fields on C . Fix a basis $\{Y_1, \dots, Y_m\}$ of \mathfrak{L} . Then any real analytic vector field Y on C can be uniquely expressed as

$$Y = \sum_{i=1}^m c_i(p) Y_i$$

with real analytic functions $c_i(p)$ on C . This is clear because for any point p in C , $\{(Y_1)_p, \dots, (Y_m)_p\}$ is a basis of the tangent space $T_p C$ of C at p . Let $\{H_1, \dots, H_\ell\}$ be the dual basis of σ with respect to $\mathbb{F} = \{\alpha_1, \dots, \alpha_\ell\}$, that is, $\alpha_i(H_j) = \delta_{ij}$. For $\lambda \in \Sigma^{+}$, we fix a basis $\{X_{\lambda_i}; 1 \leq i \leq m(\lambda)\}$ of \mathfrak{g}^{λ} , where $m(\lambda) = \dim \mathfrak{g}^{\lambda}$ and put $X_{-\lambda_i} = -\theta(X_{\lambda_i})$.

Lemma 3. Let \tilde{X}_{Θ} be the homogeneous space $G/P_{\Theta}(K)$. Fix an element g in G and identify $N^{-} \times A(\Theta)$ with the open dense submanifold of \tilde{X}_{Θ} by the map $(n, a) \mapsto gnaP_{\Theta}(K)$ (cf. Lemma 1). For an element Y in \mathfrak{g} , let $Y|_{\tilde{X}_{\Theta}}$ be the vector field on \tilde{X}_{Θ} corresponding to the 1-parameter group which is defined by the action $\exp(tY)$ on \tilde{X}_{Θ} ($t \in \mathbb{R}$). Then at any point $p = (n, a)$ in

$N^- \times A(\mathbb{Q})$, the vector field is expressed as

$$(2.3) \quad \begin{aligned} (Y|\tilde{X}_{\mathbb{Q}})_p &= \sum_{\lambda \in \Sigma^+} \sum_{i=1}^{m(\lambda)} c_{-\lambda_i}(g,n) (X_{-\lambda_i})_p \\ &+ \sum_{\lambda \in \langle \mathbb{Q} \rangle^+} \sum_{i=1}^{m(\lambda)} c_{\lambda_i}(g,n) e^{-2\lambda \log a} (X_{-\lambda_i})_p \\ &+ \sum_{\alpha_i \in \mathbb{Q}} c_i(g,n) (H_i)_p \end{aligned}$$

by the identification $T_{nN^- \oplus T_a A(\mathbb{Q})} \simeq T_p(N^- \times A(\mathbb{Q})) \simeq T_{\text{gnaP}_{\mathbb{Q}}(K)} \tilde{X}_{\mathbb{Q}}$. Here the real analytic functions $c_{\pm \lambda_i}(g,n)$ and $c_i(g,n)$ are determined by the equation

$$(2.4) \quad \begin{aligned} \text{Ad}^{-1}(gn)Y &= \sum_{\lambda \in \Sigma^+} \sum_{i=1}^{m(\lambda)} (c_{\lambda_i}(g,n) X_{\lambda_i} + c_{-\lambda_i}(g,n) \\ &X_{-\lambda_i}) + \sum_{i=1}^l c_i(g,n) H_i + M(g,n), \quad M(g,n) \in \mathfrak{m}. \end{aligned}$$

Proof. Assume $|t|$ is sufficiently small. Then the direct sum decompositions $\mathfrak{g} = \mathfrak{n}^- + \mathfrak{a} + \mathfrak{n}^+ + \mathfrak{m} = \mathfrak{n}^- + \mathfrak{a}(\mathbb{Q}) + \mathfrak{m}_{\mathbb{Q}}(K) + \mathfrak{a}_{\mathbb{Q}} + \mathfrak{n}_{\mathbb{Q}}^+$ and the relation $[\mathfrak{a}, \mathfrak{n}^-] \subset \mathfrak{n}^-$ show that we can put

$$(2.5) \quad \begin{aligned} \exp(tY) gn &= gn \exp N_1^-(t) \exp A_1(t) \exp N_1^+(t) \exp M_1(t), \\ \exp N_1^+(t) a &= a \exp N_2^-(t) \exp A_2(t) \exp P_2(t), \\ \exp N_1^-(t) \exp A_1(t) a \exp N_2^-(t) a^{-1} &= \exp N_3^-(t) \exp A_1(t), \end{aligned}$$

where $N_1^-(t) \in \mathfrak{n}^-$ ($i=1,2,3$), $N_1^+(t) \in \mathfrak{n}^+$, $A_1(t) \in \mathfrak{a}$, $A_2(t) \in \mathfrak{a}(\mathbb{Q})$, $M_1(t) \in \mathfrak{m}$ and $P_2(t) \in \mathfrak{m}_{\mathbb{Q}}(K) + \mathfrak{a}_{\mathbb{Q}} + \mathfrak{n}_{\mathbb{Q}}^+$. Hence we have

$$(2.6) \quad \exp(tY) \text{gnaP}_{\mathbb{Q}}(K) = gn \exp N_3^-(t) a \exp(A_1(t) + A_2(t)) P_{\mathbb{Q}}(K).$$

Put $(\partial N_1^-(t)/\partial t)(0) = N_1^-$ ($i=1,2,3$), $(\partial N_1^+(t)/\partial t)(0) = N_1^+$ and $(\partial A_j(t)/\partial t)(0) = A_j$ ($j=1,2$). Then (2.5) shows that

$$(2.7) \quad \begin{aligned} \text{Ad}^{-1}(gn)Y &\equiv N_1^- + A_1 + N_1^+ \pmod{\mathfrak{m}}, \\ \text{Ad}^{-1}(a)N_1^+ &\equiv N_2^- + A_2 \pmod{\mathfrak{m}_{\mathbb{Q}}(K) + \mathfrak{a}_{\mathbb{Q}} + \mathfrak{n}_{\mathbb{Q}}^+}, \\ N_1^- + \text{Ad}(a)N_2^- &= N_3^-. \end{aligned}$$

