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Homomorphism Theorems in Local Dynamical Systems Theory

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Introduction: The purpose of this lecture is to announce and explain results recently obtained by myself to establish Homomorphism Theorems for continuous local dynamical systems. The theorems depend on what we shall consider as homomorphisms. I. Kimura and myself have introduced various categories of local dynamical systems. If a homomorphism theorem holds, the smaller the category, the easier the problem. As we are going to see, in the smallest category GH(-1), the problem is trivial and in the category GH(0), this is not too difficult. In other larger categories, we need some additional conditions on homomorphisms.

To establish a homomorphism theorem in the topological group theory, first we must study this in the abstract group theory. In the latter, we have a beautiful theorem, i.e. without any additional condition on homomorphisms, the theorem holds (see [8]). However, if we refer to [8], the theorem holds for topological groups, under an additional condition on homomorphisms. (This is, of course, caused by the weak definition of a homomorphism. If we give a stronger definition to it, we need not any additional condition, as done in [3]. In fact in [3], the definition of a homomorphism is given in such a way that for it, the homomorphism theorem holds.) We shall keep the definitions of various GH-categories and find a necessary and sufficient (to some extent) condition on a homomorphism in these categories, in order that we have the homomorphism theorems.

We study the problem first for abstract local dynamical systems, which were introduced by Hájek and studied by him and by myself, and then for continuous local dynamical systems, as we do at the study of topological groups.

Standing Notation and Notational Conventions:

R: the set of real numbers, i.e. the unique totally ordered field satisfying the continuity axiom, unique up to isomorphisms considering this structure. More exactly, R denotes a representative of the class of sets having this structure. We shall identify the members of this class, in other words, two sets having this structure will not be considered as different.

R_0: R endowed with the usual topology.

X and Y: sets.

O and G: the topologies or the families of open sets defining the topologies on X and Y, respectively. When X and Y are endowed with topologies O and G, then (X, O) and (Y, G) denote the topological spaces of which the carrier sets are X and Y and the topologies are O and G, respectively.

For $A \subset X$, if we write $A(\subset (X, O))$, then we understand

A is endowed with the subspace topology induced by (X, O) i.e. (A, O_A), where O_A is the trace of O in A.

Let $D \subset X \times R$ and $x \in X$. D_x denotes the section of D at x, i.e. $D_x = \{t | (x, t) \in D\}$. Let $f: D \to Y$ be a map and $x \in X$, then f_x denotes the map: $D_x \to Y$ defined by $f_x(t) = f(x, t)$.

§1. Abstract and Continuous Local Dynamical Systems and their Germs.

1. Definitions. (AG)(AS) Abstract Systems and their Germs.

Definition 1. Let $\mathcal{D} \subset X \times R$. $\mathcal{D}$ is said to be abstract-dynamically admissible over X iff for every $x \in X$, $\mathcal{D}_x$ is an open interval containing 0.

Definition 2 ([5],[11]). Let $\mathcal{D} \subset X \times R$, and $\mu: \mathcal{D} \to X$ be a map. $(X, \mathcal{D}, \mu)$ is called an elementary dynamical system on the phase set X with domain $\mathcal{D}$ iff

(ES0) $\mathcal{D}$ is abstractly admissible

(ESI) Identity Axiom: for every $x \in X$, $\mu(x, 0) = x$

(ESII) Homomorphism Axiom: if (x, t), $(x, t+s)$, $(\mu(x,t), s) \in \mathcal{D}$, then

$$\mu(x, t+s) = \mu(\mu(x, t), s).$$

Definition 3 ([11]). An elementary system $(X, \mathcal{D}, \pi)$ is called a germ of an abstract local dynamical system (abbreviated as an abstract germ) on the phase set X with domain $\mathcal{D}$ iff

(AGN) No-interseciton Axiom: if (x, t), $(x_1, t) \in \mathcal{D}$ and $\mu(x, t) = \mu(x_1, t)$, then $x = x_1$.

Definition 4 (]5], [11], [12]). An elementary dynamical system (X, D, π) is called an abstract local dynamical system (abbreviated as an abstract system) on the phase set X with domain D iff

(ASN) Nonextendability Axiom: for every $(x, t) \in D$,

$$D_{\pi(x,t)} = D_x - t.$$

An abstract system is said to be global and called an abstract global (dynamical) system iff for every $x \in X$, $D_x = R$ or equivalently $D = X \times R$ (which implies (ASN)).

Proposition 1 ([11]). An abstract system is an abstract germ.

Theorem 1 ([11]). Let $(X, \mathcal{D}, \mu)$ be an elementary system, then there exists a unique abstract system (X, D, π) with $D \supset \mathcal{D}$ such that $\mu = \pi|\mathcal{D}$, iff $(X, \mathcal{D}, \mu)$ is an abstract germ.

Definition 5 ([11]). The uniquely determined abstract system by an abstract germ (see Theorem 1) is called the abstract system generated by the abstract germ.

Remark 1. Let $(X, \mathcal{D}, \mu)$ and (X, D, π) be an abstract germ and an abstract system. If there arises no ambiguity, we shall omit some data, e.g. we shall express them by (an abstract germ) μ and (an abstract system) π.

Definition 6. Let $(X, \mathcal{D}_1, \mu_1)$, $(X, \mathcal{D}_2, \mu_2)$ be two abstract germs on the same phase set X. μ_1 is called an abstract subgerm of μ_2 iff $\mathcal{D}_1 \subset \mathcal{D}_2$ and $\mu_1 = \mu_2|\mathcal{D}_1$.

The abstract germs μ_1 and μ_2 are said to be equivalent iff μ_1 and μ_2 have a common abstract subgerm. (Actually this defines an equivalence relation.)

Proposition 2. Equivalent abstract germs generate the same abstract system.

(CG)(CS) Continuous Systems and their Germs.

Definition 1. Let $\mathcal{D} \subset X \times R$. $\mathcal{D}$ is said to be continuous-dynamically admissible over $(X, \mathcal{O})$ iff $\mathcal{D}$ is abstractly admissible over X and $\mathcal{D}$ is a neighborhood of $X \times \{0\}$ in $(X, \mathcal{O}) \times R_0$.

