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Non-existence of a normal conditional expectation in a continuous crossed product

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1. Indtroduction. In the study of a discrete crossed product, we are able to make use of next two general principles, one is "any element in a discrete crossed product can be represented as an operator valued function", the other is "there exists a faithful normal conditional expectation of a discrete crossed product onto the original von Neumann algebra". Unfortunately these principles are no more valid in the case of a continuous crossed product. M. Takesaki [8] succeeded in expressing any factor of type III as a crossed product of a von Neumann algebra of type II_∞ by the modular action.

H. Takai [7] verified that any operator in the crossed product with the continuous action of real numbers can be expressed in a form of a distribution on some space instead of a matrix-form.

U. Haagerup [4], on the other side, has mentioned that if, in place of a faithful normal conditional expectation, one would like to have a normal semi-finite operator valued weight from a crossed product into the original von Neumann algebra, then such an object always can be found.

In this paper, we will show the fact that there is no normal conditional expectation in a crossed product with a continuous action of a locally compact connected group.

2. Notation and Preliminary. Let $\mathfrak{A}$ be a von Newmann algebra on a Hilbert space $\mathfrak{H}$ and G be a locally compact group. The triple $(\mathfrak{A}, G, \alpha)$ is said a continuous W^*-dynamical system if the mapping α of G into the group Aut $(\mathfrak{A})$ of all automorphisms of $\mathfrak{A}$ is a homomorphism and the function $G \to \omega \circ \alpha_g(x)$ is continuous on G for any $x \in \mathfrak{A}$ and $\omega \in \mathfrak{A}_*$. ($\mathfrak{A}_*$ is the predual of $\mathfrak{A}$.)

$G \otimes_\alpha \mathfrak{A}$ is the von Neumann algebra generated by the family of the operator $\{ \pi_\alpha(x), \lambda(g) \; ; \; x \in \mathfrak{A}, g \in G\}$;

$$(\pi_\alpha(x)\zeta)(h) = \alpha_{h^{-1}}(x)\zeta(h) \; , \quad \zeta \in L^2(G, \mathfrak{H})$$
$$(\lambda(g)\zeta)(h) = \zeta(g^{-1}h) \quad , \quad \zeta \in L^2(G, \mathfrak{H}).$$

Π_α is then a normal isomorphism of $\mathfrak{A}$ onto $\pi_\alpha(\mathfrak{A})$, and we often identify the von Neumann algebra of $\mathfrak{A}$ with the von Neumann algebra $\pi_\alpha(\mathfrak{A})$ (See [8]).

Let T be a mapping of a von Neumann algebra M onto a von Meumann subalgebra N of M.

Definition 2.1. T is called a conditional expectation of M onto N if T has the following properties;

(i) $T(1) = 1$, where 1 is the identity operator of M.

(ii) $T(axb) = a(Tx)b$, for all $a, b \in N$, $x \in M$.

Moreover T is called normal if ${}^tT(N_*) \subset M_*$

Let ϕ be an automorphism of a von Neumann algebra M.

Definition 2.2. ϕ is freely action if the element x of M with the property that $x\phi(y) = yx$ for any $y \in M$ cannot be other than zero.

For each automophism ψ of M, there is a unique central projection q of M such that

(i) $\psi(q) = q$

(ii) $\psi|_{M_q}$ is an inner automorphism of M_q

3. Main results.

Theorem 3.1. Let $(\mathfrak{A}, G, \alpha)$ be a W*-dynamical system with $\sup\{p(\alpha_g) ; g \in G, g \neq e\} \neq 1$, where e is the identity of G.

The following statements are equivalent ;

(i) G is a discrete group

(ii) there exists a normal conditional expectation of $G \otimes_\alpha \mathfrak{A}$ onto $\mathfrak{A}$.

Remark 3.2. If G is a discrete group, there is a faithful normal conditional expectation of $G \otimes_\alpha \mathfrak{A}$ onto $\mathfrak{A}$. (cf. [2] Proposition 1,4,6)

Before we prove the implication of (i) from (ii), we will give two lemmas, one of them (Lemma 3.3.) will have a repeated use in the whole of our study.

Lemma 3.3. Let T be a conditional expectation of $G \otimes_\alpha \mathfrak{A}$ onto $\mathfrak{A}$. We then have $T(\lambda(g))_{(1 - p(\alpha_g))} = 0$ for any $g \in G$.

Proof. For $y \in \mathfrak{A}_{(1 - p(\alpha_g))}$, we have ;

$$y\,T(\lambda(g)^*) = T(y\lambda(g)^*) = T(\lambda(g)^*\lambda(g)y\lambda(g)^*),$$

since $\lambda(g)y\lambda(g)^* = \alpha_g(y)$ is an element of ,

$$y\,T(\lambda(g)^*) = T(\lambda(g)^*)\alpha_g(y) .$$

Therefore $T(\lambda(g)^*)_{(1 - p(\alpha_g))} = 0$ becouse α_g is a freely acting automorphism of $\mathfrak{A}_{(1 - p(\alpha_g))}$.