If $\lambda \in \langle \mathbb{Q} \rangle^+$, we have

$$\begin{aligned} \text{Ad}^{-1}(a)X_{\lambda_i} &= e^{-\lambda \log a} X_{\lambda_i} \\ &= e^{-\lambda \log a} (X_{\lambda_i} - X_{-\lambda_i}) + e^{-\lambda \log a} X_{-\lambda_i} \end{aligned}$$

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$$\equiv e^{-2\lambda \log a} \text{Ad}^{-1}(a)X_{-\lambda_i} \pmod{\mathfrak{m}_{\mathbb{H}}(\mathbb{K})}.$$

On the other hand, if $\lambda \in \Sigma^+ - \langle \mathbb{H} \rangle^+$, we have

$$\text{Ad}^{-1}(a)X_{\lambda_i} = e^{-\lambda \log a} X_{\lambda_i} \in \mathfrak{n}_{\mathbb{H}}^+.$$

Then $A_2 = 0$ and

$$\begin{aligned} N_1^- + \text{Ad}^{-1}(a)N_2^- &= \sum_{\lambda \in \Sigma^+} \sum_{i=1}^m c_{-\lambda_i}(g,n)X_{-\lambda_i} \\ &\quad + \sum_{\lambda \in \langle \mathbb{H} \rangle^+} \sum_{i=1}^m c_{\lambda_i}(g,n)e^{-2\lambda \log a} X_{-\lambda_i}, \end{aligned}$$

$$A_1 + A_2 \equiv \sum_{\alpha_i \in \mathbb{H}} c_i(g,n)H_i \pmod{\mathfrak{a}_{\mathbb{H}}}.$$

Thus we obtain (2.3) by (2.6) and (2.7).

q.e.d.

Let \hat{X} be the product manifold $G \times N^- \times \mathbb{R}$ and let $\hat{x} = (g, n, t)$ be a point in \hat{X} ($g \in G$, $n \in N^-$, $t = (t_1, \dots, t_l) \in \mathbb{R}^l$). Then G acts on \hat{X} by the correspondence $(g', (g, n, t)) \mapsto (g'g, n, t)$ for $g' \in G$. Put $\text{sgn } \hat{x} = (\text{sgn } t_1, \dots, \text{sgn } t_l) \in \{-1, 0, 1\}^l$, $\mathbb{H}_{\hat{x}} = \{\alpha_i \in \mathbb{F}; t_i \neq 0\}$ and $a(\hat{x}) = \exp(-\sum_{t_i \neq 0} H_i \log |t_i|) \in A(\mathbb{H}_{\hat{x}})$, where $\text{sgn } s = s/|s|$ for $s \in \mathbb{R} - \{0\}$ and $\text{sgn } 0 = 0$. We will define an equivalence relation for points in \hat{X} .

Definition 4. Two points $\hat{x} = (g, n, t)$ and $\hat{x}' = (g', n', t')$ in \hat{X} are equivalent, which will be denoted by $\hat{x} \sim \hat{x}'$, if and only if the following two conditions hold.

$$(2.8) \quad \text{sgn } \hat{x} = \text{sgn } \hat{x}'.$$

$$(2.9) \quad gna(\hat{x})P_{\mathbb{H}_{\hat{x}}}(\mathbb{K}) = g'n'a(\hat{x}')P_{\mathbb{H}_{\hat{x}'}}(\mathbb{K}) \text{ in } \tilde{X}_{\mathbb{H}_{\hat{x}}}.$$

Then we denote by \tilde{X} the quotient space of \hat{X} with the quotient topology defined by the equivalence relation.

Since the action of G on \hat{X} is compatible with the equivalence

lence relation, G also acts on \tilde{X} . Let π be the natural projection of \hat{X} onto \tilde{X} . Put $\tilde{U}_g = \pi(\{g\} \times N^- \times \mathbb{R}^l)$ for $g \in G$. Then the map $(n, t) \mapsto ((g, n, t))$ defines the bijection (cf. Lemma 1)

$$(2.10) \quad \gamma_g : N^- \times \mathbb{R}^l \xrightarrow{\sim} \tilde{U}_g.$$

Theorem 5. The quotient space \tilde{X} has the following properties.

i) \tilde{X} is a simply connected, compact, real analytic manifold without boundary.

$$\text{ii) } \quad \tilde{X} = \bigcup_{w \in W} \tilde{U}_{m_w} = \bigcup_{g \in G} \tilde{U}_g.$$

Here \tilde{U}_g is an open submanifold of \tilde{X} with the topology such that the map (2.10) is a real analytic diffeomorphism. Moreover

$\tilde{X} - \tilde{U}_g$ is a union of a finite number of submanifolds of \tilde{X} whose codimensions in \tilde{X} are not lower than 2.