Definition 2 ([1], [5], [9]). Let $(X, \mathcal{D}, \mu)$ be an elementary system. $((X, \mathcal{O}), \mathcal{D}, \mu)$ is called a germ of a continuous local dynamical system (abbreviated as a continuous germ) on the phase space X with domain $\mathcal{D}$ iff

(CG1) $\mathcal{D}$ is continuous-dynamically admissible over $(X, \mathcal{O})$

(CG2) μ: $\mathcal{D}(\subset (X, \mathcal{O}) \times R_0) \to (X, \mathcal{O})$ is continuous.

Proposition 1 ([11]). If $((X, \mathcal{O}), \mathcal{D}, \mu)$ is a continuous germ, then $(X, \mathcal{D}, \mu)$ is an abstract germ.

Definition 3. Let (X, D, π) be an abstract system. $((X, \mathcal{O}), D, \pi)$ is called a continuous local dynamical system (abbreviated as a continuous system) on the phase space X with domain D iff $((X, \mathcal{O}), D, \pi)$ is a continuous germ. A continuous system is said to be global and called a continuous global (dynamical) system iff $D = X \times R$.

Proposition 2 ([1], [5], [11], [12]). Let $D \subset X \times R$ be abstract-dynamically admissible over X and π: $D \to R$ a map. $((X, \mathcal{O}), D, \pi)$ is a continuous system iff

(CSI) Openness Axiom: D is open in $(X, \mathcal{O}) \times R_0$.

(CSII) Continuity Axiom: π: $D(\subset (X \times O) \times R_0) \to (X, O)$ is continuous

(ESI) for every $x \in X$, $\pi(x, 0) = x$

(ESII) if $(x, t), (x, t+s), (\pi(x, t), s) \in D$, then

$$\pi(x, t+s) = \pi(\pi(x, t), s)$$

(ASN) for every $(x, t) \in D$, $D_{\pi(x,t)} = D_x - t$.

Remark 1 ([5], [9], [11], [12]). Put $D_x = (a_x, b_x)$. (ASN) is equivalent to

(CSN) if a_x (or b_x) is finite, then the cluster set of $\pi_x(t)$ as $t \downarrow a_x$ (or $\uparrow b_x$) is empty.

(CSI) is equivalent to ($\bar{R}_0$ denoting the usual compactification of R_0)

(CSI_a) the maps a and b: $(X, O) \to \bar{R}_0$ defined by $x \to a_x$ and $x \to b_x$ are upper and lower semicontinuous, respectively.

Theorem 1. Let $((X, O), \mathcal{D}, \mu)$ be a continuous germ, then there exists a unique continuous system $((X, O), D, \pi)$ with $D \supset \mathcal{D}$ such that $\mu = \pi|_{\mathcal{D}}$.

Definition 4. The uniquely determined continuous system by a continuous germ (see Theorem 1) is called the continuous system generated by the continuous germ.

Remark 2. The similar abbreviation will be applied to a continuous germ and a continuous system as explained for an abstract germ and an abstract system in (AG)(AS) Remark 1. Thus a continuous germ $((X, O), \mathcal{D}, \mu)$ and a continuous system $((X, O), D, \pi)$ are expressed (a continuous germ) μ and (a continuous system) π and then $(X, \mathcal{D}, \mu)$ and (X, D, π) are expressed by the abstract germ μ and the abstract system π, respectively.

Definition 5. Let $((X, \mathcal{O}), \mathcal{D}_1, \mu_1)$ and $((X, \mathcal{O}), \mathcal{D}_2, \mu_2)$ be two continuous germs on the same phase space $(X, \mathcal{O})$. μ_1 is called a continuous subgerm of μ_2 iff $\mathcal{D}_1 \subset \mathcal{D}_2$ and $\mu_1 = \mu_2 | \mathcal{D}_1$.

Two continuous germs μ_1 and μ_2 are said to be equivalent iff μ_1 and μ_2 have a common continuous subgerm. (Actually, this defines an equivalence relation.)

Proposition 3. Equivalent continuous germs generate the same continuous system.

2. *Some Fundamental Concepts, Restrictions of Systems.*

(AS) Abstract Systems.

In the following, (X, D, π) is an abstract system.

Definition 1. For every $x \in X$, $\pi_x(D_x)$ is called the orbit through (or of) x (w. r. to π) and denoted by $C_\pi(x)$.

Definition 2. A subset Y of X is said to be quasi-invariant (w. r. to π) iff for every $x \in Y$, $C_\pi(x) \subset Y$.

A quasi-invariant set Y is said to be invariant iff for every $x \in Y$, $D_x = R$. (If we assume further $Y \neq \phi$, this definition reduces to the classical notion of invariance.)

Definition 3. A subset Y of X is said to be abstractly admissible w. r. to π iff there exists $E \subset Y \times R$ such that $(Y, E, \pi|_E)$ is an abstract system. Then $\pi|_E$ is denoted by $\pi|_Y$ and $(Y, E, \pi|_Y)$ is called the restriction of π to Y.

Proposition 1. If Y is abstractly admissible w. r. to the abstract system π, the abstract system $\pi|_Y$ is uniquely determined.

Proposition 2. A subset Y of X is abstractly admissible w. r. to π iff for every $y \in Y$, there exists an open interval J_y containing 0 and contained in D_y such that $\pi_y(J_y) \subset Y$.

Corollary 1. Every quasi-invariant set is abstractly admissible.

Corollary 2. Every quasi-invariant set w. r. to a global system is invariant.

Definition 4. Let $x \in X$. x and the orbit $C_\pi(x)$ are said to be

global if $D_x = R$

strictly local if $D_x \neq R$.

Definition 5. Let $x \in X$. x and the orbit $C_\pi(x)$ are said to be singular iff $C_\pi(x) = \{x\}$. S_π denotes the set of singular points

Proposition 3. A singular point (or orbit) is global.

(CS) Continuous Systems.

In the following, $((X, \mathcal{O}), D, \pi)$ is a continuous system, expressed by (the continuous system) π.

Definition 1. For every $x \in X$, the orbit $C_\pi(x)$ w. r. to the abstract system π is also called the orbit through (or of) x w. r. to the continuous system π and denoted by the same symbol $C_\pi(x)$.

Definition 2. A quasi-invariant set w. r. to the abstract system π is said to be quasi-invariant w. r. to the continuous system π. An invariant set w. r. to the continuous system is defined in the same way.

Definition 3. A subset Y of X is said to be continuously admissible w. r. to (the continuous system) π iff there exists $E \subset Y \times R$ such $((Y, O_Y), E, \pi|_E)$ is a continuous system. Then, $\pi|_E$ is denoted by $\pi|_Y$, and $((Y, O_Y), E, \pi|_Y)$ is called the restriction of the continuous system π to Y.