Lemma 3.4. $\sup\{p(\alpha_g) ; g \in G, g \neq e\}$ is a G-invariant central projection of $\mathfrak{A}$.

Proof. For any $x \in \mathfrak{A}$, g, h $\in$ G with $g \neq e$, we have

$$\alpha_{hgh^{-1}}(x\alpha_h(p(\alpha_g))) = \alpha_h(U)x\alpha_h(p(\alpha_g))\alpha_h(U)^* ,$$

where $\alpha_g|\mathfrak{A}_{p(\alpha_g)} = AdU$, $U^*U = p(\alpha_g)$, $UU^* = p(\alpha_g)$.

Therefore we get $\alpha_h(p(\alpha_g)) \leq p(\alpha_{hgh^{-1}})$, so that

$$\alpha_h(\sup\{p(\alpha_g) ; g \in G, g \neq e\}) \leq \sup\{p(\alpha_g) ; g \in G, g \neq e\} .$$

Hence $\sup\{p(\alpha_g) ; g \in G, g \neq e\}$ is a G-invariant central projection of $\mathfrak{A}$.

[The proof of Theorem 3.1.] Lemma 3.4 implies that it is sufficient to prove the theorem in the case where $P(\alpha_g) = 0$ for all $g \in G$ except the identity e.

Let T be a normal conditional expectation of $G \otimes_\alpha \mathfrak{A}$ onto $\mathfrak{A}$. By [5] Lemma 2.12, there is a left Hilbert algebra $\mathfrak{U}$ such that the crossed product $G \otimes_\alpha \mathfrak{A}$ is the left von Neumann algebra $\mathcal{L}(\mathfrak{U})$ of $\mathfrak{U}$, more precisely

$$\pi_\ell(\xi) = \int_G \pi_\alpha(x(g))\lambda(g)dg \quad , \qquad \xi \in \mathfrak{U}$$

where $x(g) = \alpha_g(\pi_\ell(\xi(g)))$ is an element of $\mathfrak{A}$ and the integration is with respect to the left Haar measure of G which will also be denoted by μ.

Since T is normal and $T(\lambda(g)) = 0$ for any $g \in G$ except e, we have

$$T(\pi_\ell(\xi)) = \int_G \pi_\alpha(x(g))T(\lambda(g))dg = \pi_\alpha(x(e))\mu(e).$$

As $\pi_\ell(\mathfrak{U})$ is a σ-weakly dense *-subalgebra of $G \otimes_\alpha \mathfrak{A}$, $\mu(e)$ must be a positive number, so G must be a discrete group.

Remark 3.5. Let $(\mathfrak{A}, G, \alpha)$ be a W*-dynamical system. U is a strongly continuous unitary representation of G on the same Hilbert space $\mathfrak{H}$ that the von Neumann algebra $\mathfrak{A}$ acts on, we suppose that $\alpha_g = AdU_g$, $U_g \in \mathfrak{A}$ for any $g \in G$.

We define then the unitary V on $L^2(G, \mathfrak{H})$ by

$$(V\xi)(g) = U_g \xi(g)$$

for $\xi \in L^2(G, \mathfrak{H})$.

We get ;

$$V\pi_\alpha(x)V^* = x \otimes 1 \qquad \text{for any } x \in \mathfrak{M}$$

$$V\lambda(g)V^* = U_g \otimes \rho(g) \qquad \text{for any } g \in G$$

where ρ is the left regular representation of G.

We therefore, get

$$V(G \otimes_\alpha \mathfrak{M})V^* = \mathfrak{M} \otimes \rho(G)''$$

$$V\pi_\alpha(\mathfrak{M})V^* = \mathfrak{M} \otimes 1 .$$

We thus know that there are many normal conditional expectations of $G \otimes_\alpha \mathfrak{M}$ onto $\mathfrak{M}$, according to the result of [9] Theorem 1.1. that there are many left slice mappings on $\mathfrak{M} \otimes \rho(G)''$.

The above example shows that the condition $\sup \{p(\alpha_g) ; g \in G, g \neq e\} \neq 1$ is neccesary to prove theorem 3.1.

Next we impose some restriction on the group in a W*-dynamical system, in order to get rid of the condition $\sup \{p(\alpha_g) ; g \in G, g \neq e\} \neq 1$ in Theorem 3.1., and we will give a decisive result about the existence of a normal expectation.

Theorem 3.5. Let G be a locally compact connected group and $(\mathfrak{M}, G, \alpha)$ be a W*-dynamical system. If there is an element g_0 in G such that α_{g_0} is an outer automorphism of $\mathfrak{M}$, there then does not exist any normal conditional expectation of $G \otimes_\alpha \mathfrak{M}$ onto $\mathfrak{M}$.