iii) The action of G on \tilde{X} is real analytic and for a point \hat{x} in \hat{X} , the G -orbit of $\pi(\hat{x})$ is isomorphic to the homogeneous space $G/P_{\mathbb{Q}(\hat{x})}(K)$ and for points \hat{x} and \hat{x}' in \hat{X} , the G -orbits of $\pi(\hat{x})$ and $\pi(\hat{x}')$ are coincide if and only if $\text{sgn } \hat{x} = \text{sgn } \hat{x}'$. Hence the orbital decomposition of \tilde{X} with respect to the action of G is of the form

$$\tilde{X} \simeq \bigcup_{\mathbb{Q} \subset \mathbb{F}} 2^{\#\mathbb{Q}} (G/P_{\mathbb{Q}}(K)) \quad (\text{disjoint union}),$$

where $\#\mathbb{Q}$ is the number of the elements of \mathbb{Q} and $2^{\#\mathbb{Q}} (G/P_{\mathbb{Q}}(K))$ is the disjoint union of $2^{\#\mathbb{Q}}$ copies of $G/P_{\mathbb{Q}}(K)$.

iv) Identify the open G -orbit $\pi(\{\hat{x} \in \hat{X}; \text{sgn } \hat{x} = (1, \dots, 1)\})$ with the Riemannian symmetric space G/K and the orbit of the lowest dimension $\pi(\{\hat{x} \in \hat{X}; \text{sgn } \hat{x} = 0\})$ with its Martin boundary G/P .

Let $\mathcal{D}(\tilde{X})$ be the totality of G -invariant differential operators on \tilde{X} whose coefficients are real analytic functions. Then the

natural restriction

$$D(\tilde{X}) \simeq D(G/K)$$

is bijective. For any homomorphism χ of $D(\tilde{X})$ to \mathbb{C} as algebras, the system of differential equations on \tilde{X}

$$\mathcal{M}_\chi : (D - \chi(D))u = 0 \quad \text{for } D \in D(\tilde{X})$$

has regular singularity along the set of walls $\tilde{X}_i = \pi(\{(g, n, t) \in \hat{X}; t_i = 0\})$ with the edge G/P in the sense of Definition 5.1 in [9].

Remark 6. Since

$$(2.11) \quad \dim \tilde{X} - \dim G/P_\Theta(K) = l - \#\Theta,$$

the open G -orbits in \tilde{X} are isomorphic to G/K and the number of them equals 2^l and that of all the G -orbits equals 3^l . The decomposition of \tilde{X} into G -orbits is of "normal crossing type" in the following sense:

For every point in \tilde{X} , there exists a local coordinate system $(x_1, \dots, x_k, y_1, \dots, y_l)$ on a neighbourhood of the point such that if $\text{sgn } y_j = \text{sgn } y'_j$ for $j=1, \dots, l$, two points $(x_1, \dots, x_k, y_1, \dots, y_l)$ and $(x'_1, \dots, x'_k, y'_1, \dots, y'_l)$ belong to the same G -orbit.

For example, put $G = SL(2, \mathbb{R})$, $N^- = \left\{ \begin{pmatrix} 1 & \\ x & 1 \end{pmatrix}; x \in \mathbb{R} \right\}$, $A = \left\{ \begin{pmatrix} 1/\sqrt{t} & \\ & \sqrt{t} \end{pmatrix}; t \in \mathbb{R}_+ \right\}$ and $z = x + \sqrt{-1}t$. Then we can easily show that \tilde{X} is isomorphic to the 1-dimensional complex projective space $\mathbb{P}_{\mathbb{C}}^1 = \mathbb{C} \cup \{\infty\}$ with the action of G

$$G \times \mathbb{P}_{\mathbb{C}}^1 \ni \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z \right) \mapsto \frac{c + dz}{a + bz} \in \mathbb{P}_{\mathbb{C}}^1$$

and that

$$\tilde{X} = U_1 \cup U_{m_w^*} \simeq \underset{z \mapsto -1/z}{\mathbb{C}} \cup \mathbb{C} \simeq \mathbb{P}_{\mathbb{C}}^1.$$

For the first step to prove Theorem 5, we prepare

Lemma 7. The map

$$(2.12) \quad \varphi_{g'}^{-1} \circ \varphi_g : \varphi_g^{-1}(\tilde{U}_g \cap \tilde{U}_{g'}) \rightarrow \varphi_{g'}^{-1}(\tilde{U}_g \cap \tilde{U}_{g'})$$

is an analytic diffeomorphism between the open subsets of $N^- \times \mathbb{R}^l$.

Proof. Let Y be an element of \mathcal{Y} . By the identification

$$\begin{array}{ccccccc} G/K & \simeq & N^- \times A & \xrightarrow{\sim} & N^- \times \mathbb{R}_+^l & \hookrightarrow & N^- \times \mathbb{R}^l \xleftarrow{\varphi_g^{-1}} \tilde{U}_g \\ \downarrow & & \downarrow & & \downarrow & & \\ \text{gnaK} & \leftarrow & (n, a) & \mapsto & (n, e^{-\alpha_1 \log a}, \dots, e^{-\alpha_l \log a}) & = & (n, t) \end{array}$$

the vector field $Y|N^- \times \mathbb{R}_+^l$ corresponding to the 1-parameter group defined by the action $\exp(sY)$ on G/K for $s \in \mathbb{R}$ is expressed as

$$(2.13) \quad \begin{aligned} Y|N^- \times \mathbb{R}_+^l &= \sum_{\lambda \in \Sigma^+} \sum_{i=1}^m c_{\lambda_i}(g, n) t_i^{2\lambda} + c_{-\lambda_i}(g, n) X_{-\lambda_i} \\ &\quad - \sum_{i=1}^l c_i(g, n) t_i \partial / \partial t_i. \end{aligned}$$

Here we denote by $t^{2\lambda}$ the function $t_1^{2\lambda(H_1)} \dots t_l^{2\lambda(H_l)}$ and the functions $c_{\pm\lambda_i}(g, n)$ and $c_i(g, n)$ are those which are determined by (2.4) (see Lemma 3). Since $\lambda(H_i)$ are non-negative integers for $\lambda \in \Sigma^+$, the vector field $Y|N^- \times \mathbb{R}_+^l$ is analytically extended to a vector field $Y|N^- \times \mathbb{R}^l$ on $N^- \times \mathbb{R}^l$.