Proposition 1. If Y is continuously admissible w. r. to π, then the continuous system $\pi|_Y$ is uniquely determined.

Proposition 2. If Y is an open set of X, or Y is quasi-invariant w. r. to π, then Y is continuously admissible.

To the author's knowledge, "only if" in the following is new.

Theorem 1. A subset Y of X is continuously admissible w. r. to π iff there exists an open set 0 of X such that Y is quasi-invariant w. r. to $\pi|_0$.

Definition 4, 5. Globality of x $(\in X)$ and $C_\pi(x)$ w. r. to the continuous system π is defined as that w. r. to the abstract system π. The same for singularness.

Proposition 3. (AG)(AS) Proposition 3 holds for continuous systems.

Standing Notation:

In the sequel throughout the paper, in (AS) of each section, (X, D, π) and (Y, E, ρ) denote abstract systems, and π and ρ are their abbreviations, and in (CS) of each section, $((X, O), D, \pi)$ and $((Y, G), E, \rho)$ denote continuous systems, and π and ρ are their abbreviations, unless otherwise stated.

Ens: the category of sets and maps.

ens: the category of sets and inclusions.

Top: the category of topological spaces and continuous maps.

top: the category of topological spaces and continuous inclusions.

Let C be a category.

obj C: the class of objects in C.

mor C: the class of morphisms in C.

Let $a, b \in$ obj C.

$[a, b]_C$: the set of morphisms: $a \to b$ in C.

n (or n') runs over -1, 0, 1, 2, 4, and

? runs over θ, B, I, BI, unless otherwise stated.

(n-θ means n.)

m (or m') runs over 0, 25, 5, unless otherwise stated.

(n means n•0 and n•m-θ means n•m.)

§2. Categories of Abstract and Continuous Local Dynamical Systems and those of their Germs.

1. Morphisms of Local Dynamical Systems.

(AS) Abstract Systems.

Definition 1 (Cf.[11]). Let $h: X \to Y$ be a map and $\phi: D \to R$ a map such that for every $x \in X$, $\phi_x: D_x(\subset R_0) \to R_0$ is continuous and $\phi(x, 0) = 0$. (h, ϕ) is called a GH-morphism: $\pi \to \rho$ iff for every $(x,t) \in D$, $h \circ \pi(x,t) = \rho(h(x),\phi(x,t))$.

A GH-morphism (h, ϕ) is said to be of type 2 iff for every $x \in X$, $\phi_x\colon D_x \to R$ is a linear map.

A type 2 GH-morphism (h, ϕ) is said to be of type 1 iff for every $x \in X$, the linear map $\phi_x\colon D_x \to R$ is independent of x, or equivalently there exists $c \in R$ such that for every $(x, t) \in D$, $\phi(x, t) = ct$.

A type 1 GH-morphism (h, ϕ) is said to be of type 0 iff for every $x \in X$, $\phi_x\colon D_x \to E_x$ $(\subset R)$ is the identity. (If so, ϕ will be denoted by id.)

A type 0 GH-morphism (h, id) is said to be of type -1 iff $X \subset Y$ and h is the inclusion.

For convenience of exposition, a GH-morphism (h, ϕ) is said to be of type 4.

Remark 1. In [11], we introduced the notions of a type n GH-isomorphism and a type n-C GH-isomorphism. "Type n GH" in this paper corresponds to "type n-C GH" in [11]. We do not consider "type 3" in contrast to [11]. This is done because, in the sequel, if we treat type n GH-morphisms in general, and type 3-C GH-morphisms in the sense of [11], it would make our arguments confusing and would not give any interesting result (cf. [11] Remark 11).

Definition 2 ([11]). Let $(h, \phi)\colon \pi \to \rho$ be a type n GH-morphism. (h, ϕ) is said to be of type n-B iff for (x, t), $(x, t+s)$, $(\pi(x, t), s) \in D$, we have $\phi(x, t+s) = \phi(\pi(x, t), s)$.

(h, ϕ) is said to be of type n-I iff for every $x \in X$, $\phi_x\colon D_x \to R$ is either strictly increasing or strictly decreasing. If (h, ϕ) is of type n-B and of type n-I, then we shall say (h, ϕ) is of type n-BI.

Proposition 1. A type n GH-morphism (h, ϕ): $\pi \to \rho$ is of type n-B iff for every $x \in h^{-1}(S_\rho)$, ϕ_x: $D_x \to R$ is a linear map, remaining the same along every orbit in $h^{-1}(S_\rho)$. If (h, ϕ) is of type n-BI, then for every $x \in h^{-1}(S_\rho)$, $\phi_x: D_x \to R$ is toplinear.

Remark 2. A type 0 GH-morphism is always of type 0-BI. A type 1 GH-morphism (h, ϕ) is always of type 1-B, and is of type 1-BI iff $\phi(x, t) = ct$ with nonzero c. A type 2 GH-morphism (h, ϕ) is of type 2-B iff when we write $\phi(x, t) = c(x)t$, $c(x)$ is constant along every orbit in $h^{-1}(S_\rho)$, and is of type 2-I iff $c(x) \neq 0$ for every $x \in X$. (One observes, if (h, ϕ) is of type 2, then $c(x)$ is constant along every orbit outside $h^{-1}(S_\rho)$.)

(CS) Continuous Systems.

Definition 1 ([6], [10]). Let (h, ϕ) be a type n GH-morphism: $(X, D, \pi) \to (Y, E, \rho)$. (h, ϕ) is called a type n·5 GH-morphism: $((X, \mathcal{O}), D, \pi) \to ((Y, \mathcal{G}), E, \rho)$ iff h: $(X, \mathcal{O}) \to (Y, \mathcal{G})$ is continuous.

A type n·5 GH-morphism: $\pi \to \rho$ is said to be of type n·0 or of type n·25 iff ϕ: $D(\subset (X, \mathcal{O})) \to R_0$ is continuous in D or in $D - h^{-1}(S_\rho) \times R$, respectively.

Definition 2. The notions of type B and type I for a GH-morphism are defined as those for a GH-morphism.

Proposition 1. (AS) Proposition 1 holds for type n·m GH-morphisms

(AG)(CG) Abstract and Continuous Germs.

Omitted.

2. *Categories of Local Dynamical Systems.*

(AS) Abstract Systems.