Proof. Let T be a normal conditional expectation of $G \otimes_\alpha \mathfrak{M}$ onto $\mathfrak{M}$, which we suppose to exist.

Assume first that there is an element h in G such that h is on a one-parameter subgroup x(t) at t = s and $\alpha_h = \alpha_{x(s)}$ is an outer automosphism of $\mathfrak{N}$, $p(\alpha_{x(s)})$ is then a central projection of $\mathfrak{N}$ which is not the identity operator of $\mathfrak{N}$.

For any $n \in \mathbb{N}$, we get ;

$$P(\alpha_{x(\frac{s}{n})}) \leq p(\alpha_{x(s)})$$

because of $(\alpha_{x(\frac{s}{n})})^n = \alpha_{x(s)}$.

It follows from Lemma 3.1. that $T(\lambda(x(\frac{s}{n})))_{(1 - P(\alpha_{x(\frac{s}{n})}))} = 0$

so that $T(\lambda(x(\frac{s}{n})))_{(1 - P(\alpha_{x(s)}))} = 0$ for any $n \in \mathbb{N}$.

Therefore we get $T(\lambda(e))_{(1 - p(\alpha_{x(s)}))} = 0$ so that $1 = p(\alpha_{x(s)})$ which is a contradiction. So the assumed situation does never take place.

When an element g in G is on a one-parameter subgroup in G, we write $e \sim g$. By the above arguement, α_h must be an inner automorphism of $\mathfrak{N}$ for any h in $\{g \in G, e \sim g\}$. Now, G is the closed group K generated by $\{g \in G, e \sim g\}$. Indeed, suppose that there is an element g in G and an open neighborhood U of e in G such that the intersection of g U and K is empty. By [10] Theorem 4.6. there exists in U a compact normal subgroup H such that G/H is a Lie group. Then there is a neighborhood V of e in G such that V contains H and each point of V/H is on a one-parameter subgroup in G/H. Since G/H is also a connected group, G/H is the group generated by V/H, so that there are a finite subset $\{g_iH, i = 1, 2, \ldots, n\}$ in G/H, and one-parameter subgroup $x_i(t)$ $(i = 1, 2, \ldots, n)$ in G/H such that $\prod_{i=1}^{n} g_iH = gH$ and g_iH is on the one-parameter subgroup $x_i(t)$ at $t = s_i$ $(i = 1, 2, \ldots, n)$. By [10] Theorem 4.15. there are one-parameter groups $y_i(t)$ $(i = 1, 2, \ldots, n)$ in G such that $y_i(t)H = x_i(t)$ for any $t \in \mathbb{R}$ $(i = 1, 2, \ldots, n)$. The element $g^{-1} \prod_{i=1}^{n} y_i(s_i)$ is contained in $H \subset U$ because of $\prod_{i=1}^{n} y_i(s_i)H = gH$. Then the intersection of K and gV is not empty since $\prod_{i=1}^{n} y_i(s_i)$ is contained in both K and gV, which is a contradiction.

As the group generated by $\{g \in G, e \sim g\}$ was shown to be dense in G, there is a net $\{g_i\}_{i \in I}$ in this group such that it converges to g_0 in G, g_0 being the element in the statement of the theorem. Since α_h is an inner automorphism of $\mathfrak{M}$ for any h in $\{g \in G, e \sim g\}$, α_{g_i} are also inner automorphisms of $\mathfrak{M}$ for all $i \in I$.

Then we get ;

$$P(\alpha_{g_i^{-1}g}) = p(\alpha_{g_0})$$

$$T(\lambda(g_i^{-1}g))(1 - P(\alpha_{g_i^{-1}g})) = T(\lambda(g_i^{-1}g))(1 - P(\alpha_{g_0})) = 0$$

, so that $1 - P(\alpha_{g_0}) = 0$, which is not the case.

We get thus a contradiction and the proof is complete.

Remark 3.6. If the group is not supposed connected, there are W*-dynamical system with a non-discrete locally compact group such that there is an element h in G with a freely acting automorphism α_h of $\mathfrak{M}$ and there is a normal conditional expectation of $G \otimes_\alpha \mathfrak{M}$ onto $\mathfrak{M}$.

Remark 3.7. In this paper we study the existence of *normal* conditional expections. But by using [1], [3] we have an example of W*-dynamical system $(\mathfrak{M}, G, \alpha)$ where the von Neumann algebra $\mathfrak{M}$ is a hyperfinite factor of type II_1 such that α_g is freely acting automorphisms of $\mathfrak{M}$ for any $g \in G$ expect e and such that there exists a conditional expectation (non-normal) of $G \otimes_{\alpha} \mathfrak{A}$ onto $\mathfrak{A}$.

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