For every point $\hat{x} = (g, \hat{n}, \hat{t})$ in \hat{X} , put $B_{\hat{x}} = \{(t_1, \dots, t_l) \in \mathbb{R}^l; \text{sgn } t_i = \text{sgn } \hat{t}_i \text{ for } 1 \leq i \leq l\}$ and define the identification

$$\begin{array}{ccccccc} G/P_{\hat{x}} & \simeq & N^- \times A(\hat{x}) & \xrightarrow{\sim} & N^- \times B_{\hat{x}} & \hookrightarrow & N^- \times \mathbb{R}^l \xleftarrow{\varphi_g^{-1}} \tilde{U}_g \\ \downarrow & & \downarrow & & \downarrow & & \\ \text{gnaP}_{\hat{x}} & \leftarrow & (n, a) & \mapsto & (n, \text{sgn } \hat{t}_1 e^{-\alpha_1 \log a}, \dots, \text{sgn } \hat{t}_l e^{-\alpha_l \log a}) & & \end{array}$$

Since (2.13) shows $(Y|N^- \times \mathbb{R}^l)_q \in T_q(N^- \times B_{\hat{x}})$ for $q \in N^- \times B_{\hat{x}}$, we can restrict the vector field $Y|N^- \times \mathbb{R}^l$ on $N^- \times B_{\hat{x}}$. Then, using the above identification and comparing (2.3) and (2.13), we see that its restriction on $N^- \times B_{\hat{x}}$ is the same one defined by (2.3).

Hence by Definition 4 we have the following claim:

Suppose $\hat{x} = (g, \hat{n}, \hat{t})$ in \hat{X} and Y in \mathcal{Y} satisfying that $g\hat{n}a(\hat{x}) \in \exp(sY)gN^{-A}(\mathbb{H}_{\hat{x}})P_{\mathbb{H}_{\hat{x}}}(K)$ for $0 \leq s \leq 1$. Then there exists an open subset V of $N^{-} \times \mathbb{R}^l$ containing (\hat{n}, \hat{t}) such that $\varphi_{(\exp Y)g}^{-1} \circ \varphi_g$ defines an analytic diffeomorphism of V to an open subset of $N^{-} \times \mathbb{R}^l$.

For any $\hat{x} = (g, \hat{n}, \hat{t})$, there exist $Y_1, \dots, Y_k \in \mathfrak{n}^{-} + \sigma(\mathbb{H}_{\hat{x}})$ such that $\hat{n}a(\hat{x}) = \exp Y_k \exp Y_{k-1} \dots \exp Y_1$. Put $y(s) = \exp\{(s - [s]) \text{Ad}(g)Y_{[s]+1} \dots \exp(\text{Ad}(g)Y_{[s]}) \dots \exp(\text{Ad}(g)Y_1)\}$ for $0 \leq s \leq k$, where $[s]$ is the largest integer satisfying $[s] \leq s$. Then $y(s)gN^{-A}(\mathbb{H}_{\hat{x}})P_{\mathbb{H}_{\hat{x}}}(K) = gN^{-A}(\mathbb{H}_{\hat{x}})P_{\mathbb{H}_{\hat{x}}}(K)$ and $y(k)g = g\hat{n}a(\hat{x})$. Applying the above claim to $y(s)$ in place of $\exp(sY)$, we see that $\varphi_{g\hat{n}a(\hat{x})}^{-1} \circ \varphi_g$, which equals $(\varphi_{y(k)g}^{-1} \circ \varphi_{y(k-1)g}) \circ \dots \circ (\varphi_{y(2)g}^{-1} \circ \varphi_{y(1)g}) \circ (\varphi_{y(1)g}^{-1} \circ \varphi_g)$, defines an analytic diffeomorphism of a suitable neighbourhood of (\hat{n}, \hat{t}) to a neighbourhood of $(1, \text{sgn } \hat{x})$.

Let \tilde{q} be an arbitrary point in $\tilde{U}_g \cap \tilde{U}_g$. Then there exist $\hat{x} = (g, \hat{n}, \hat{t})$ and $\hat{x}' = (g', \hat{n}', \hat{t}')$ satisfying $\pi(\hat{x}) = \pi(\hat{x}') = \tilde{q}$. We denote by $P_{\mathbb{H}_{\hat{x}}}(K)_0$ the connected component of $P_{\mathbb{H}_{\hat{x}}}(K)$ containing 1. Then $P_{\mathbb{H}_{\hat{x}}}(K) = P_{\mathbb{H}_{\hat{x}}}(K)_0 M$. Since $g\hat{n}a(\hat{x})P_{\mathbb{H}_{\hat{x}}}(K) = g'\hat{n}'a(\hat{x}')P_{\mathbb{H}_{\hat{x}'}}(K)$ we have $(g\hat{n}a(\hat{x}))^{-1}g'\hat{n}'a(\hat{x}') = \hat{p}\hat{m}$ with $\hat{m} \in M$ and $\hat{p} \in P_{\mathbb{H}_{\hat{x}}}(K)_0$.