Theorem 1 ([11]). Put

obj $GH(n - ?)$ = the class of abstract systems

mor $GH(n - ?)$ = the class of type n-? GH-morphisms.

For $(h, \phi) \in [\pi, \rho]_{GH(n-?)}$, $(k, \psi) \in [\rho, \sigma]_{GH(n-?)}$, $(\ell, \lambda) = (k, \psi)\circ(h, \phi)$ is defined as follows:

$\ell = k\circ h$ (composition in Ens)

$\lambda: D \longrightarrow R$ is defined by

$(x, t) \longmapsto \psi(h(x), \phi(x, t))$.

Then $GH(n - ?) = \{\text{obj } GH(n - ?), \text{mor } GH(n - ?), \circ\}$ is a category.

Proposition 1. The category $GH(-1)$ is an ordered class.

Remark 1. $GH(-1) = GH(-1 - B) = GH(-1 - I) = GH(-1 - BI)$

$GH(0) = GH(0 - B) = GH(0 - I) = GH(0 - BI)$

$GH(1 - I) = GH(1 - BI)$ is a subcategory of $GH(1) = GH(1 - B)$.

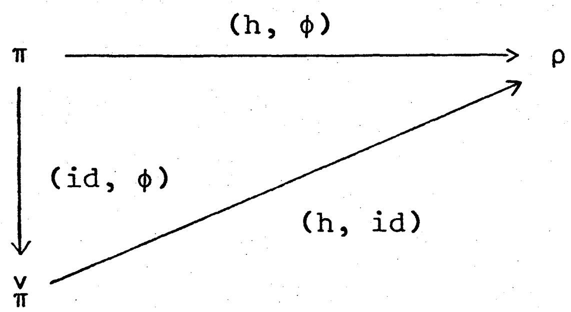
In general, $GH(n - BI)$ is a subcategory of $GH(n - B)$ and of $GH(n - I)$, both of which are subcategories of $GH(n)$. If $n \leq n'$, then $GH(n - ?)$ is a subcategory of $GH(n' - ?)$.

Theorem 2. $(h, \phi) \in [\pi, \rho]_{GH(n - ?)}$ is an isomorphism in $GH(n - ?)$ if $(h, \phi) \in$ mor $GH(n - BI)$ and $h: X \to Y$ is an isomorphism in ens for $n = -1$ and in Ens for $n \neq -1$, and iff $(h, \phi) \in$ mor $GH(n)$, for every $x \in S_\pi$, $\phi_x: R_0 \to R_0$ is an isomorphism in Top and $h: X \to Y$ is an isomorphism in ens for $n = -1$ and in Ens for $n \neq -1$ $(? = \theta, I)$.

Corollary 1. $(h, \phi) \in [\pi, \rho]_{GH(0)}$ or $[\pi, \rho]_{GH(n-I)}$ is an isomorphism in $GH(0)$ or in $GH(n - I)(n = 1, 2)$ iff $h: X \to Y$

is a bijection.

Theorem 3 (First Decomposition Theorem). $(h, \phi) \in [\pi, \rho]_{GH(n-BI)}$ is decomposed in a way that the diagram



commutes, where $\check{\pi}$ is the abbreviation of $(X, \check{D}, \check{\pi})$ with $\check{D} = \bigcup_{x \in X} \{x\} \times \phi_x(D_x)$, $(id, \phi): \pi \to \rho$ is an isomorphism in $GH(n - BI)$, and $(h, id): \check{\pi} \to \rho$ is a morphism in $GH(0)$.

(CS) Continuous Systems.

Theorem 1. Put

obj GH(n•m - ?) = the class of continuous systems

mor GH(n•m - ?) = the class of type n•m - ? GH-morphisms.

Define the composition of two morphisms in mor GH(n•m - ?) as in GH(n - ?) ((AS) Definition 1).

Then GH(n•m - ?) = {obj GH(n•m - ?), mor GH(n•m - ?), ∘} is a category.

Proposition 1. The category GH(-1) is an ordered class.

Remark 1.

GH(-1) = GH(-1 - B) = GH(-1 - I) = GH(-1 - BI)
= GH(-1•25) = ••••••••••
= GH(-1•5) = ••••••••••

GH (0) = GH(0 - B) = GH(0 - I) = GH(0 - BI)
= GH(0•25) = ••••••••••

$= GH(0 \cdot 5) = \cdots\cdots\cdots\cdots$

In general, $GH(n \cdot m - BI)$ is a subcategory of $GH(n \cdot m - B)$ and of $GH(n \cdot m - I)$, both of which are subcategories of $GH(n \cdot m)$.

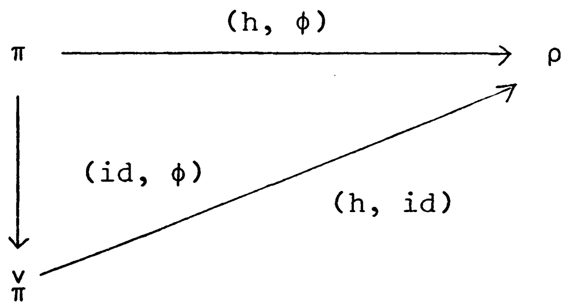
If $n \leq n'$, $m \leq m'$, then $GH(n \cdot m - ?)$ is a subcategory of $GH(n' \cdot m' - ?)$.

Theorem 2. $(h, \phi) \in [\pi, \rho]_{GH(n \cdot m - ?)}$ is an isomorphism in $GH(n \cdot m - ?)$ if $(h, \phi) \in \mathrm{mor}\, GH(n \cdot m - BI)$ and $h: (X, \mathcal{O}) \to (Y, G)$ is an isomorphism in top for $n = -1$ and in Top for $n \neq -1$, and iff $(h, \phi) \in \mathrm{mor}\, GH(n \cdot m)$, for every $x \in S_\pi$, ϕ_x: $R_0 \to R_0$ is an isomrophism in Top and h: $(X, O) \to (Y, G)$ is an isomorphism in top for $n = -1$ and in Top for $n \neq -1$ ($? = \theta$, I).

Corollary 1. $(h, \phi) \in [\pi, \rho]_{GH(0)}$ or $[\pi, \rho]_{GH(n \cdot m - I)}$ is an isomorphism in $GH(0)$ or in $GH(n \cdot m - I)$ $(n = 1, 2)$ iff h: $(X, O) \to (Y, G)$ is a homeomorphism.

Theorem 3 (First Decomposition Theorem).