Since we can choose Y'_1, \dots, Y'_k in $\mathfrak{n}_{\mathbb{H}}(K) + \sigma_{\mathbb{H}} + \mathfrak{n}_{\mathbb{H}}^{+}$ so that $\hat{p} = \exp Y'_k \exp Y'_{k-1} \dots \exp Y'_1$, we see by the same argument as in the case of $\varphi_{g\hat{n}a(\hat{x})}^{-1} \circ \varphi_g$ that $\varphi_{g\hat{n}a(\hat{x})\hat{p}}^{-1} \circ \varphi_{g\hat{n}a(\hat{x})}$ defines an analytic diffeomorphism between suitable neighbourhoods of $(1, \text{sgn } \hat{x})$.

Moreover, since $\varphi_{g\hat{n}a(\hat{x})\hat{p}\hat{m}}^{-1} \circ \varphi_{g\hat{n}a(\hat{x})\hat{p}}((n, t)) = (\hat{m}^{-1}n\hat{m}, t)$,

$\varphi_{g'\hat{n}'a(\hat{x}')}^{-1} \circ \varphi_{g\hat{n}a(\hat{x})\hat{p}}$ is an analytic diffeomorphism of $N^{-} \times \mathbb{R}^l$.

Thus we have proved that $\varphi_g^{-1} \circ \varphi_{g\hat{n}a(\hat{x})}, \varphi_{g'}^{-1} \circ \varphi_{g'\hat{n}'a(\hat{x}')}^{-1}$,

$\varphi_{g\hat{n}a(\hat{x})\hat{p}}^{-1} \circ \varphi_{g\hat{n}a(\hat{x})}$ and $\varphi_{g'\hat{n}'a(\hat{x}')^{-1} \circ \varphi_{gna(\hat{x})\hat{p}}$ define analytic diffeomorphisms of suitable open neighbourhoods of $(1, \text{sgn } \hat{x})$ to open subsets of $N^- \times \mathbb{R}^l$. Combining these maps and their inverse, we see that $\varphi_{g'}^{-1} \circ \varphi_g$ defines an analytic diffeomorphism of an open set containing (\hat{n}, \hat{t}) to an open set containing (\hat{n}', \hat{t}') , which implies $\varphi_g^{-1}(U_g \cap U_{g'})$ and $\varphi_{g'}^{-1}(U_g \cap U_{g'})$ are open in $N^- \times \mathbb{R}^l$ and that the map (2.12) is an analytic local diffeomorphism. But the map is bijective, so we have the claim of Lemma 7. q.e.d.

Proof of Theorem 5. First we remark that the proof of

Lemma 7 shows that

$$(2.14) \quad \psi_g: \pi^{-1}(\tilde{U}_g) \ni (g', n', t') \mapsto \varphi_{g'}^{-1} \circ \varphi_g, ((n', t')) \in N^- \times \mathbb{R}^l$$

defines a real analytic map of the open subset $\pi^{-1}(\tilde{U}_g)$ of \hat{X} to $N^- \times \mathbb{R}^l$. Therefore for any open subset V of $N^- \times \mathbb{R}^l$, $\pi^{-1} \circ \varphi_g(V) = \psi_g^{-1}(V)$ is open in \hat{X} . On the other hand, for any open subset \hat{V} of \hat{X} , $\varphi_g^{-1} \circ \pi(\hat{V})$ is clearly open in $N^- \times \mathbb{R}^l$. Hence the map (2.10) is a homeomorphism.

For points x and x' in \tilde{X} , there exists g in G such that \tilde{U}_g contains x and x' because Lemma 1 shows that $\{g \in G; \tilde{U}_g \ni x\}$ and $\{g \in G; \tilde{U}_g \ni x'\}$ are open dense in G . Since φ_g is homeomorphic and $N^- \times \mathbb{R}^l$ is Hausdorff, \tilde{X} is also Hausdorff.

Thus we see that X is a connected real analytic manifold. The claims ii) and iii) are clear from what we have proved. The claim concerning $\tilde{X} - \tilde{U}_g$ immediately follows from Iwasawa decomposition (1.2) and Lemma 1.

Whitney's transversality theorem says that for any submanifold \tilde{Y} of \tilde{X} satisfying $\text{codim}_{\tilde{X}} \tilde{Y} \geq 2$ and for any differentiable map $\gamma: S^1 (= \text{the unit circle}) \rightarrow \tilde{X}$, there exists a differentiable map $\gamma': S^1 \rightarrow \tilde{X} - \tilde{Y} \subset \tilde{X}$ such that γ is homotopic to γ' .

Therefore the fundamental group of \tilde{X} equals that of \tilde{U}_g . Since the fundamental group of \tilde{U}_g is trivial, \tilde{X} is simply connected.

Consider the compact subset $B = K \times \{1\} \times [-1, 1]^l$ of \hat{X} . Then $\pi(B)$ is also compact because it is the image of a compact set under the continuous map. Since $\{\exp(-\sum_{j=1}^l H_j \log t_j); 0 < t_j \leq 1 \text{ for } 1 \leq j \leq l\}$ equals \overline{A}_+ , Cartan decomposition (1.1) shows that $\pi(B)$ contains all the open G -orbits of \tilde{X} . Therefore the compact set $\pi(B)$ is open dense in \tilde{X} , which implies $\pi(B) = \tilde{X}$ and that \tilde{X} is compact.

To prove the claim in iv), we prepare the following:

Lemma 8. Let Y be an element of the Lie algebra $\mathfrak{o} + \mathfrak{n}^-$.

Then by the identification

$$\begin{array}{ccc} N^-A & \xleftarrow{\sim} & N^- \times \mathbb{R}_+^l \hookrightarrow N^- \times \mathbb{R}^l, \\ \downarrow & & \downarrow \\ na = n \exp(-\sum_{j=1}^l H_j \log t_j) & \longleftarrow & (n, t) \end{array}$$

the left invariant vector field $Y|_{N^- \times \mathbb{R}_+^l}$ on the Lie group N^-A corresponding to Y is expressed as

$$Y|_{N^- \times \mathbb{R}_+^l} = \sum_{\lambda \in \Sigma^+} \sum_{i=1}^{m(\lambda)} c_{-\lambda_i} t^\lambda X_{-\lambda_i} - \sum_{j=1}^l c_j t_j \partial / \partial t_j,$$

where

$$Y = \sum_{\lambda \in \Sigma^+} \sum_{i=1}^{m(\lambda)} c_{-\lambda_i} X_{-\lambda_i} + \sum_{j=1}^l c_j H_j.$$

Therefore $Y|_{N^- \times \mathbb{R}_+^l}$ can be analytically extended to a vector field on $N^- \times \mathbb{R}^l$.

Proof. For $a = \exp(-\sum_{j=1}^l H_j \log t_j)$, we have

$$\text{Ad}(a)X_{-\lambda_i} = e^{-\lambda \log a} X_{-\lambda_i} = t^\lambda X_{-\lambda_i},$$

which proves the claim (cf. the proof of Lemma 7). q.e.d.

Now we will prove iv). For a Lie subalgebra \mathfrak{h} of \mathfrak{g} , we denote by $U(\mathfrak{h})$ the universal enveloping algebra of $\mathfrak{h}_\mathbb{C}$ and naturally identify $U(\mathfrak{h})$ with a subalgebra of $U(\mathfrak{g})$. Let

$\mu : \mathbb{D}(G)^K \rightarrow \mathbb{D}(G/K)$ be the natural surjective map with the kernel $\mathbb{D}(G)^K \cap U(\mathfrak{g})\mathbb{k}$. Then for $D \in \mathbb{D}(G)^K$, there exists a unique element $D' \in U(\sigma + \pi^-)$ such that $D' \equiv D \pmod{U(\mathfrak{g})\mathbb{k}}$ because of the Iwasawa decomposition $\mathfrak{g} = \mathbb{k} + \sigma + \pi^-$. Since $D' - D \in U(\mathfrak{g})\mathbb{k}$, Lemma 8 proves that $\mu(D)$ can be analytically extended to a differential operator on \tilde{U}_g for every $g \in G$. Therefore we have the analytic extension \tilde{D} of $\mu(D)$ on \tilde{X} because \tilde{X} is simply connected. Let τ_g be the transformation on \tilde{X} corresponding to the action of $g \in G$. Since $\tau_g^* \tilde{D} - \tilde{D}$ vanishes on the open subset G/K of \tilde{X} , we have $\tau_g^* \tilde{D} = \tilde{D}$ on \tilde{X} , which shows $\tilde{D} \in \mathbb{D}(\tilde{X})$. Hence the map $\mathbb{D}(\tilde{X}) \rightarrow \mathbb{D}(G/K)$ is surjective and the injectivity of the map is clear because G/K is open in \tilde{X} .

Now we remember the concept of regular singularity in [9] and the structure of $\mathbb{D}(G/K)$ (cf. Chapter X in [6]). Let $(x_1, \dots, x_n, t_1, \dots, t_l)$ be a local coordinate system of \tilde{X} such that \tilde{X}_j is defined by $t_j = 0$ for every $j = 1, \dots, l$. Put $\mathcal{V}_j = t_j \partial / \partial t_j$, $\mathcal{V} = (\mathcal{V}_1, \dots, \mathcal{V}_l)$ and $tD_x = (t_1 \partial / \partial x_1, t_1 \partial / \partial x_2, \dots, t_l \partial / \partial x_n)$. Let P_j be differential operators of order r_j ($j = 1, \dots, l$) on \tilde{X} whose coefficients are real analytic functions. Then the system of differential equations

$$\mathcal{M} : P_j u = 0 \quad \text{for } j = 1, \dots, l$$

is said to have regular singularity along the set of walls $\{\tilde{X}_1, \dots, \tilde{X}_l\}$ if the following conditions hold:

[RS-0] There are differential operators $Q_{j,k}^i$ of order $< r_j + r_k - r_i$ such that

$$[P_j, P_k] = \sum_{i=1}^l Q_{j,k}^i P_i \quad \text{for } j, k = 1, \dots, l.$$

[RS-1] For any j , P_j is of the form

$$P_j = P_j(t, x, \mathcal{V}, tD_x).$$

[RS-2] Put $a_j(x, s) = P_j(0, x, s, 0)$ and let $\hat{a}_j(x, t)$

be its homogeneous part of degree r_j with respect to s .

Then the solution of the system of equations

$$\dot{a}_1(x, s) = \dots = \dot{a}_l(x, s) = 0$$

is only the origin $s = 0 \in \mathbb{C}^l$ for any x .

For $D \in \mathbb{D}(G)^K$, let D'_σ be a unique element of $U(\sigma)$ defined by the equation

$$(2.15) \quad D - D'_\sigma \in \mathfrak{n}^- U(\mathfrak{g}) + U(\mathfrak{g}) \mathfrak{k}$$

and put

$$(2.16) \quad D_\sigma = e^\rho \cdot D'_\sigma \cdot e^{-\rho}$$

where e^ρ is the function on A defined by $e^\rho(a) = e^{\rho \log a}$ for $a \in A$.