$(h, \phi) \in [\pi, \rho]_{GH(n \cdot 0 - BI)}$ is decomposed in a way that the diagram



commutes, where $\check{\pi}$ is the abbreviation of $((X, \mathcal{O}), \check{D}, \check{\pi})$ with $\check{D} = \bigcup_{x \in X} \{x\} \times \phi_x(D_x)$, (id, ϕ): $\pi \to \check{\pi}$ is an isomorphism in $GH(n \cdot 0 - BI)$, and (h, id): $\check{\pi} \to \rho$ is a morphism in $GH(0)$.

(AG)(CG) Abstract and Continuous Germs. Omitted.

3. Some Functors.

Theorem 1. $G(n \cdot m - ?)$: $GH(n \cdot m - ?) \longrightarrow GH(n - ?)$

$$((X, O), D, \pi) \longmapsto (X, D, \pi)$$

$$(h, \phi)\left(\in [\pi, \rho]_{GH(n \cdot m - ?)}\right) \longmapsto (h, \phi)\left(\in [\pi, \rho]_{GH(n-?)}\right)$$

is a functor (forgetful functor).

Theorem 2 (To prove, one uses results in [12]).

(1) Every functor $G(n \cdot m - ?)$ is surjective but not injective as a map of the objects to the objects.

(2) Every functor $G(n \cdot m - ?)$ is surjective but not injective as a map of the morphisms to the morphisms.

(3) Every functor $GH(n \cdot m - ?)$ is faithful but not full, $(n \neq -1)$.

Remark 1. By (3) above, every functor in Theorem 1 may not reflect limits and colimits of diagrams in the origin category, however the image by the functor of a diagram in the origin category gives necessary conditions for existence of the limit and the colimit of the diagram.

§3. Quotients of Local Dynamical Systems.

1. Monomorphisms and Epimorphisms.

Omitted.

2. Coproducts.

Theorem 1 (Cf. [5] II, 3·4 and IV, 3·6). $GH(n-?)$ and $GH(n \cdot m - ?)$ have coproducts, except $n = -1, 1$.

Details are omitted.

3. Quotient Local Dynamical Systems.

(AS) Abstract Systems.

Definition 1 ([5] II, 3•5). An equivalence relation $\sim$ in X is said to be compatible with π iff for every pair $(x_1, x_2) \in X \times X$, with $x_1 \sim x_2$, we have $\pi(x_1, t) \sim \pi(x_2, t)$ for all $t \in D_{x_1} \cap D_{x_2}$.

As far as the author knows, the following is new.

Theorem 1. For every equivalence relation $\approx$ in X, we have the smallest equivalence relation in X compatible with π and including $\approx$.

In the following theorem, as far as the author knows, "if" was proved by Hájek in [5] II, 3•5, but "only if" is new.

Theorem 2. Let $\sim$ be an equivalence relation in X. There exists a unique abstract system $\tilde{\pi}$ on $\tilde{X} = X/\sim$ such that $(pr, id): \pi \to \tilde{\pi}$ is a morphism in $GH(0)$, where pr denotes the canonical projection: $X \to \tilde{X}$, iff $\sim$ is compatible with π. Then, $(pr, id): \pi \to \tilde{\pi}$ is an epimorphism in $GH(0)$.

This theorem is proved by establishing the following four lemmas.

Lemma 1. Let $\sim$ be an arbitrary equivalence relation in X, then

$$\mathcal{D} = D/(\sim\times=)_D = \bigcup\left\{\{\tilde{x}\} \times \mathcal{D}_{\tilde{x}} \mid \tilde{x} \in \tilde{X}\right\}$$

with

$$\mathcal{D}_{\tilde{x}} = \bigcup \{D_x \mid x \in \tilde{x}\},$$

where $\tilde{x}$ denotes an equivalence class in X w. r. to $\sim$ and also an element $\tilde{x} = pr(x)$ in $\tilde{X}$, and $(\sim\times=)_D$ denotes the restriction to D of $\sim\times=$.

Lemma 2. Let $\sim$ be as in Lemma 1. The map $\pi: D \to X$ is compatible with the equivalence relations $(\sim\times=)_D$ and $\sim$ (in terms of [2] II, 6,5) iff $\sim$ is compatible with the abstract system π.

Lemma 3. Let $\sim$ be as in Lemma 1, then there exists a map $\tilde{\mu}$: $\tilde{\mathcal{D}} = D/(\sim\times=)_D \to \tilde{X}$ such that the diagram

$$\begin{array}{ccc} D & \xrightarrow{\quad\pi\quad} & X \\ {\scriptstyle (pr,\ id)}\downarrow & & \downarrow{\scriptstyle pr} \\ \tilde{\mathcal{D}} & \xrightarrow{\quad\tilde{\mu}\quad} & \tilde{X} \end{array}$$

commutes, iff $\sim$ is compatible with the abstract system π.

Lemma 4. Let $\sim$ be as in Lemma 1. If $\sim$ is compatible with the abstract system π, $(X, \tilde{\mathcal{D}}, \tilde{\mu})$ is an abstract germ, where $\tilde{\mathcal{D}}$ and $\tilde{\mu}$ are as defined in Lemmas 1 and 3.

(CS) Continuous Systems.

In the following throughout this section, $\sim$ is an equivalence relation in $(X, \mathcal{O})$ and pr: $X \to \tilde{X} = X/\sim$ is the canonical projection. If $\sim$ is compatible with the abstract system $\pi(= (X, D, \pi))$, then $(\tilde{X}, \tilde{\mathcal{D}}, \tilde{\mu})$ and $(\tilde{X}, \tilde{D}, \tilde{\pi})$ denote the abstract germ and the abstract system defined by (X, D, π) and $\sim$ in (AS), and $(\tilde{X}, \tilde{\mathcal{O}})$ denotes the topological space $(X, \mathcal{O})/\sim$.

Theorem 1. $((\tilde{X}, \tilde{\mathcal{O}}), \tilde{D}, \tilde{\pi})$ is a continuous system such that (pr, id): $\pi \to \tilde{\pi}$ is an epimorphism in GH(O) iff

(I) $\sim$ is compatible with the abstract system π

(II_0) $\tilde{D}$ is open in $\dfrac{(X, \mathcal{O}) \times R_0}{\sim\times=}$

and

(II_1) $\tilde{\pi}$: $\tilde{\mathcal{D}}(\subset \frac{(X, \mathcal{O}) \times R_0}{\sim\times=}) \to (\tilde{X}, \tilde{\mathcal{O}})$ is continuous.

To prove this theorem, we use Proposition 10 in [3] I, 3,6 and the following Proposition A which is a direct consequence of a known theorem (first proved by the author, as far as he knows, [12] 5, Theorem A).

Proposition A. Let $\sim$ be an equivalence relation in a topological space $(X, \mathcal{O})$, then the quotient topology T_q of $(X \times R)/\sim\times=$ coincides with the product topology $(X/\sim) \times R$, or in short

$$\frac{(X, \mathcal{O}) \times R_0}{\sim\times =} = \frac{(X, \mathcal{O})}{\sim} \times R_0.$$

Theorem 2. $((\tilde{X}, \tilde{\mathcal{O}}), \tilde{D}, \tilde{\pi})$ is a continuous system such that (pr, id): $\pi \to \tilde{\pi}$ is an epimorphism in GH(0), iff

(I) $\sim$ is compatible with the abstract system π

$(\tilde{\mathrm{II}}_{0,a})$ D contains an open neighborhood of $X \times \{0\}$ in $(X, \mathcal{O}) \times R_0$ saturated w. r. to $(\sim\times=)_D$ and

$(\tilde{\mathrm{II}}_{1,a})$ every open subset of $D(\subset (X, \mathcal{O}) \times R_0)$ saturated w. r. to $(\sim\times=)_D$ is the intersection of D and an open subset of $(X, \mathcal{O}) \times R_0$ saturated w. r. to $\sim\times=$.

To prove Theorem 2, we establish the following Lemmas 1 and 2.

Lemma 1. (II_0) and (II_1) in Theorem 1 are equivalent to

$(\tilde{\mathrm{II}}_0)$ $\tilde{\mathcal{D}} = D/(\sim\times=)_D$ is a neighborhood of $\tilde{X} \times \{0\}$ in $(X \times R)/\sim\times=$

and

($\tilde{II}_1$) The quotient topology of $\tilde{\mathcal{D}} = D/(\sim\times=)_D$ $(D \subset (X, \mathcal{O}) \times R_0)$ coincides with the subspace topology of it induced by the quotient topology of $((X, \mathcal{O}) \times R_0)/\sim\times=$.

Lemma 2. ($\tilde{II}_1$) in Lemma 1 is equivalent to ($\tilde{II}_{1,a}$) in Theorem 2.

Corollary 1. If $\sim$ is compatible with the abstract system π and is open, then $((\tilde{X}, \tilde{\mathcal{O}}), \tilde{D}, \tilde{\pi})$ is a continuous system such that (pr, id): $\pi \to \tilde{\pi}$ is an epimorphism in GH(0).

Definition 1 ([5] II, 3·5). An equivalence relation $\sim$ in X is said to be strongly compatible with the abstract system π iff $\sim$ is compatible with the abstract system π and $x_1 \sim x_2$ implies $D_{x_1} = D_{x_2}$.

Corollary 2 (Cf. [5] IV, 3·8). If $\sim$ is strongly compatible with π, then $((\tilde{X}, \tilde{\mathcal{O}}), \tilde{D}, \tilde{\pi})$ is a continuous system such that (pr, id): $\pi \to \tilde{\pi}$ is an epimorphism in GH(0).

Remark 1. Even for an equivalence relation in X compatible with π, its openness and strong compatibility are independent.

Further, compatibility with π of an equivalence relation in X implies neither openness nor strong compatibility with π.

4. *Coequalizers.*

(AS) Abstract Systems.

Theorem 1. GH(-1) and GH(0) have coequalizers, but other GH(n - ?) does not, ? = BI, I.

Theorem 2. GH(0) is cocomplete, but other GH(n - ?) is not cocomplete, ? = BI, I.

Details are omitted.

(CS) Continuous Systems.

Definition 1. Let $GH(0)_{op}$ be the subcategory of $GH(0)$ such that mor $GH(0)_{op}$ consists of type 0 GH-morphisms (h, id): $\pi \to \rho$ with open map h: $(X, \mathcal{O}) \to (Y, \mathcal{O})$.

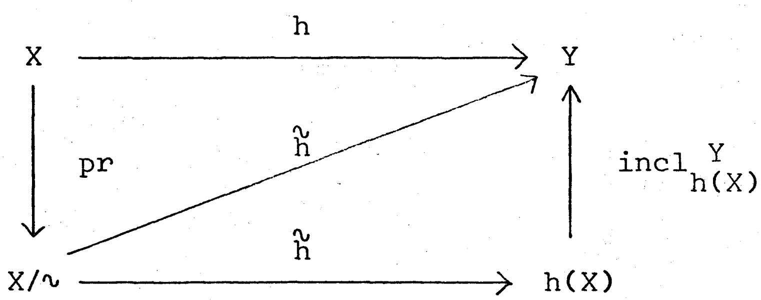
Theorem 1. $GH(-1)$ and $GH(0)_{op}$ have coequalizers.

Theorem 2. $GH(0)_{op}$ is cocomplete.

Details are omitted.

§4. Homomorphism Theorems.

Definition and Standing Notation: Let X and Y be sets, and h: $X \to Y$ a map. The equivalence relation $\sim$ in X defined by $x_1 \sim x_2$ iff $h(x_1) = h(x_2)$ is called the equivalence relation in X associated to h. In the canonical decomposition of h: $X \to Y$ into



the uniquely determined bijective map: $X/\sim \to h(X)$, called the map induced by h on passing to the quotient space ([2] II, 6,5, C57), is denoted by $\tilde{h}$, where $incl^{Y}_{h(X)}$ is the inclusion: $h(X) \to Y$. For simplicity, $incl^{Y}_{h(X)} \circ \tilde{h}$ is also denoted by $\tilde{h}$.

1. *Second Decomposition Theorems.*

(AS) Abstract Systems.

Definition 1. A morphism (h, ϕ): $\pi \to \rho$ in $GH(n - ?)$ is said to be of type n - ? - II* iff

(II*) for every pair $(x_1, x_2)(\in X \times X)$ such that $h(x_1) = h(x_2)$, w

have $\phi_{x_1}(t) = \phi_{x_2}(t)$ for all $t \in D_{x_1} \cap D_{x_2}$.