Then denoting by $U(\sigma)^W$ the subalgebra $\{D \in U(\sigma) ; \text{Ad}(m_w)D = D \text{ for } w \in W\}$ of $U(\sigma)$, the map

$$\begin{array}{ccc} \tilde{\gamma} : \mathbb{D}(G)^K & \longrightarrow & U(\sigma) \\ \downarrow & & \downarrow \\ D & \longmapsto & D_\sigma \end{array}$$

defines a surjective homomorphism of $\mathbb{D}(G)^K$ onto $U(\sigma)^W$ with the kernel $\mathbb{D}(G)^K \cap U(\mathfrak{g})\mathfrak{k}$. Therefore it induces the isomorphism

$$(2.17) \quad \gamma : \mathbb{D}(G/K) \simeq \mathbb{D}(G)^K / \mathbb{D}(G)^K \cap U(\mathfrak{g})\mathfrak{k} \xrightarrow{\sim} U(\sigma)^W.$$

Here the order of $\gamma(D)$ equals that of D for $D \in \mathbb{D}(G/K)$ and $U(\sigma)^W$ is known to be a polynomial ring over \mathbb{C} with l algebraically independent homogeneous elements $p_1(H_1, \dots, H_l), \dots, p_l(H_1, \dots, H_l)$.

Now we will verify the conditions [RS-0], [RS-1] and [RS-2] for the system \mathcal{M}_χ , which is expressed as

$$\mathcal{M}_\chi : (D_j - \chi(D_j))u = 0 \quad \text{for } j=1, \dots, l,$$

where $D_j = \gamma^{-1}(p_j)$. Since $\mathbb{D}(G/K)$ is a commutative ring, [RS-0] is clear. Moreover Lemma 8 shows [RS-1] and that in [RS-2]

$$(2.18) \quad a_j(x, s_1, \dots, s_l) = p_j(\rho(H_1) - s_1, \dots, \rho(H_l) - s_l) - \chi(D_j).$$

Therefore the system of equations $\dot{a}_j(x, s) \equiv p_j(-s) = 0$ for $j=1, \dots, l$ implies $s = 0$.

Thus we complete the proof of Theorem 5.

The following proposition will be used in a subsequent paper.

Proposition 9. We denote by τ_{\oplus} the involutive automorphism of \tilde{X} induced by the map of $\hat{X} : (g, n, t) \mapsto (g, n, s)$, where $s_i = t_i$ if $\alpha_i \notin \oplus$ and $s_j = -t_j$ if $\alpha_j \in \oplus$. Then τ_{\oplus} and the action of G are commutative mutually and $\tau_{\oplus}^* D = D$ for any $D \in \mathbb{D}(\tilde{X})$.

Proof. The commutativity is clear seeing Definition 4.

For $\tilde{D} \in U(\mathfrak{g})$, we denote by \tilde{D}' the unique element in $U(\mathfrak{n} + \mathfrak{n}^-)$ satisfying $\tilde{D} - \tilde{D}' \in U(\mathfrak{g})\mathfrak{k}$. This correspondence induces the identification

$$\mathbb{D}(\tilde{X}) \simeq \mathbb{D}(G/K) \hookrightarrow U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{k} \simeq U(\mathfrak{n} + \mathfrak{n}^-).$$

Consider in the open submanifold \tilde{U}_1 of \tilde{X} . The totality of left N^-A -invariant differential operators on \tilde{U}_1 is naturally identified with $U(\mathfrak{n} + \mathfrak{n}^-)$ (cf. Lemma 8). Since $\tau_{\oplus}(\tilde{U}_1) = \tilde{U}_1$, τ_{\oplus} induces an involutive automorphism τ_{\oplus}^* of $U(\mathfrak{n} + \mathfrak{n}^-)$, which satisfies

$$(2.19) \quad \begin{cases} \tau_{\oplus}^*(H_j) = H_j, \\ \tau_{\oplus}^*(X_{-\lambda_1}) = (-1)^{\sum_{\alpha_j \in \oplus} \lambda(H_j)} X_{-\lambda_1}. \end{cases}$$

Using the identifications, D in $\mathbb{D}(\tilde{X})$ can be expressed as

$$D = (D - D'_{\mathfrak{n}}) + D'_{\mathfrak{n}},$$

where $D - D'_{\mathfrak{n}} \in \mathfrak{n}^- U(\mathfrak{n} + \mathfrak{n}^-)$ and $e^{\mathfrak{p}} \cdot D'_{\mathfrak{n}} \cdot e^{-\mathfrak{p}} \in U(\mathfrak{a})^W$. Since $\tau_{\oplus}^* D'_{\mathfrak{n}} = D'_{\mathfrak{n}}$ and $\tau_{\oplus}^* D \in \mathbb{D}(\tilde{X})$, we have $\tau_{\oplus}^* D - D \in \mathfrak{n}^- U(\mathfrak{n} + \mathfrak{n}^-) \cap \mathbb{D}(G/K)$.

Therefore the isomorphism (2.17) proves $\tau_{\oplus}^* D - D = 0$. q.e.d.

Put $\hat{X}_0 = G \times \mathbb{R}^l$ and identify \hat{X}_0 with the closed submanifold $G \times \{1\} \times \mathbb{R}^l$ of \hat{X} . Then \hat{X}_0 has the analytic action of G and the equivalence relation \sim induced by those on \hat{X} . We remark here that the analytic map $\pi|_{X_0} : \hat{X}_0 \rightarrow \tilde{X}$, which will be denoted by