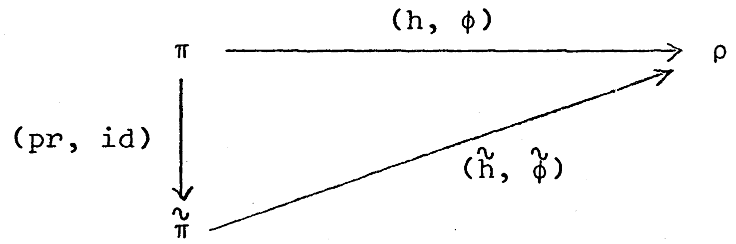
The class of type n - ? -II* GH-morphisms is denoted by mor GH(n - ? - II*)($\subset$ mor GH(n - ?)).

Remark 1. For n = -1, 0, 1, mor GH(n - ? - II*) = mor GH(n - ?). However for n = 2, 4, mor GH(n - ? - II*) does not define a subcategory of GH(n - ?). On the other hand

mor GH(0) = mor GH(0 - ? - II*) $\subset$ mor GH(1 - ?)

= mor GH(1 - ? - II*) $\subset$ mor GH(2 - ? - II*) $\subset$ mor GH(4 - ? - II*).

Theorem 1 (Second Decomposition Theorem). Let (h, ϕ): $\pi \to \rho$ be in mor GH(n - ? - II*). Then the equivalence relation $\sim$ in X associated to h is compatible with π, so that a unique abstract system $(\tilde{X}, \tilde{D}, \tilde{\pi})$ exists such that $\tilde{X} = X/\sim$ and (pr, id): $\pi \to \tilde{\pi}$ is an epimorphism in GH(0). Further, there exists a unique monomorphism $(\tilde{h}, \tilde{\phi})$: $\tilde{\pi} \to \rho$ in GH(n - ?) such that



commutes (the second decomposition).

(CS) Continuous Systems.

Definition 1. mor GH(n·m - ? - II*) is understood without any further explanation.

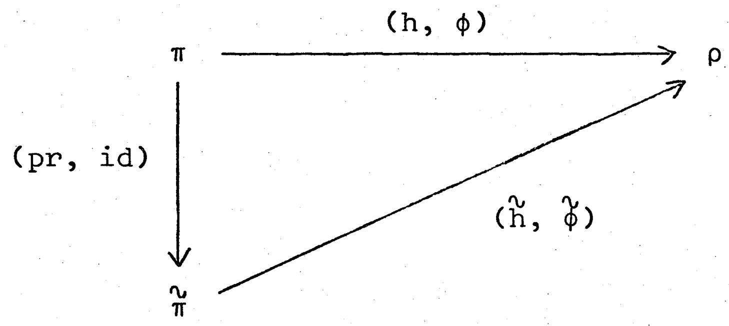
Theorem 1 (Second Decomposition Theorem). Let (h, ϕ): $\pi \to \rho$ be in mor GH(n·m - ? - II*) and $\sim$ the equivalence relation in X associated to h: $X \to Y$. There exists a continuous system

$\tilde{\pi}$ on $(\tilde{X}, \tilde{\mathcal{O}}) = (X, \mathcal{O})/\sim$ such that (pr, id): $\pi \to \tilde{\pi}$ is an epimorphism in GH(0) iff

$(\widetilde{II}_{0,a})$ D contains an open neighborhood of $X \times \{0\}$ in $(X, \mathcal{O}) \times R_0$ saturated w. r. to $(\sim x=)_D$ and

$(\widetilde{II}_{1,a})$ every open saturated subset of D ($\subset (X, \mathcal{O}) \times R_0$) w. r. to $(\sim x=)_D$ is the intersection of D and an open subset of $(X, \mathcal{O}) \times R_0$ saturated w. r. to $\sim x=$ (in $X \times R$).

If such $\tilde{\pi}$ exists, then it is unique and there exists $\tilde{\phi}$: $\tilde{D} \to R$ such that $(\tilde{h}, \tilde{\phi})$: $\tilde{\pi} \to \rho$ is a monomorphism in GH(n•m′ - ?) and the diagram



commutes (the second decomposition). Here in general $m' = 5$ and if $h(X)(\subset (Y, \mathcal{G}))$ is Hausdorff, then $m' = 25$.

Remark 1. In general, $(\widetilde{II}_{0,a})$ and $(\widetilde{II}_{1,a})$ may not be satisfied, and are independent.

Corollary 1. Let (h, ϕ): $\pi \to \rho$ be as in Theorem 1. If h: $(X, \mathcal{O}) \to (Y, \mathcal{G})$ is open, then we have the second decomposition of (h, ϕ). (This corollary will be improved later, see Corollary 1 to Theorem 1 in no. 2, (CS) below.)

Corollary 2. Let (h, ϕ): $\pi \to \rho$ be as in Theorem 1. If for every pair (x_1, x_2) ($\in X \times X$), with $h(x_1) = h(x_2)$, we have $D_{x_1} = D_{x_2}$, then we have the second decomposition of (h, ϕ).

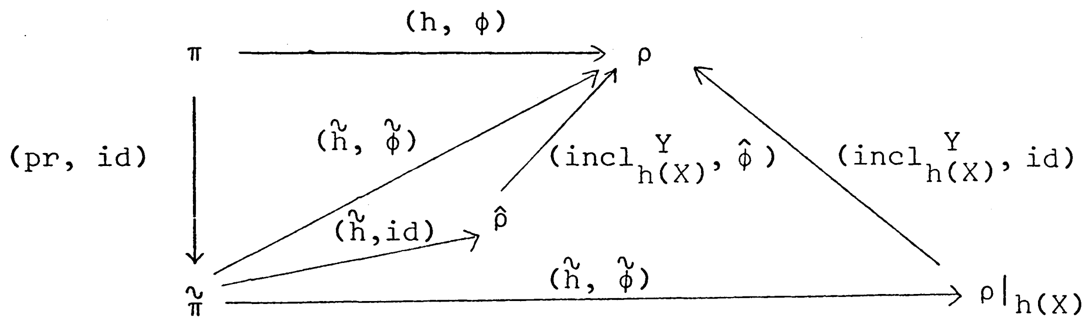
2. *Homomorphism Theorems.*

(AS) Abstract Systems.

Proposition 1. If $(h, \phi): \pi \to \rho$ is a morphism in $GH(n - I)$, then $h(X)$ is an abstract-dynamically admissible subset of Y w. r. to ρ.

Theorem 1 (Homomorphism Theorem I). Let $(h, \phi): \pi \to \rho$ be in mor $GH(n - BI - II^*)$, then $\rho|_{h(X)}$ is an abstract system and there exists a unique abstract system $\tilde{\pi}$ on $\tilde{X} = X/\sim$ and an abstract system $\hat{\rho}$ on $\hat{Y} = h(X)$ such that the diagram

(D I)

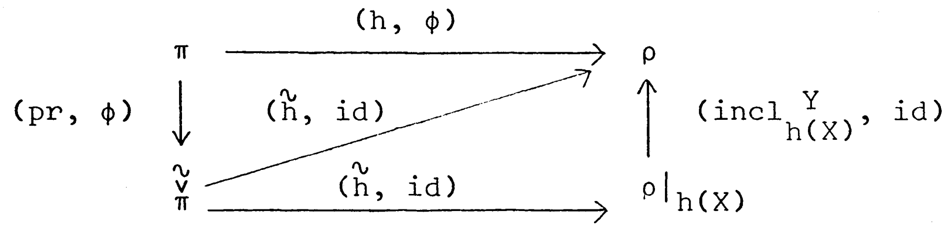


commutes, where $(\tilde{h}, id): \tilde{\pi} \to \hat{\rho}$ is an isomorphism in $GH(0)$, $(incl^Y_{h(X)}, id): \rho|_{h(X)} \to \rho$ is a morphism in $GH(-1)$ and $(\tilde{h}, \tilde{\phi}): \tilde{\pi} \to \rho|_{h(X)}$ is an isomorphism in $GH(n - BI)$.

If further, (h, ϕ) is an epimorphism in $GH(n - BI - II^*)$, then $(\tilde{h}, \tilde{\phi}): \pi \to \rho$ is a bimorphism in $GH(n - BI)$ and if further, $h: X \to Y$ is surjective, then $(\tilde{h}, \tilde{\phi}): \tilde{\pi} \to \rho$ is an isomorphism in $GH(n - BI)$.

Theorem 2 (Homomorphism Theorem II). Let $(h, \phi): \pi \to \rho$ be in mor $GH(n - BI)$, then $\rho|_{h(X)}$ is an abstract system and there exists an abstract system $\tilde{\check{\pi}}$ on $\tilde{X} = X/\sim$ such that the diagram

(D II)



commutes, where $(\tilde{h}, id): \pi \to \rho|_{h(X)}$ is an isomorphism in $GH(0)$, $(incl^{Y}_{h(X)}, id): \rho|_{h(X)} \to Y$ is a morphism in $GH(-1)$ and $(pr, \phi): \pi \to \check{\tilde{\pi}}$ is in mor $GH(n - BI)$.

If, further, (h, ϕ) is an epimorphism in $GH(n - BI)$, then $(\tilde{h}, id): \check{\tilde{\pi}} \to \rho$ is an epimorphism in $GH(0)$, and if, further, $h: X \to Y$ is surjective, $(\tilde{h}, id): \tilde{\pi} \to \rho$ is an isomorphism in $GH(0)$.

Remark 1. If $n = -1$, these theorems are superfluous. If $n = 0$, mor $GH(n - BI - II^*)$ equals to mor $GH(0)$ and if $n = 1$, mor $GH(n - BI)$ equals to mor $GH(1 - I)$.

(CS) Continuous Systems.

Theorem 1 (Homomorphism Theorem I). Let $(h, \phi): \pi \to \rho$ be in mor $GH(n \cdot m' - BI - II^*)$ and $\sim$ the equivalence relation in X associated to $h: X \to Y$. There exist a unique continuous system $\tilde{\pi}$ on $(\tilde{X}, \tilde{O}) = (X, O)/\sim$, a unique continuous system $\hat{\rho}$ on $h(X)(\subset (Y, G))$, and a unique isomorphism $(\tilde{h}, id): \tilde{\pi} \to \tilde{\rho}$ in $GH(0)$ and a unique isomorphism $(\tilde{h}, \tilde{\phi}): \tilde{\pi} \to \rho|_{h(X)}$ in $GH(n \cdot m' - BI)$ such that the diagram (D I) in (AS) commutes, regarded as a diagram in $GH(n \cdot m')$, $(incl^{Y}_{h(X)}, \hat{\phi}): \hat{\rho} \to \rho$ being in mor $GH(n \cdot m' - BI)$ and $(incl^{Y}_{h(X)}, id): \rho|_{h(X)} \to \rho$ being in mor $GH(-1)$, iff $(\tilde{II}_{0,a})$ and $(\tilde{II}_{1,a})$ in Theorem 1 of no. 1 (CS) and the following conditions (S) and $(\tilde{III}_h)$ are satisfied:

(S) $h(X)$ is a quasi-invariant set of $\rho|_V$ for some $V \in G$.

$(\tilde{III}_h)$ For every $O \in O$ saturated w. r. to $\sim$, $h(O)$ is open in $h(X)(\subset (Y, G))$.

Here in general $m' = 5$, and if $h(X)(\subset (Y, G))$ or equivalently $(X, O)/\sim$ is Hausdorff, then $m' = 25$.

Corollary 1. Let $(h, \phi): \pi \to \rho$ and $\sim$ be as in the preceeding theorem. If $h: (X, \mathcal{O}) \to (Y, \mathcal{G})$ is open, then we have the same decomposition of (h, ϕ) as in theorem 1.

Theorem 2 (Homomorphism Theorem II). Let $(h, \phi): \pi \to \rho$ be in mor $GH(n\cdot 0 - BI)$ and $\sim$ the equivalence relation in X associated to $h: X \to Y$. There exist a continuous system $\check{\tilde{\pi}}$ on $(\tilde{X}, \tilde{\mathcal{O}}) = (X, \mathcal{O})/\sim$, and an isomorphism $(\tilde{h}, id): \check{\tilde{\pi}} \to \rho|_{h(X)}$, $h(X)$ being continuously admissible w. r. to ρ such that the diagram (D II) in (AS) commutes, regarded as a diagram in $GH(n\cdot 0)$, where $(pr, \phi): \pi \to \check{\tilde{\pi}}$ is an epimorphism in $GH(n\cdot 0 - BI)$ and $(incl^{Y}_{h(X)}, id): \rho|_{h(X)} \to \rho$ is a morphism in $GH(-1)$ iff the same conditions as in Theorem 1 are satisfied.

Corollary 1. Let $(h, \phi): \pi \to \rho$ and $\sim$ be as in the preceeding theorem. If $h: (X, \mathcal{O}) \to (Y, \mathcal{G})$ is open, then we have the same decomposition of (h, ϕ) as in Theorem 2.

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