π_0 , induces a homeomorphism of the quotient space \hat{X}_0/\sim with the quotient topology onto \tilde{X} because the map $r: \hat{X} \ni (g, n, t) \mapsto (gn, t) \in \hat{X}_0$ satisfies $r(\hat{x}) = \hat{x}$ for $\hat{x} \in \hat{X}$ (cf. Bourbaki [3]). Let $\hat{x} = (g, t)$ be a point of \hat{X}_0 . Then by the natural identifications $T_{\hat{x}}\hat{X}_0 \simeq \mathfrak{g} + T_t\mathbb{R}^l$ and $T_{\pi(\hat{x})}\tilde{U}_g \simeq \mathfrak{n}^- + T_t\mathbb{R}^l$, the differential $(d\pi_0)_{\hat{x}}$ is expressed as

$$(2.20) \quad \begin{aligned} (d\pi_0)_{\hat{x}}(\partial/\partial t_j) &= \partial/\partial t_j, \quad j=1, \dots, l, \\ (d\pi_0)_{\hat{x}}(Y) &= \sum_{\lambda \in \Sigma^+} \sum_{i=1}^{m(\lambda)} (c_{\lambda_i}(g)t^{2\lambda_i} + c_{-\lambda_i}(g))X_{-\lambda_i} \\ &\quad - \sum_{j=1}^l c_j(g)t_j \partial/\partial t_j, \quad Y \in \mathfrak{g}, \end{aligned}$$

where

$$(2.21) \quad \begin{aligned} \text{Ad}(g^{-1})Y &= \sum_{\lambda \in \Sigma^+} \sum_{i=1}^{m(\lambda)} (c_{\lambda_i}(g)X_{\lambda_i} + c_{-\lambda_i}(g)X_{-\lambda_i}) \\ &\quad + \sum_{j=1}^l c_j(g)H_j \pmod{\mathfrak{m}}, \end{aligned}$$

(cf. (2.13)). Therefore π_0 is smooth, that is, $(d\pi_0)_{\hat{x}}$ is surjective for any $\hat{x} \in \hat{X}_0$. Moreover \tilde{X} has the following universal property.

Proposition 10. Given an analytic map f of \hat{X}_0 to a real analytic manifold \tilde{Y} such that $f(\hat{x}) = f(\hat{x}')$ if $\hat{x} \sim \hat{x}'$ in \hat{X}_0 , then there is a unique analytic map \bar{f} of \tilde{X} to \tilde{Y} such that the following diagram is commutative:

$$\begin{array}{ccc} \hat{X}_0 & \xrightarrow{f} & \tilde{Y} \\ \pi_0 \downarrow & \nearrow \bar{f} & \\ \tilde{X} & & \end{array}$$

Proof. We have only to prove the analyticity of \bar{f} . Let s_g be the analytic map of \tilde{U}_g to X_0 defined by $(n, t) \mapsto (g, n, t)$. Since $\bar{f}|_{\tilde{U}_g} = \bar{f} \circ \pi_0 \circ s_g|_{\tilde{U}_g} = f \circ s_g|_{\tilde{U}_g}$ for $g \in G$, \bar{f} is also analytic.

q.e.d.

In [8] another realization of G/K and G/P is given. The following proposition shows the relation between the realization in [8] and \tilde{X} .

Proposition 11. The natural map

$$K \times (-1, 1)^l \hookrightarrow G \times \mathbb{R}^l \xrightarrow{\pi_0} \tilde{X}$$

induces an analytic diffeomorphism

$$\gamma : (K/M) \times (-1, 1)^l \rightarrow \tilde{X}$$

onto an open dense submanifold of \tilde{X} which contains G/P .

Proof. Let $\hat{x} = (k, t)$ be a point of $K \times (-1, 1)^l$. Then the following

$$\begin{cases} (d\pi_0)_{\hat{x}}(\partial/\partial t_j) = \partial/\partial t_j, & j=1, \dots, l, \\ (d\pi_0)_{\hat{x}}(\text{Ad}(k)(X_{\lambda_i} - X_{-\lambda_i})) = (t^{2\lambda_i} - 1)X_{-\lambda_i}, & \lambda_i \in \Sigma^+, i=1, \dots, m(\lambda), \end{cases}$$

shows that the map $d\pi_0: T_{\hat{x}}(K \times (-1, 1)^l) \rightarrow T_{\pi(\hat{x})}\tilde{X}$ is surjective

because $\text{Ad}^{(k)}(X_{\lambda_i} - X_{-\lambda_i}) \in \mathfrak{k}$ and $t^{2\lambda_i} - 1 \neq 0$. Moreover, since

$(k, t) \sim (km, t)$ for any $m \in M$, which is clear because $kma(\hat{x})P_{\hat{x}}(K) = ka(\hat{x})P_{\hat{x}}(K)$, we obtain the smooth analytic map $\gamma : (K/M) \times (-1, 1)^l \rightarrow \tilde{X}$. Comparing the dimensions of the manifolds, we see that γ is analytic local diffeomorphism.

Here we note that Cartan decomposition (1.1) induces the analytic diffeomorphism

$$\begin{array}{ccc} K/M \times A^+ & \rightarrow & G/K \\ \downarrow & & \downarrow \\ (kM, a) & \mapsto & kaK \end{array}$$

onto an open dense submanifold of G/K . Therefore putting $\tilde{Z} = K/M \times \{(-1, 1) - \{0\}\}^l$, we see that the restriction $\gamma|_{\tilde{Z}}$ is injective and $\gamma(\tilde{Z})$ is open dense in \tilde{X} . Since \tilde{Z} is open dense in $(K/M) \times (-1, 1)^l$ and $\gamma|_{\tilde{Z}}$ is injective and γ is an analytic local diffeomorphism, we can conclude that γ is injective. Thus we can

identify $(K/M) \times (-1, 1)^l$ with an open dense submanifold of \tilde{X} .
 Moreover, since K acts transitively on G/P , we have $K/M \times \{0\}^l \xrightarrow{\sim} G/P$ by Definition 4 and Theorem 5. q.e.d.

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Toshio OSHIMA
Department of Mathematics
Faculty of Science
University of Tokyo
Hongo, Bunkyo-ku
Tokyo, Japan